Toeplitz Operators on Poly-analytic Spaces via Time-Scale Analysis

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Abstract. This is a review paper based on the series of our papers devoted to a structure of true-poly-analytic Bergman function spaces over the upper half-plane in the complex plane and to a detailed study of properties of Toeplitz operators with separate symbols acting on them via time-scale analysis approach.

1 Introduction

Analytic functions are the main object of classical complex analysis. One definition of analytic functions is in terms of Cauchy-Riemann operator

$$\partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \quad z = x + \mathrm{i}y.$$

Then f is analytic in some (simply connected bounded or unbounded) domain Ω of the complex plane $\mathbb{C}$ if $\partial_{\overline{z}} f = 0$ on Ω. A natural extension of this definition is to iterate the Cauchy-Riemann operator which yields a notion of poly-analytic function, i.e., f is poly-analytic (or, analytic of order n) in Ω if $\partial_{\overline{z}}^n f = 0$ on Ω. In this paper we will consider $\Omega = \Pi$ – the upper half-plane in the complex plane. Poly-analytic functions of order n which are not poly-analytic of any other lower order are called true-poly-analytic of order n, see the two visionary papers of Vasilevski [16] and [17]. For further reading on poly-analytic functions we refer to book of Balk [5] as well as a review of some recent developments provided by Abreu and Feichtinger [3].

It was recently observed by Abreu in [2] that *the true-poly-analytic Bergman space of order* $k+1$ (over the upper half-plane) *may be alternatively viewed as the space of wavelet transforms with Laguerre functions of order* k. This perhaps unexpected result enables us to see poly-analytic spaces and related operators of complex analysis from a new perspective of time-scale (or, more generally, time-frequency) analysis, cf. [3]. In this paper we are especially interested in Toeplitz operators on (true)-poly-analytic Bergman spaces. A recent interest of this topic may be seen e.g. in [7]. We provide a survey of results from [11]-[14], where further results, details, and proofs can be found. The majority of presented results and examples are based on the ideas, techniques, and results by N. Vasilevski and his coauthors developed for the case of the Bergman spaces and Toeplitz operators on these spaces.

Outline The main ingredients of the whole theory (on the hand of time-scale analysis) are the affine group $\mathbb{G}$ of orientation-preserving linear transformations of the real line, and a parameterized family of admissible affine coherent states (wavelets) $\{\psi^{(k)}\}_{k\in\mathbb{Z}_+}$ whose Fourier transform is related to Laguerre functions $\ell_k(x) = \mathrm{e}^{-x/2}L_k(x)$ with L_k being the Laguerre polynomial of degree $k \in \mathbb{Z}_+$. In this context we first describe the structure of the space $A^{(k)}$ of wavelet transforms (related to wavelet $\psi^{(k)}$ and Hardy-space functions) inside the Hilbert space $L_2(\mathbb{G}, \mathrm{d}\nu_L)$, and link this construction with the intertwining property of induced representation of $\mathbb{G}$. This allows us to describe the direct and natural connection between wavelet subspaces and Hardy spaces. Indeed, this study provides a time-scale approach to poly-analytic spaces as explained in [2]. These results are then applied to the study of behavior of a parameterized family of Toeplitz operators acting on wavelet subspaces (i.e., true-poly-analytic Bergman spaces on Π). Given a function (symbol) $a = a(\zeta)$, $\zeta \in \mathbb{G}$, the Toeplitz operator $T_a^{(k)}$ acting on $A^{(k)}$ is defined as usual by $T_a^{(k)} f = P^{(k)}(af)$, $f \in A^{(k)}$, where $P^{(k)}$ is the orthogonal projection

[1] *Mathematics Subject Classification (2010):* Primary 47B35, Secondary 47G30, 47L80, 42C40,
Key words and phrases: Wavelet, poly-analytic function, Toeplitz operator, operator algebra, Bergman space
Acknowledgement: The first author acknowledges a partial support of grant VEGA 1/0164/14.

of $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ onto $A^{(k)}$. The following result (cf. Theorem 3.1) gives an easy and direct access to important properties of Calderón-Toeplitz operators:

The Calderón-Toeplitz operator $T_a^{(k)}$ with a symbol $a = a(u)$, $u \in \mathbb{R}_+$, acting on wavelet subspace $A^{(k)}$ is unitarily equivalent to the multiplication operator $\mathfrak{A}_a^{(k)} = \gamma_{a,k} I$ acting on $L_2(\mathbb{R}_+)$, where the function $\gamma_{a,k} : \mathbb{R}_+ \to \mathbb{C}$ has the explicit form

$$\gamma_{a,k}(x) = \int_{\mathbb{R}_+} a\left(\frac{u}{2x}\right) \ell_k^2(u)\,\mathrm{d}u, \quad x \in \mathbb{R}_+. \tag{1}$$

