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ON A CERTAIN CLASS OF SUBMEASURES BASED ON TRIANGULAR NORMS

ONDREJ HUTNÍK

Institute of Mathematics, Faculty of Science,
Pavol Jozef Šafárik University in Košice, Jesenná 5, 041 54 Košice, Slovakia
ondrej.hutnik@upjs.sk

RADKO MESIAR

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering,
Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia; and
IRAFM University of Ostrava, 701 03 Ostrava, Czech Republic.
mesiar@math.sk

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In this paper we study a generalization of a submeasure notion which is related to a probabilistic concept, especially to Menger spaces where triangular norms play a crucial role. The resulting notion of a τ_T-submeasure is suitable for modeling those situations in which we have only probabilistic information about the measure of the set. We characterize a class of universal τ_T-submeasures (i.e., τ_T-submeasures for an arbitrary t-norm T) and give explicit formulas for τ_T-submeasures for some classes of t-norms. Also, transformations and aggregations of τ_T-submeasures are discussed.

Keywords: probabilistic metric space, submeasure, triangular norm, additive generator, Menger space

1. Introduction

Non-additive set functions have appeared naturally earlier in classical measure theory dealing with countable additive set functions or more general finite additive set functions (charges), e.g., outer measures, semi-variations of vector measures, and capacities are widely known examples of such functions. Many authors have investigated different kinds of non-additive set functions, e.g., triangular and k-triangular set functions [2], null-additive set functions [19], fuzzy measures and integrals [11,18]. Nowadays, they are extensively used in many areas of mathematics. Most of the naturally arising non-additive set functions satisfy some subadditivity (or, more restrictive, submodular) conditions which fairly well recompense the lack of additivity. Much attention was paid there to develop a theory of submeasures [8] (which will be called numerical submeasures in this paper), Dobrakov submeasures [6] and semimeasures [7] (which need not be subadditive in general!), and their various generalizations and extensions [10,13,14,20]. The main motivation for developing a theory of submeasures is that submeasures are used as a convenient tool in investigating some properties of measures, e.g., uniform σ-additivity, equi-continuity, absolute continuity, etc. Also, many set functions arising naturally in the study of group-valued measures and covering problems are submeasures.

Let Σ be a ring of subsets of a fixed (non-empty) set Ω and $\overline{\mathbb{R}}_+ = [0, \infty]$ be the extended positive real line. A mapping $\eta : \Sigma \to \overline{\mathbb{R}}_+$ which is non-decreasing, subadditive and vanishing on the empty set is said to be a *numerical submeasure*. More precisely,

(i) $\eta(\emptyset) = 0$;
(ii) $\eta(A) \leq \eta(B)$ for $A, B \in \Sigma$ such that $A \subset B$;
(iii) $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \Sigma$.

A pair (Σ, η) will be called a *numerical submeasure ring*.

In this paper we investigate a submeasure notion which is related to probabilistic metric spaces [17], in particular to a class of functions known as triangular norms [16] (which arose naturally from Menger's ideas in probabilistic geometry). Replacing the Fréchet distance $d(p, q)$ between p, q on a non-empty set Ω by a real function $F_{p,q}(x)$ (interpreted as the probability that distance between elements p, q of a non-empty set is less than x) one obtains the following:

$$p = q \Rightarrow F_{p,q} = 1, \quad x > 0, \tag{1}$$

$$p \neq q \Rightarrow F_{p,q} < 1 \quad \text{for some } x > 0, \tag{2}$$

$$F_{p,q} = F_{q,p}. \tag{3}$$

This led Menger [17] to introduce the so called 'statistical metric spaces' (later called probabilistic metric spaces) as a set Ω together with a family of probability functions $F_{p,q}$ with $F_{p,q}(0) = 0$ satisfying (1), (2), (3) and the 'probabilistic analogue' of the triangle inequality expressed by

$$F_{p,r}(x + y) \geq T(F_{p,q}(x), F_{q,r}(y)), \tag{4}$$

which holds for all $p, q, r \in \Omega$ and all real x, y. Here T is meant as a binary operation on $[0, 1]$ which is symmetric, non-decreasing in each argument, fulfilling $T(a, 0) > 0$ whenever $a > 0$ and the boundary condition $T(1, 1) = 1$.

The inequality (4) has fallen in oblivion until its rediscovery by Schweizer and Sklar [23] involving a particular class of functions T, namely triangular norms (t-norms for short), see [3,15,16,24] for an excellent overview on the mentioned topics and its developments.

The inequality (4) will serve as a starting point for our study. Especially, we are interested in such a mapping γ from Σ to Δ^+ (the class of all distance distribution functions) which is closely related to (4) and may be understood as an extension of a submeasure notion in a probabilistic context. This notion of a submeasure is suitable for modeling those situations in which we have only probabilistic information about the measure of the set. Since it depends on a t-norm T we propose to call it a τ_T *-submeasure.*

In the next section some necessary notions and definitions from probabilistic metric spaces and triangular norms theory are summarized. In Section 3 the notion of a τ_T-submeasure is given and a class of universal τ_T-submeasures (i.e., τ_T-submeasures for an arbitrary t-norm T) is characterized. Some explicit formulas for τ_T-submeasures related to the class of Archimedean t-norms having an additive generator are stated. Also, the properties of τ_T-submeasures in connection with φ-transformations of a t-norm T are investigated. Several examples of τ_T-submeasures are given. Aggregation of τ_T-submeasures is discussed in the last section.

2. Basic definitions

Let Δ be the family of all distribution functions on the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ (i.e., $F : \overline{\mathbb{R}} \to [0,1]$ is non-decreasing, left continuous on $\mathbb{R}$, $F(-\infty) = 0$ and $F(\infty) = 1$) equipped with the usual pointwise order $\leq$ (i.e., for $F, G \in \Delta$, $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in \overline{\mathbb{R}}$). Denote by ε_a the unit steps in Δ, i.e., the distribution functions defined for $a \in [-\infty, +\infty[$ by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{for } x \leq a, \\ 1, & \text{for } x > a, \end{cases}$$

while for $a = +\infty$,

$$\varepsilon_\infty(t) = \begin{cases} 0, & \text{for } t < +\infty, \\ 1, & \text{for } t = +\infty. \end{cases}$$

A *distance distribution function* is a distribution function whose support is a subset of $\overline{\mathbb{R}}_+$, i.e., a distribution function $F : \overline{\mathbb{R}} \to [0,1]$ with $F(0) = 0$. The class of all distance distribution functions will be denoted by Δ^+. Clearly, $\varepsilon_a \in \Delta^+$ if and only if $a \geq 0$, and ε_0 is the maximal element of Δ^+. Distance distribution functions are a proper tool for measuring distances in probabilistic metric spaces [24]. For this purpose it is necessary to introduce the following notion which is particularly important in the generalization of the triangle inequality of metric spaces to probabilistic metric spaces, see (4).

Definition 1. A *triangle function* is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ which is symmetric, associative, non-decreasing in each variable and has ε_0 as the identity.

A mapping $T : [0,1]^2 \to [0,1]$ is called a *triangular norm* (t-norm for short) if it is symmetric, associative, non-decreasing in each argument and has 1 as the identity. A dual *t-conorm* S to a t-norm T is defined as $S(x,y) = 1 - T(1-x, 1-y)$. This duality allows to translate many properties of t-norms to the corresponding properties of t-conorms. For an extensive overview on t-norms and t-conorms see [3,15,16].

Example 1. The most important t-norms are the minimum M, the product Π, the Łukasiewicz t-norm W and the drastic product D given by:

$$\begin{aligned} M(x,y) &:= \min\{x,y\}; \\ \Pi(x,y) &:= xy; \\ W(x,y) &:= \max\{x+y-1,0\}; \\ D(x,y) &:= \begin{cases} \min\{x,y\}, & \max\{x,y\}=1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In the literature these t-norms are often denoted by T_M, T_P, T_L and T_D, respectively.

A t-norm T is *continuous* if it is continuous as a two-place function. A continuous t-norm T is *Archimedean* if $T(x,x) < x$ for all $x \in]0,1[$. A t-norm T has *zero-divisors* if $T(x,y) = 0$ for some $x,y \in]0,1[$. A t-norm is *strictly increasing* if $T(x,y) < T(x,z)$ whenever $x,y,z \in]0,1[$ and $y < z$. A continuous Archimedean t-norm with zero-divisors is called *nilpotent*. A continuous Archimedean t-norm is called *strict* if it is strictly increasing.

Definition 2. Let Ω be a non-empty set, $\mathcal{F} : \Omega \times \Omega \to \Delta^+$ a function which assigns to each pair $(p,q) \in \Omega \times \Omega$ a distance distribution function $F_{p,q} \in \Delta^+$, and $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ a triangle function. The triple $(\Omega, \mathcal{F}, \tau)$ is called a *probabilistic metric space* (in the sense of Šerstnev [25]) if the following properties hold for all $p,q,r \in \Omega$:

(1) $F_{p,q} = \varepsilon_0$ if and only if $p = q$;
(2) $F_{p,q} = F_{q,p}$;
(3) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$.

Our considerations of a submeasure notion are closely related to probabilistic metric space associated with a particular type of triangle function, namely

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u),G(v)), \tag{5}$$

where T is a left-continuous t-norm (however, this is not necessary, e.g., if $T = D$, then τ_T is a triangle function [22,24]). The triple $(\Omega, \mathcal{F}, \tau_T)$ is called a *Menger probabilistic metric space under T* (a Menger space, for short [24]). Clearly, different t-norms lead to different triangle functions (and also to spaces with different geometrical and topological properties). Indeed, for all $p,q,r \in \Omega$ and for all $x,y \in \overline{\mathbb{R}}_+$ we obtain the inequality (4). Note that the inequality (4) may be interpreted in the way of the classical metric spaces as follows: the third side in a triangle depends on the other two sides in the sense that if the knowledge of two sides increases than also the knowledge of third side increases or that knowing the upper bounds of two sides we have an upper bound for the third side (thus the attribute 'triangle inequality' is legitimate).

3. τ_T-submeasures

Motivated by the inequality (4) let us define notion of a submeasure as a mapping from a ring of subsets of M to the space of all distance distribution functions as follows:

Definition 3. Let $T : [0,1]^2 \to [0,1]$ be a t-norm, and Σ a ring of subsets of Ω. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(A)$ is denoted by γ_A) such that

(a) if $A = \emptyset$, then $\gamma_A(x) = \varepsilon_0(x)$, $x > 0$;
(b) if $A \subset B$, then $\gamma_A(x) \geq \gamma_B(x)$, $x > 0$;
(c) $\gamma_{A\cup B}(x+y) \geq T(\gamma_A(x), \gamma_B(y))$, $x, y > 0$, $A, B \in \Sigma$,

is said to be a τ_T*-submeasure* and the triple (Σ, γ, T) is called a τ_T*-submeasure ring*.

Remark 1. We prefer to use the notion of τ_T-submeasure instead of the more natural formulation of T-submeasure to underline the very close relationship between property (c) and the triangle inequality (4) where in the context of Menger spaces the triangle function (5) plays a crucial role. Indeed, the property (c) means that $\gamma_{A\cup B} \geq \tau_T(\gamma_A, \gamma_B)$. The second reason is the fact that the concept of a triangular norm-based measure (or T-measure) was introduced in [4] and we want to avoid possible misunderstandings when speaking about measures in our case, see Section 5.

Remark 2. In virtue of known results on triangle functions presented in [22] it is also possible to define and study a more general notion of submeasures related to probabilistic metric spaces with the (triangle) function

$$\tau_{L,f}(F,G)(x) = \sup_{L(u,v)=x} f(F(u), G(v)).$$

Here f is a mapping from $[0,1]^2$ to $[0,1]$, and L is a binary operation on $\overline{\mathbb{R}}_+$ which is non-decreasing in both components, continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$, except possibly at the points $(0,\infty)$ and $(\infty,0)$, and $\mathrm{Ran}_L = \overline{\mathbb{R}}_+$. The most usual example of such operation is

$$L_k(x,y) = (x^k + y^k)^{\frac{1}{k}}, \quad k > 0,$$

see [22,24]. A mapping $\gamma : \Sigma \to \Delta^+$ will be called a $\tau_{L,f}$*-submeasure*, if it satisfies (a), (b) and

(c*) $\gamma_{A\cup B}(L(x,y)) \geq f(\gamma_A(x), \gamma_B(y))$, $x, y > 0$, $A, B \in \Sigma$.

Clearly, if $L = L_1$ and f is a t-norm, then a notion of $\tau_{L,f}$-submeasure reduces to a notion of τ_T-submeasure. It is easy to observe that every τ_{L,f_1}-submeasure is also a τ_{L,f_2}-submeasure if $f_1 \geq f_2$.

For instance, $L(x,y) = L_\infty(x,y) = \max\{x,y\}$ and $f = T$ (a t-norm) yields the inequality

$$F_{p,r}(\max\{x,y\}) \geq T(F_{p,q}(x), F_{q,r}(y)),$$

where the triplet $(\Omega, \mathcal{F}, \tau_{\max,T})$ is called a *non-Archimedean Menger space* [12,15]. However, we do not consider this general case of Menger spaces and related submeasure notions in this paper.