As it can be seen, the function $\gamma_{a,k}$ is obtained by integrating a dilation of a symbol $a = a(u)$ of an operator $T_a^{(k)}$ against a Laguerre function of order k. This result extends the result of Vasilevski for the Toeplitz operators acting on the Bergman space (i.e., the case $k = 0$ in our notation) in very interesting way which differs from the case of Toeplitz operators acting on weighted Bergman spaces studied in paper [9], and then summarized in Vasilevski book [18]. Moreover, a number of results following immediately from this equivalency may be derived including the spectral-type representation of Toeplitz operators whose symbols depend only on imaginary coordinate in the upper half-plane, as well as formulas for the Wick symbols and the star product in terms of function $\gamma_{a,k}$.

Since the function $\gamma_{a,k}$ given by (1) is responsible for many interesting features and behavior of the corresponding Toeplitz operator $T_a^{(k)}$ acting on $A^{(k)}$, we describe its basic properties, e.g., the limit behavior of higher order derivatives of $\gamma_{a,k}$ for bounded as well as integrable symbols a, sufficient conditions for $\gamma_{a,k}$ to be continuous on the whole $[0, +\infty]$: a question which is closely related to the behavior of a symbol $a = a(u)$, $u \in \mathbb{R}_+$, at a neighborhood of points 0 and $+\infty$. The result of Theorem 3.15 states that the limit at infinity and at zero of the function $\gamma_{a,k}$ is independent of parameter k. In fact, it depends only on a limit of the corresponding symbol a, but not on the particularly chosen Laguerre functions, thus providing an interesting information about asymptotic behavior of true-poly-analytic Bergman spaces.

In connection with the above mentioned results some operator algebras and a functional dependence of Toeplitz operators may be described. In particular, if we consider a symbol $a = a(u) \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ (bounded on $\mathbb{R}_+$ having limits at the endpoints of $[0, +\infty]$) such that the corresponding function $\gamma_{a,k}$ separates the points of $[0, +\infty]$, then we construct a function such that each Toeplitz operator $T_a^{(k)}$ acting on $A^{(k)}$ with a symbol $a = a(u) \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ is the function of Toeplitz operator acting on the Bergman space $A_2(\Pi)$ with symbol $\chi_{[0,\lambda]}(u)$. Interpretation and applicability of these results from the viewpoint of localization in the time-frequency analysis are discussed.

An interesting and important feature of Toeplitz operators on wavelet subspaces is that they can be bounded for symbols that are unbounded near the boundary. We show that for unbounded symbols $a = a(u)$, $u \in \mathbb{R}_+$, the behavior of certain iterated means rather than the behavior of symbol a itself plays a crucial role in the boundedness properties. Contrary to the case of Toeplitz operators on weighted Bergman spaces these means do not depend on a weight parameter k. We present a number of examples and construct wide families of unbounded symbols for which the Toeplitz operator is not only bounded, but also belongs to the algebra of bounded Toeplitz operators generated by $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$-symbols.

Furthermore, for the case of symbols $b = b(v)$ depending on horizontal variable v of upper-half plane Π (or, for general symbols as well) the Toeplitz operator $T_b^{(k)}$ is unitarily equivalent to certain pseudo-differential operator of the form

$$\left(\mathfrak{B}_b^{(k)} f\right)(x) = \int_{\mathbb{R}_+} \frac{2\sqrt{xy}}{x+y} P_k\left(\frac{8xy}{(x+y)^2} - 1\right) \hat{b}(x-y) f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}_+$$

with P_n being the Legendre polynomial of degree $n \in \mathbb{Z}_+$. This class of operators is interesting itself, because it extends and generalizes the class of pseudo-differential operators considered in [6]. Therefore, having the unitary equivalent images of $T_a^{(k)}$, $a = a(u) \in L_\infty(\mathbb{R}_+)$, and $T_b^{(k)}$, $b = b(v) \in C(\overline{\mathbb{R}})$, we describe the Fredholm symbol algebras of the Toeplitz operator algebras.

An interesting feature of each of these algebras studied in this case is that they are algebras with compact commutator and non-compact semi-commutator property.