Let us mention that to any τ_T-submeasure ring (Σ, γ, T) one can assign a numerical submeasure ring (Σ, η_γ) as follows:

Proposition 1. *Let T be a t-norm, and Σ be a ring of subsets of Ω. If $\gamma : \Sigma \to \Delta^+$ is a τ_T-submeasure, then the set function $\eta_\gamma : \Sigma \to \overline{\mathbb{R}}_+$ given by*

$$\eta_\gamma(A) = \sup\{x \in \overline{\mathbb{R}}_+;\ \gamma_A(x) < 1\} \tag{6}$$

is a numerical submeasure.

The first natural question is whether for a given t-norm T there exists a corresponding τ_T-submeasure and whether such a τ_T-submeasure is unique for T. For this purpose we distinguish two cases:

(1) submeasures γ which are τ_T-submeasures for arbitrary t-norm T. Such submeasures will be called *universal τ_T-submeasures*;
(2) τ_T-submeasures whose structure, or representation, depend on the inner structure of T (e.g., additive generator, φ-transformation, etc.).

In the next two subsections we study these two classes of τ_T-submeasures.

3.1. *Universal τ_T-submeasures*

In this part we deal with the characterization of universal τ_T-submeasures which is closely related to solving a special functional inequality. Firstly, let us make some easy observations.

The question on existence and uniqueness may be partially solved by the comparison of t-norms. The partial (non-linear) order $\leq$ on the class of all t-norms may be defined as follows: $T_2 \leq T_1$ if and only if the inequality $T_2(x,y) \leq T_1(x,y)$ holds for all $(x,y) \in [0,1]^2$. In this case we say that t-norm T_1 is *stronger* than T_2 (or, equivalently, T_2 is weaker than T_1). Immediately, if T_1 is stronger than T_2, then each τ_{T_1}-submeasure is a τ_{T_2}-submeasure as well (i.e., a class of τ_{T_1}-submeasures is a subset of a class of τ_{T_2}-submeasures). Because of this observation in the case of universal τ_T-submeasures it is enough to find a τ_M-submeasure γ (because M is the pointwise largest, i.e., the strongest t-norm) and the question on existence is solved to the positive.

In what follows let us make the following convention: since Δ^+ is the set of all distribution functions with support $\overline{\mathbb{R}}_+$ it is sufficient to state the expression for a

τ_T-submeasure $\gamma : \Sigma \to \Delta^+$ just for values $x > 0$. In case $x \leq 0$ we always suppose $\gamma_\cdot(x) = 0$.

Our first result demonstrates the close relationship between the notion of a (universal) τ_T-submeasure and the probabilistic concept. Indeed, if (Σ, η) is a numerical submeasure ring, from the probabilistic point of view, the number $\gamma_A(x)$ in the following theorem may be interpreted as the probability that the submeasure $\eta(A)$ of A is less than x. From it follows that every numerical submeasure ring may be viewed as a τ_T-submeasure ring (of a special kind).

Theorem 1. *Let (Σ, η) be a numerical submeasure ring. Then*

$$\gamma_A(x) = \varepsilon_0(x - \eta(A)), \quad x > 0, \tag{7}$$

is a universal τ_T-submeasure.

Proof. It is easy to see that (a) and (b) of Definition 3 are satisfied. For (c) it is necessary to prove that $\gamma_{A\cup B}(x + y) \geq T(\gamma_A(x), \gamma_B(y))$ for an arbitrary t-norm T. According to the above remarks it is enough to prove it for the strongest t-norm M. Then

$$\gamma_{A\cup B}(x + y) = \varepsilon_0(x + y - \eta(A \cup B)) \geq \varepsilon_0(x - \eta(A) + y - \eta(B)).$$

If $x - \eta(A) < 0$ and $y - \eta(B) < 0$, then the inequality holds trivially because $\gamma_A(x) = \gamma_B(y) = \gamma_{A\cup B}(x + y) = 0$. If $x - \eta(A) \geq 0$ and $y - \eta(B) < 0$, then $M(\gamma_A(x), \gamma_B(y)) = 0$ and obviously $\gamma_{A\cup B}(x+y) \geq 0$. Analogously for $x - \eta(A) < 0$ and $y - \eta(B) \geq 0$. The only non-trivial case is $x - \eta(A) \geq 0$ and $y - \eta(B) \geq 0$.

Let us suppose that $0 \leq x - \eta(A) \leq y - \eta(B)$. Then $M(\gamma_A(x), \gamma_B(y)) = \gamma_A(x)$ and

$$\begin{aligned}\gamma_{A\cup B}(x + y) &\geq \varepsilon_0(x - \eta(A) + y - \eta(B)) \\ &\geq \varepsilon_0(x - \eta(A)) = \gamma_A(x) = M(\gamma_A(x), \gamma_B(y)).\end{aligned}$$

Analogously for $0 \leq y - \eta(B) \leq x - \eta(A)$. Thus,

$$\gamma_{A\cup B}(x + y) \geq M(\gamma_A(x), \gamma_B(y)),$$

i.e., γ is a τ_M-submeasure (and therefore a universal τ_T-submeasure). □

Remark 3. Observe that for any numerical submeasure $\eta : \Sigma \to \overline{\mathbb{R}}_+$ we have $\eta = \eta_\gamma$, where γ is given by formula (7) and η_γ by (6).

In what follows we deal with a more detailed characterization of the structure of universal τ_T-submeasures. For this purpose we describe our problem in terms of functional equations (functional inequalities, in fact). We have already mentioned that property (c) only depends on the t-norm T. In the case of universal τ_T-submeasures this property reads as follows

$$\gamma_{A\cup B}(x + y) \geq \min\{\gamma_A(x), \gamma_B(y)\}, \quad A, B \in \Sigma,\ x, y > 0.$$

By virtue of Definition 3 we want to find a function $f : [0, \infty[\times]0, \infty[\to [0, 1]$ satisfying the following properties:

(a') $f(0, x) = 1$ for all $x > 0$;
(b') $f(a, x)$ is non-increasing in its first and non-decreasing in its second component;
(c') f is a solution of the functional inequality

$$f(a + b, x + y) \geq \min\{f(a, x), f(b, y)\}, \tag{8}$$

for all $a, b \geq 0$, $x, y > 0$. Clearly, the functional inequality (8) characterizes the class of all universal τ_T-submeasures and all solutions of (8) satisfying (a') and (b') will be appropriate candidates for universal τ_T-submeasures. Note that both restrictions (a') and (b') are necessary to obtain universal τ_T-submeasures. Easily, we may observe that all functions additive in the sense $f(a+b, x+y) = f(a, x)+f(b, y)$ (more generally, superadditive) fulfil condition (8), but the first two conditions cannot be satisfied.

Theorem 2. *Let (Σ, η) be a numerical submeasure ring, and $\Phi \in \Delta$. Then a mapping $\gamma : \Sigma \to \Delta^+$ given by*

$$\gamma_A(x) = \Phi\left(\frac{cx}{\eta(A)}\right), \quad c > 0,\, x > 0,$$

is a parametric family of universal τ_T-submeasures (with convention $\frac{\infty}{\infty} = \infty$).

Proof. Conditions (a) and (b) of Definition 3 are satisfied immediately. Thus, for $A, B \in \Sigma$ and $x, y > 0$ suppose that $\gamma_A(x) = M(\gamma_A(x), \gamma_B(y))$, i.e., $\frac{x}{\eta(A)} \leq \frac{y}{\eta(B)}$. Then

$$\gamma_{A\cup B}(x + y) = \Phi\left(\frac{c(x + y)}{\eta(A \cup B)}\right) \geq \Phi\left(\frac{c(x + y)}{\eta(A) + \eta(B)}\right) \geq \Phi\left(\frac{cx}{\eta(A)}\right),$$

where the last inequality follows from the fact that

$$\frac{c(x + y)}{\eta(A) + \eta(B)} \geq \frac{cx}{\eta(A)} \Leftrightarrow \frac{y}{\eta(B)} \geq \frac{x}{\eta(A)},$$

which proves the theorem. □

In what follows we present few examples of parametric families of universal τ_T-submeasures related to this result.

Example 2. Let (Σ, η) be a numerical submeasure ring.

(i) A most trivial universal τ_T-submeasure is given by $\gamma_A(x) = 1$, $x > 0$, which corresponds to $\Phi(z) = \varepsilon_0(z)$.
(ii) A mapping $\gamma_A(x) = \min\{\frac{cx}{\eta(A)}, 1\}$, $x > 0$, is a universal τ_T-submeasure for all $c > 0$. It corresponds to distribution function of random variable uniformly distributed over $[0, 1]$.

(iii) For the function

$$\Phi(z) = \begin{cases} \frac{z}{1+z}, & z \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

we get the mapping

$$\gamma_A(x) = \frac{cx}{cx + \eta(A)}, \quad c > 0,\, x > 0, \tag{9}$$

which is clearly a parametric family of universal τ_T-submeasures.
Note that for a fixed numerical submeasure η, the numerical submeasure η_γ defined as in (6), where γ and η are related by means of (9), is given by

$$\eta_\gamma(A) = \begin{cases} 0, & \eta(A) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 4. Note that the mapping γ in (iii) is moreover a $\tau_{L,D}$-submeasure for each $c > 0$ and each binary operation L on $\overline{\mathbb{R}}_+$ (see Remark 2) with neutral element 0. Indeed, if $\eta(A) \neq 0$, $\eta(B) \neq 0$, then $D(\gamma_A(x), \gamma_B(y)) = 0$, $x, y > 0$, and $\gamma_{A\cup B}(L(x,y)) \geq 0$. If $\eta(B) = 0$, then $\eta(A \cup B) = \eta(A)$ and $D(\gamma_A(x), \gamma_B(y)) = \gamma_A(x)$. Since $x, y > 0$, and $L(x,y) \geq x$, then

$$\gamma_{A\cup B}(L(x,y)) = \frac{cL(x,y)}{cL(x,y) + \eta(A \cup B)} \geq \frac{cx}{cx + \eta(A)} = D(\gamma_A(x), \gamma_B(y)),$$

i.e., γ is a $\tau_{L,D}$-submeasure.

As it was already stated the class of solutions of functional inequality (8) satisfying (a') and (b') is surely wider than the class obtained in Theorem 2. For instance, it is easy to verify that a mapping

$$\gamma_A(x) = \max\{0, \min\{x - \eta(A), 1\}\}, \quad x > 0,$$

is also a universal τ_T-submeasure which is not covered by Theorem 2. As we will see in the next section the class of τ_T-submeasures corresponding to a class of t-norms having additive generator has very similar form to that given in Theorem 1.

3.2. *τ_T-submeasures related to the generated t-norm T*

Now we are interested in such τ_T-submeasures which are 'dependent' on some (structural, or other) properties of t-norm T and especially we want to find an appropriate formula for the corresponding τ_T-submeasure γ. In Theorem 3 we solve the problem for a class of t-norms having additive generator.

Recall that an *additive generator* $t : [0,1] \to \overline{\mathbb{R}}$ of a t-norm T is a strictly decreasing function, right-continuous in 0 satisfying $t(1) = 0$, such that for all $(x,y) \in [0,1]^2$ we have

$$\begin{aligned} t(x) + t(y) \;&\in\; \mathrm{Ran}(t) \cup [t(0), \infty], \\ T(x,y) \;&=\; t^{(-1)}(t(x) + t(y)), \end{aligned}$$

where $t^{(-1)} : \overline{\mathbb{R}} \to [0,1]$ is the pseudo-inverse function to t, i.e.,

$$t^{(-1)}(z) = \begin{cases} 1, & z < 0, \\ t^{-1}(z), & z \in [0, t(0)], \\ 0, & z > t(0). \end{cases}$$

Example 3. We list here some functions $t : [0,1] \to \overline{\mathbb{R}}$ which may be used as additive generators and the t-norms T induced by them, see [16].

(i) The additive generator of D is given by

$$t_D(x) = \begin{cases} 2 - x, & x \in [0,1[, \\ 0, & x = 1. \end{cases}$$

(ii) The Schweitzer-Sklar family T_λ^{SS} with $\lambda \in [-\infty, \infty]$, is given by

$$T_\lambda^{SS}(x,y) = \begin{cases} M(x,y), & \lambda = -\infty, \\ \Pi(x,y), & \lambda = 0, \\ D(x,y), & \lambda = +\infty, \\ (\max\{0, x^\lambda + y^\lambda - 1\})^{\frac{1}{\lambda}}, & \lambda \in]-\infty, 0[\cup]0, \infty[, \end{cases}$$

with additive generators

$$t_\lambda^{SS}(x) = \frac{1 - x^\lambda}{\lambda}, \quad x \in [0,1],$$

for $\lambda \in]-\infty, 0[\cup]0, \infty[$, and

$$t_0^{SS}(x) = \begin{cases} -\log x, & x \in]0,1] \\ \infty, & x = 0. \end{cases}$$

(iii) The family of Yager t-norms T_λ^Y is defined for $\lambda \in [0, \infty]$ as follows

$$T_\lambda^Y(x,y) = \begin{cases} D(x,y), & \lambda = 0, \\ M(x,y), & \lambda = +\infty, \\ \max\left\{1 - ((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}, 0\right\}, & \lambda \in]0, \infty[. \end{cases}$$

For $\lambda \in]0, \infty[$ its additive generator has the form

$$t_\lambda^Y(x) = (1-x)^\lambda, \quad x \in [0,1].$$

Theorem 3. *Let (Σ, η) be a numerical submeasure ring. If t is an additive generator of a t-norm T, then $\gamma : \Sigma \to \Delta^+$ given by*

$$\gamma_A(x) = t^{(-1)}(\eta(A) - x), \quad x > 0,$$

is a τ_T-submeasure.