2 Time-scale approach to poly-analytic spaces: a description

Affine group and its induced representations. The *affine group* $\mathbb{G}$ consists of all transformations $A_{u,v}$ of the real line $\mathbb{R}$ of the type $A_{u,v}(x) := ux + v$, $x \in \mathbb{R}$, where $u > 0$, $v \in \mathbb{R}$. Indeed, writing

$$\mathbb{G} = \{\zeta = (u, v);\ u > 0, v \in \mathbb{R}\},$$

one has the multiplication law on $\mathbb{G}$ of the form $\zeta_1 \diamond \zeta_2 = (u_1, v_1) \diamond (u_2, v_2) = (u_1 u_2, u_1 v_2 + v_1)$. With respect to the multiplication $\diamond$ the group $\mathbb{G}$ is non-commutative with the identity element $e = (1, 0)$, and a locally compact Lie group on which the left-invariant Haar measure is given by $\mathrm{d}\nu_L(\zeta) = u^{-2}\,\mathrm{d}u\mathrm{d}v$. The usual identification of the group $\mathbb{G}$ with the upper half-plane $\Pi = \{\zeta = v + \mathrm{i}u;\ v \in \mathbb{R}, u > 0\}$ in the complex plane $\mathbb{C}$ equipped with the hyperbolic metric and the corresponding (hyperbolic) measure $\mathrm{d}\nu_L$ will also be used. Then $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ denotes the Hilbert space of all square-integrable complex-valued functions on $\mathbb{G}$ with respect to the measure $\mathrm{d}\nu_L$.

The affine group $\mathbb{G}$ may be decomposed as a semi-direct product $\mathbb{G} = \mathsf{N} \ltimes \mathsf{A}$, where $\mathsf{N} = \{(1, v); v \in \mathbb{R}\}$ is the abelian normal closed subgroup, and the quotient group A is isomorphic to the one-parameter closed subgroup $\{(u, 0); u > 0\} \cong \mathbb{R}_+$. Thus, if H is a closed subgroup of $\mathbb{G}$ and $X = \mathbb{G}/H$ is the corresponding left-homogeneous space, we may induce representations of $\mathbb{G}$ in the subspaces which depend on $X = \mathbb{G}/H$ with $H = \{e\}$, $H = \mathsf{A}$ and $H = \mathsf{N}$, respectively. Indeed,

(i) $X = \mathbb{G}/\{e\} = \mathbb{G}$ – a character of the subgroup $\{e\}$ induces a left-regular representation of $\mathbb{G}$ on $L_2(X) = L_2(\mathbb{G}, \mathrm{d}\nu_L)$ in the form

$$[\Lambda(u, v)F](x, y) := F\left(\frac{x}{u}, \frac{y - v}{u}\right);$$

(ii) $X = \mathbb{G}/\mathsf{N} \cong \mathsf{A} = \mathbb{R}_+$ – a character of the subgroup N induces a co-adjoint representation of $\mathbb{G}$ on $L_2(\mathbb{G}/\mathsf{N}) = L_2(\mathbb{R}_+)$ in the form

$$[\rho\,(u, v)\,f]\,(x) := \mathrm{e}^{-2\pi\mathrm{i}\frac{v}{x}} f\left(\frac{x}{u}\right);$$

(iii) $X = \mathbb{G}/\mathsf{A} \cong \mathsf{N} = \mathbb{R}$ – a character of the subgroup A induces a quasi-regular representation of $\mathbb{G}$ on $L_2(\mathbb{G}/\mathsf{A}) = L_2(\mathbb{R})$ in the form

$$[\pi\,(u, v)\,f]\,(y) := \frac{1}{\sqrt{u}} f\left(\frac{y - v}{u}\right);$$

whenever $(u, v), (x, y) \in \mathbb{G}$.

The Hilbert space $L_2\,(\mathbb{R})$ under the action π contains precisely two closed proper invariant subspaces $H_2\,(\mathbb{R})$ and $H_2\,(\mathbb{R})^{\perp}$, called the Hardy and conjugate Hardy spaces, respectively, such that $L_2(\mathbb{R}) = H_2(\mathbb{R}) \oplus H_2(\mathbb{R})^{\perp}$. Thus, π is a reducible representation on $L_2\,(\mathbb{R})$, and we can decompose it into two irreducible representations, such that $\pi(u, v) = \pi^+(u, v) \oplus \pi^-(u, v)$. From it follows that only the Hardy space $H_2(\mathbb{R})$ is considered, although the discussion and further results are equally valid for the conjugate Hardy space $H_2\,(\mathbb{R})^{\perp}$. From the action on signals (we identify a signal with an element $f \in L_2(\mathbb{R})$) we observe that $\mathbb{G}$ consists precisely of the transformations we apply to a signal: *translation* (time-shift) by an amount v, and *zooming in or out* by the factor u. Hence, the group $\mathbb{G}$ naturally relates to the geometry of signals.