Proof. (a) Let $A = \emptyset$ and $x > 0$. Then $\eta(A) = 0$ and

$$\gamma_A(x) = t^{(-1)}(\eta(A) - x) = t^{(-1)}(-x) = \varepsilon_0(x).$$

(b) If $A \subset B$, $x > 0$, then $\eta(A) - x \leq \eta(B) - x$. Since the cases when $\eta(A) - x$ and $\eta(B) - x$ do not belong to $\mathrm{Ran}(t)$ yield the validity of the inequality $\gamma_A(x) \geq \gamma_B(x)$, let us suppose that $\eta(A) - x \in \mathrm{Ran}(t)$ and $\eta(B) - x \in \mathrm{Ran}(t)$. Then using the monotonicity of $t^{(-1)}$ on $\mathrm{Ran}(t)$ yields

$$\gamma_A(x) = t^{(-1)}(\eta(A) - x) \geq t^{(-1)}(\eta(B) - x) = \gamma_B(x).$$

(c) Let $A, B \in \Sigma$, $x, y > 0$. Then

$$\begin{aligned} T(\gamma_A(x), \gamma_B(y)) &= t^{(-1)}(t(\gamma_A(x)) + t(\gamma_B(y))) \\ &= t^{(-1)}(t(t^{(-1)}(\eta(A) - x)) + t(t^{(-1)}(\eta(B) - y))). \end{aligned}$$

Again, the cases when $\eta(A) - x$ and $\eta(B) - y$ do not belong to $\mathrm{Ran}(t)$ are trivial and therefore we consider only the case $\eta(A) - x \in \mathrm{Ran}(t)$ and $\eta(B) - y \in \mathrm{Ran}(t)$. Since $\eta(A \cup B) \leq \eta(A) + \eta(B)$ and $t^{(-1)}$ is strictly decreasing, then

$$\begin{aligned} T(\gamma_A(x), \gamma_B(y)) &= t^{(-1)}(\eta(A) + \eta(B) - (x + y)) \\ &\leq t^{(-1)}(\eta(A \cup B) - (x + y)) = \gamma_{A \cup B}(x + y), \end{aligned}$$

which completes the proof. □

Remark 5. Since an additive generator t of a t-norm T is uniquely determined up to a positive multiplicative constant, by Theorem 3 we obtain a parametric family of corresponding τ_T-submeasures γ. It is also possible to consider a parameter $c > 0$ in the expression of γ as in Theorem 2 to obtain a two-parametric family of τ_T-submeasures.

Example 4. Here we give examples of τ_T-submeasures corresponding to t-norms T from Example 3.

(i) A τ_D-submeasure is given by

$$\gamma_A^D(x) = \max\Big\{\min\{2 + x - \eta(A),\, 1\},\, 0\Big\}, \quad x > 0.$$

(ii) A family of τ_T-submeasures corresponding to T_λ^{SS}, $\lambda \in]-\infty, 0[\cup]0, \infty[$, is given by

$$\gamma_A^{SS,\lambda}(x) = \min\left\{\sqrt[\lambda]{1 + \lambda(x - \eta(A))},\, 1\right\}, \quad x > \max\left\{\eta(A) - \frac{1}{\lambda}, 0\right\}.$$

For $\lambda = 0$ we get the τ_Π-submeasure in the form

$$\gamma_A^\Pi(x) = \min\{e^{x - \eta(A)}, 1\}, \quad x > 0.$$

(iii) The $\tau_{T_\lambda^Y}$-submeasures are given by

$$\gamma_A^{Y,\lambda}(x) = \min\left\{1 - \sqrt[\lambda]{\eta(A) - x},\, 1\right\}, \quad x > \max\{\eta(A) - 1, 0\}.$$

From it follows that

$$\gamma_A^{Y,1}(x) = \min\{1 + x - \eta(A),\, 1\}, \quad x > \max\{\eta(A) - 1, 0\},$$

is a τ_W-submeasure.

In Proposition 1 a numerical submeasure η_γ of the form (6) is associated to a τ_T-submeasure. Having a notion of comparison and additive generator of t-norms we may state the following result.

Theorem 4. *Let (Σ, γ, T_1) be a τ_{T_1}-submeasure ring. If t is an additive generator of a t-norm T such that $T \leq T_1$, then a mapping $\eta_{\gamma,t} : \Sigma \to \mathbb{R}_+$ given by*

$$\eta_{\gamma,t}(A) = \sup\{z \in \mathbb{R}_+;\, t(\gamma_A(z)) \geq z\}$$

is a numerical submeasure.

Proof. Applying Drewnowski approach [8] (which was used to study semimetrizable Fréchet-Nikodym topology generated by a numerical submeasure) we define

$$\rho(A, B) = \sup\{x \in \mathbb{R}_+;\, t(\gamma_{A\triangle B}(x)) \geq x\}, \quad A, B \in \Sigma,$$

where $A\triangle B$ is a symmetrical difference of sets, and show that ρ is a pseudometric on Σ. Obviously, if $A = B$, then $\rho(A, A) = 0$, and $\rho(A, B) = \rho(B, A)$ for each $A, B \in \Sigma$. To prove triangle inequality observe that $\rho(E, F) < z$ if and only if $t(\gamma_{E\triangle F}(z)) < z$. Therefore, let us suppose $\rho(A, C) < x$ and $\rho(C, B) < y$, for $x, y > 0$ and $A, B, C \in \Sigma$. Then

$$\begin{aligned}\gamma_{A\triangle B}(x + y)) &\geq \gamma_{(A\triangle C)\cup(C\triangle B)}(x + y)) \geq T_1(\gamma_{A\triangle C}(x), \gamma_{C\triangle B}(y))) \\ &\geq t^{(-1)}(t(\gamma_{A\triangle C}(x)) + t(\gamma_{C\triangle B}(y))).\end{aligned}$$

The case when $t(\gamma_{A\triangle C}(x)) + t(\gamma_{C\triangle B}(y))$ does not belong to $[0, t(0)]$ is trivial, and thus we consider $t(\gamma_{A\triangle C}(x)) + t(\gamma_{C\triangle B}(y)) \in [0, t(0)]$. Then

$$t(\gamma_{A\triangle B}(x + y))) \leq t(\gamma_{A\triangle C}(x)) + t(\gamma_{C\triangle B}(y)) < x + y,$$

which shows that $\rho(A, B) < x + y$. Evidently, for any $u > \rho(A, C) + \rho(C, B)$ there is $x > \rho(A, C)$ such that $u - x > \rho(C, B)$, and hence $u > \rho(A, B)$. Consequently, $\rho(A, B) \leq \rho(A, C) + \rho(C, B)$. To finish our proof it is sufficient to put $\eta_{\gamma,t}(A) = \rho(A, \emptyset)$. □

3.3. *φ-transformations of τ_T-submeasures*

Recall that if t is an additive generator of an Archimedean t-norm T and φ is an *automorphism* (i.e., a strictly increasing bijection of the closed unit interval), then $t_\varphi = t \circ \varphi$ is an additive generator of the Archimedean t-norm

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y))), \quad x, y \in [0, 1],$$

which is called a *φ-transformation* of T. From Theorem 3 we immediately get

Corollary 1. *Let (Σ, η) be a numerical submeasure ring. If t is an additive generator of an Archimedean t-norm T and φ is an automorphism, then a mapping γ given by*

$$\gamma_A(x) = t_\varphi^{(-1)}(\eta(A) - x)), \quad x > 0,$$

is a τ_{T_φ}-submeasure.

It is well known, see [16], that $T : [0,1]^2 \to [0,1]$ has a continuous additive generator if and only if T is a continuous Archimedean t-norm. Another useful characterization is provided via semigroup isomorphisms. Recall that a t-norm T_1 is *isomorphic* to a t-norm T_2 if there exists an automorphism φ such that T_1 is φ-transformation of T_2. Immediately,

- a t-norm T is *strict* if and only if T is isomorphic to Π;
- a t-norm T is *nilpotent* if and only if T is isomorphic to W.

Therefore, when considering τ_T-submeasures corresponding to a continuous Archimedean t-norm T, it is enough to deal with the product t-norm Π and Łukasiewicz t-norm W. The corresponding τ_Π-submeasure is given in Example 4 (ii), whereas the τ_W-submeasure is given in Example 4 (iii).

Remark 6. On this place let us mention that many different τ_T-submeasures may correspond to a fixed t-norm T because the corresponding functional inequality

$$f(a + b, x + y) \geq T(f(a, x), f(b, y))$$

with restrictions (a') and (b') may have many different solutions $f : [0,\infty[\times]0,\infty[\to [0,1]$. For instance, the functional inequality characterizing all τ_Π-submeasures has the form of the usual two-dimensional Cauchy exponential inequality [1] $f(a+b, x+y) \geq f(a,x)f(b,y)$. As we have already mentioned both restrictions (a') and (b') are necessary. For instance, one solution of the two-dimensional Cauchy exponential inequality is also $f(a,x) = \min\{x^a, 1\}$, which satisfies (a'), but does not satisfy (b'), and thus it is not suitable for defining a τ_Π-submeasure.

Let φ be an automorphism. In our next results we use some well-known facts from the comparison of continuous Archimedean t-norms, see [16]: $T_\varphi \leq T$ if and only if the function $t_\varphi \circ t^{-1} : [0, t(0)] \to \mathbb{R}$ is subadditive, i.e., for all $x, y \in [0, t(0)]$ with $x + y \in [0, t(0)]$ we have

$$(t_\varphi \circ t^{-1})(x + y) \leq (t_\varphi \circ t^{-1})(x) + (t_\varphi \circ t^{-1})(y).$$

Particularly, if $T = \Pi$, then $\Pi_\varphi \leq \Pi$ if and only if the function

$$(t_\varphi \circ t^{-1})(x) = -\ln(\varphi(e^{-x})), \quad x \in \overline{\mathbb{R}}_+,$$

is subadditive, i.e.,

$$\ln(\varphi(e^{-x-y})) \geq \ln(\varphi(e^{-x})) + \ln(\varphi(e^{-y})) = \ln(\varphi(e^{-x})\varphi(e^{-y})), \quad x, y \in \overline{\mathbb{R}}_+,$$

which is equivalent to $\varphi(e^{-x}e^{-y}) \geq \varphi(e^{-x})\varphi(e^{-y})$. It means that the function φ is supermultiplicative on $[0, 1]$ and we have

Proposition 2. *Let φ be an automorphism and γ be a τ_Π-submeasure. If φ is supermultiplicative, then γ is a τ_{Π_φ}-submeasure.*

In particular, the proposition claims that a class of τ_Π-submeasures is a subset of a class of τ_{Π_φ}-submeaures.

Example 5. Let us consider the Hamacher family of t-norms [16]

$$T_\lambda^H(x, y) = \begin{cases} D(x, y), & \lambda = \infty, \\ 0, & \lambda = x = y = 0, \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)}, & \lambda \in [0, \infty[\text{ and } (\lambda, x, y) \neq (0, 0, 0). \end{cases}$$

Each t-norm T_λ^H, $\lambda \in]0, \infty[$, is a φ_λ^H-transformation of Π, where

$$\varphi_\lambda^H(x) = \frac{x}{\lambda + (1-\lambda)x}.$$

It is easy to verify that φ_λ^H is supermultiplicative on $[0, 1]$ for all $\lambda \in]0, \infty[$, and consequently each τ_Π-submeasure γ is also $\tau_{\Pi_{\varphi_\lambda^H}}$-submeasure for each $\lambda \in]0, \infty[$.

For $\lambda = 0$ and $(x, y) \neq (0, 0)$ we have

$$T_0^H(x, y) = \frac{1}{\frac{1}{x} + \frac{1}{y} - 1},$$

and the corresponding φ_0^H-transformation has the form

$$\varphi_0^H(x) = e^{\frac{x-1}{x}},$$

which clearly is not a supermultiplicative function on $]0, 1]$. Moreover, since $D \leq \Pi$, then each τ_Π-submeasure is a τ_D-submeasure. Summarizing these results we can see that each τ_Π-submeasure γ is a $\tau_{T_\lambda^H}$-submeasure for $\lambda \in]0, \infty]$.

Comparing t-norms T with W we obtain the following result.

Proposition 3. *Let φ be an automorphism, and γ be a τ_W-submeasure. If the function $\psi(x) = 1 - \varphi(1 - x)$ is subadditive, then γ is a τ_{W_φ}-submeasure.*

Example 6. Let us consider the Yager family of t-norms T_λ^Y given in Example 3 (iii). For each $\lambda \in]0, \infty[$ there exists an increasing bijection

$$\varphi_\lambda^Y(x) = 1 - (1 - x)^\lambda,$$

such that each T_λ^Y, $\lambda \in]0, \infty[$, is a φ_λ^Y-transformation of W. Since the function $\psi_\lambda^Y(x) = x^\lambda$ is clearly subadditive for all $\lambda \in]0, \infty[$, then each τ_W-submeasure γ is also $\tau_{W_{\varphi_\lambda^Y}}$-submeasure for each $\lambda \in [0, \infty[$.

Immediately, we come to a natural question: *Given a τ_T-submeasure γ, how to transform γ in order to obtain that $\varphi(\gamma)$ is a τ_{T_φ}-submeasure?* More or less surprisingly, an *involution* (i.e., a strictly increasing function $\varphi : [0,1] \to [0,1]$ with $\varphi \circ \varphi = \mathrm{Id}_{[0,1]}$) is applicable.

Theorem 5. *Let T be a t-norm and φ be an involution. Then the following statements are equivalent:*

(i) γ is a τ_T-submeasure;
(ii) $\varphi(\gamma)$ is a τ_{T_φ}-submeasure.

Proof. If γ is a τ_T-submeasure and φ is an involution, then

$$\begin{aligned}\varphi(\gamma_{A\cup B}(x+y)) &\geq \varphi(T(\gamma_A(x),\gamma_B(y)))\\ &= \varphi\left((\varphi\circ\varphi)^{-1}(T((\varphi\circ\varphi)(\gamma_A(x)),(\varphi\circ\varphi)(\gamma_B(y))))\right)\\ &= T_\varphi(\varphi(\gamma_A(x)),\varphi(\gamma_B(y)))),\end{aligned}$$

shows that $\mu_A(x) = \varphi(\gamma_A(x))$ is a τ_{T_φ}-submeasure.

Conversely, if $\varphi(\gamma)$ is a τ_{T_φ}-submeasure, then

$$\begin{aligned}\gamma_{A\cup B}(x+y) &\geq \varphi^{-1}(T_\varphi(\varphi(\gamma_A(x)),\varphi(\gamma_B(y))))\\ &= (\varphi\circ\varphi)^{-1}\left(T((\varphi\circ\varphi)(\gamma_A(x)),(\varphi\circ\varphi)(\gamma_A(x)))\right)\\ &= T(\gamma_A(x),\gamma_B(y)),\end{aligned}$$

which completes the proof. □

Similarly to additive generator the concept of a *multiplicative generator* of a t-norm T has been introduced [16]. Such a generator is a strictly increasing function $\theta : [0,1] \to \overline{\mathbb{R}}_+$ right-continuous in 0 satisfying $\theta(1) = 1$ such that for all $x, y \in [0,1]$ we have

$$\begin{aligned}\theta(x)\theta(y) &\in \mathrm{Ran}(\theta) \cup [0,\theta(0)],\\ T(x,y) &= \theta^{(-1)}(\theta(x)\cdot\theta(y)),\end{aligned}$$

where the pseudo-inverse $\theta^{(-1)} : \overline{\mathbb{R}}_+ \to [0,1]$ is defined by

$$\theta^{(-1)}(z) = \begin{cases} 0, & z \in [0,\theta(0)],\\ \theta^{-1}(z), & z \in]\theta(0),1]\\ 1, & z > 1.\end{cases}$$

The proof of the following result is analogous to the proof of Theorem 3 (or by using the relationship between an additive and multiplicative generator of a t-norm).

Theorem 6. *Let (Σ,η) be a numerical submeasure ring. If θ is a multiplicative generator of a t-norm T, then $\gamma : \Sigma \to \Delta^+$ given by*

$$\gamma_A(x) = \theta^{(-1)}\left(e^{x-\eta(A)}\right), \quad x > 0,$$

is a τ_T-submeasure.

For some classes of t-norms (continuous Archimedean t-norms) we may get the following characterizations of τ_T-submeasures.

Theorem 7. *Let T be a strict t-norm with a multiplicative generator θ. Then the following statements are equivalent:*

(i) γ is a τ_T-submeasure;
(ii) $\theta(\gamma)$ is a τ_Π-submeasure.

Proof. (i)⇒(ii) If θ is a multiplicative generator of a t-norm T, then $T(x, y) = \theta^{(-1)}(\theta(x) \cdot \theta(y))$. Since γ is a τ_T-submeasure, we get

$$\gamma_{A\cup B}(x + y) \geq T(\gamma_A(x), \gamma_B(y)) = \theta^{(-1)}(\theta(\gamma_A(x)) \cdot \theta(\gamma_B(y))).$$

If $\theta(\gamma_A(x)) \cdot \theta(\gamma_B(y))$ belongs to $]\theta(0), 1]$, then

$$\theta(\gamma_{A\cup B}(x + y)) \geq \theta(\gamma_A(x)) \cdot \theta(\gamma_B(y)) = \Pi(\theta(\gamma_A(x)), \theta(\gamma_B(y))),$$

i.e., $\theta(\gamma)$ is a τ_Π-submeasure. The other cases are trivial.

(ii)⇒(i) Let $\theta(\gamma)$ be a τ_Π-submeasure. Recall that each strict t-norm T is a θ-transformation of Π. Thus,

$$\begin{aligned} T(\gamma_A(x), \gamma_B(y)) &= \Pi_\theta(\gamma_A(x), \gamma_B(y)) = \theta^{-1}(\theta(\gamma_A(x)) \cdot \theta(\gamma_B(y))) \\ &\leq \gamma_{A\cup B}(x + y), \end{aligned}$$

which proves that γ is a τ_T-submeasure. □

Also, for the class of all nilpotent t-norms we may state the following result which is based on the fact that for each nilpotent t-norm T with additive generator t there exists a unique isomorphism $\varphi : [0, 1] \to [0, 1]$ given by $\varphi(x) = 1 - \frac{t(x)}{t(0)}$ such that $T = W_\varphi$, see [16]. Thus,

Theorem 8. *If T is a nilpotent t-norm with additive generator t, then the following statements are equivalent:*

(i) γ is a τ_T-submeasure;
(ii) $\varphi(\gamma)$ is a τ_W-submeasure, where $\varphi(x) = 1 - \frac{t(x)}{t(0)}$.

4. Aggregation of τ_T-submeasures

In this section we mention some direct consequences of the results stated in [21] which can be used for solving a question on aggregation of T-submeasures. Recall that a function

$$H : \bigcup_{n\in\mathbb{N}} [0, 1]^n \to [0, 1]$$

is called an *aggregation operator* if it is nondecreasing in each component, $H(x) = x$ for all $x \in [0, 1]$ and satisfies the boundary conditions $H(0, \ldots, 0) = 0$ and

$H(1,\ldots,1)=1$, see [5]. Each aggregation operator H can be represented by a family $(H_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, i.e., functions $H_{(n)}:[0,1]^n\to[0,1]$ given by $H_{(n)}(x_1,\ldots,x_n)=H(x_1,\ldots,x_n)$. If some n-ary operations $H_{(n)}$, $n\geq 2$, are non-decreasing in each component and fulfil the boundary conditions, then $H_{(n)}$ are referred to as *n-ary aggregation operators*.

Observe that if $\gamma^{(1)},\ldots,\gamma^{(n)}$, $n\in\mathbb{N}$, are τ_T-submeasures for some t-norm T and $\gamma:=H_{(n)}(\gamma^{(1)},\ldots,\gamma^{(n)})$ defined by $\gamma(x)=H_{(n)}(\gamma^{(1)}(x),\ldots,\gamma^{(n)}(x))$ for some n-ary aggregation operator $H_{(n)}$, then by monotonicity

$$\begin{aligned}\gamma_{A\cup B}(x+y) &= H_{(n)}(\gamma^{(1)}_{A\cup B}(x+y),\ldots,\gamma^{(n)}_{A\cup B}(x+y))\\ &\geq H_{(n)}(T(\gamma^{(1)}_A(x),\gamma^{(1)}_B(y)),\ldots,T(\gamma^{(n)}_A(x),\gamma^{(n)}_B(y))).\end{aligned}$$

For better readability put $x_i=\gamma^{(i)}_A(x)$ and $y_i=\gamma^{(i)}_B(y)$, $i=1,2,\ldots,n$. If the inequality

$$H_{(n)}(T(x_1,y_1),\ldots,T(x_n,y_n))\geq T(H_{(n)}(x_1,\ldots,x_n),H_{(n)}(y_1,\ldots,y_n)) \tag{10}$$

holds for all $x_i,y_i\in[0,1]$, $i=1,2,\ldots,n$, then clearly γ is a τ_T-submeasure as well. In [21] the property (10) is defined as the *domination* of an n-ary aggregation operator $H_{(n)}$ over the t-norm T, we denote it by $H_{(n)}\gg T$. If $H_{(n)}\gg T$ for all $n\in\mathbb{N}$, then we say that an aggregation operator H *dominates* T (we denote $H\gg T$).

From the results of paper [21] on domination we immediately get the following theorems as consequences.

Theorem 9. *Let T be a continuous Archimedean t-norm with an additive generator t, let $\gamma^{(1)},\ldots,\gamma^{(n)}$, $n\in\mathbb{N}$, be τ_T-submeasures, and $H:\bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$ be an aggregation operator. If there exists a subadditive aggregation operator $K:\bigcup_{n\in\mathbb{N}}[0,t(0)]^n\to[0,t(0)]$ such that for all $n\in\mathbb{N}$ and for all $x_i\in[0,1]$, $i=1,2,\ldots,n$,*

$$t(H(x_1,\ldots,x_n))=K(t(x_1),\ldots,t(x_n)), \tag{11}$$

then $\gamma=H(\gamma^{(1)},\ldots,\gamma^{(n)})$ is a τ_T-submeasure.

Example 7. By Example 18 in [21] the strongest subadditive aggregation operator $K:\bigcup_{n\in\mathbb{N}}[0,t(0)]^n\to[0,t(0)]$ given by

$$K(u_1,\ldots,u_n)=\begin{cases}0, & u_1=\cdots=u_n=0,\\ t(0), & \text{otherwise},\end{cases}$$

satisfies (11) for any additive generator t, for all $x_i\in[0,1]$, $i=1,2,\ldots,n$, and some $n\in\mathbb{N}$, if and only if H is the weakest aggregation operator

$$H_w(x_1,\ldots,x_n)=\begin{cases}1, & x_1=\cdots=x_n=1,\\ 0, & \text{otherwise}.\end{cases}$$

By Theorem 9, $\gamma=H_w(\gamma^{(1)},\ldots,\gamma^{(n)})$ is a τ_T-submeasure.

By Proposition 27 in [21] for any $n \in \mathbb{N}$ the class of all n-ary aggregation operators dominating M is given by

$$\mathcal{D}_M^{(n)} = \{\min_{\mathcal{F}}; \mathcal{F} = (f_1, \dots, f_n)\},$$

where $\min_{\mathcal{F}}(x_1, \dots, x_n) = \min(f_1(x_1), \dots, f_n(x_n))$, and $f_i : [0,1] \to [0,1]$ are non-decreasing functions with $f_i(1) = 1$ for all $i = 1, 2, \dots, n$, and $f_i(0) = 0$ for at least one $i \in \{1, 2, \dots, n\}$. Then we have

Theorem 10. *If $\gamma^{(1)}, \dots, \gamma^{(n)}$, $n \in \mathbb{N}$, are universal τ_T-submeasures, then the mapping $\gamma = H_{(n)}(\gamma^{(1)}, \dots, \gamma^{(n)})$ is a universal τ_T-submeasure for all n-ary aggregation operators $H_{(n)} \in \mathcal{D}_M^{(n)}$.*

Remark 7. Let $\gamma^{(1)}, \dots, \gamma^{(n)}$, $n \in \mathbb{N}$, be universal τ_T-submeasures. Since the weakest aggregation operator H_w dominates all t-norms, then clearly $\gamma = H_w(\gamma^{(1)}, \dots, \gamma^{(n)})$ is also a universal τ_T-submeasure.

5. Concluding remarks

We have discussed a submeasure notion related to triangle functions in Menger spaces and t-norms. As we have seen for each given t-norm T there exist many corresponding τ_T-submeasures. Some explicit formulas for parametric families of τ_T-submeasures corresponding to continuous Archimedean t-norms and their φ-transformations have been stated and the aggregation of τ_T-submeasures have been discussed.

Note that τ_T-submeasures can be seen as fuzzy number-valued submeasures, where the value γ_A can be seen as a non-negative LT-fuzzy number [9], and where $\tau_T(\gamma_A, \gamma_B)$ corresponds to the T-sum of fuzzy numbers γ_A and γ_B. Moreover, each universal τ_T-submeasure γ can be represented by means of a non-decreasing system $(\eta_\alpha)_{\alpha \in [0,1]}$ of numerical submeasures (compare the horizontal representation $(F_\alpha)_{\alpha \in [0,1]}$ of a fuzzy subset F), where

$$\gamma_A(x) = \sup\{\alpha \in [0,1]; \eta_\alpha(A) \le x\}.$$

In this context (and as a promising subject for the further research) let us introduce a notion of *τ-measure* on Σ, which may be naturally defined as a mapping $\mu : \Sigma \to \Delta^+$ satisfying

(i) if $A = \emptyset$, then $\mu(A) = \varepsilon_0$;
(ii) $\mu(A \cup B) = \tau(\mu(A), \mu(B))$ whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$.

For example $\tau = \tau_T$ yields a notion of τ_T-measure (which is clearly different from the notion of a T-measure in [4]).

Another direction of our research in this area is the integration with respect to τ_T- (τ-)submeasures and τ_T- (τ-)measures.

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