There is an intertwining operator between the co-adjoint representation ρ and quasi-regular representation π. This is the Fourier transform $\mathscr{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ in the form $\mathscr{F}\{f\}(\xi) :=$

$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, \mathrm{d}x$, since it uses the characters which induce those representations. Also there exists an intertwining operator between π and Λ given by the identity $[W_\psi f](u,v) := \langle f, \pi(u,v)\psi\rangle$, $f \in L_2(\mathbb{R})$, which provides a starting point for time-scale analysis on $\mathbb{R}$. Usually it is desirable to make this map unitary as well, and this is expressed by the following resolution of identity

$$\langle f, g\rangle = \int_{\mathbb{G}} \langle f, \pi(\zeta)\psi\rangle \langle \pi(\zeta)\psi, g\rangle \, \mathrm{d}\nu_L(\zeta),$$

also known as the Calderón reproducing formula. To achieve this the mother wavelet $\psi \in L_2(\mathbb{R})$ has to be *admissible*: for the affine group this is equivalent to

$$\int_{\mathbb{R}_+} |\hat{\psi}(u)|^2 \, \frac{\mathrm{d}u}{u} = 1.$$

Wavelets from Laguerre functions and their subspaces. Here we describe an alternative approach to true-poly-analytic Bergman spaces using wavelets built from Laguerre functions. Thus, for $k \in \mathbb{Z}_+$ consider the parameterized family of admissible wavelets $\psi^{(k)}$ on $\mathbb{R}$ defined on the Fourier transform side as follows

$$\hat{\psi}^{(k)}(\xi) = \chi_+(\xi)\sqrt{2\xi}\, \ell_k(2\xi),$$

where $\ell_n(x) := \mathrm{e}^{-x/2} L_n(x)$ are the (simple) Laguerre functions with $L_n(x)$ being the Laguerre polynomial of order $n \in \mathbb{Z}_+$ and χ_+ the characteristic function of the positive half-line. Then according to the Calderón reproducing formula

$$f(v) = \int_{\mathbb{R}_+} \Big(\big(D_u \psi^{(k)}\big) * \big(D_u \psi^{(k)}\big) * f \Big)(v) \, \frac{\mathrm{d}u}{u^2}$$

for all $f \in H_2(\mathbb{R})$, where $*$ is the usual convolution on $L_2(\mathbb{R})$. For each $k \in \mathbb{Z}_+$ define the subspaces $A^{(k)}$ of $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ as follows

$$A^{(k)} := \Big\{ [W_k f](u,v) = \Big(f * \big(D_u \psi^{(k)}\big)\Big)(v);\ f \in H_2(\mathbb{R}) \Big\}.$$

Indeed, $W_k f$ are exactly the continuous wavelet transforms of functions $f \in H_2(\mathbb{R})$ with respect to wavelets $\psi^{(k)}$. Consequently, $A^{(k)}$ will be referred to as *wavelet subspaces* of $L_2(\mathbb{G}, \mathrm{d}\nu_L)$. Note that it is possible to consider the "conjugate" wavelet $\hat{\bar{\psi}}^{(k)}(\xi) = \hat{\psi}^{(k)}(-\xi)$, the "conjugate" wavelet subspaces

$$\bar{A}^{(k)} := \Big\{ [W_{\bar{k}} f](u,v) = \Big(f * \big(D_u \bar{\psi}^{(k)}\big)\Big)(v);\ f \in H_2(\mathbb{R})^\perp \Big\}$$

and to build up the theory in this setting. In what follows we will state the results only for $A^{(k)}$, similar results may be stated for its "conjugate" counterpart $\bar{A}^{(k)}$.

Remark 2.1 Note that poly-analytic Bergman spaces and introduced wavelet subspaces share intriguing patterns that may prove usable. A deeper study of this connection is given in the recent paper [2]: the important and interesting observation of that paper is that for $k \in \mathbb{N}$ *the spaces $A^{(k-1)}$ of continuous wavelet transforms of Hardy space functions with respect to wavelets from Laguerre functions coincide with the true-poly-analytic Bergman spaces of order k on the upper half-plane* (symmetrically, $\bar{A}^{(k-1)}$ corresponds to the space of all true-poly-anti-analytic functions of order k from $L_2(\Pi)$). This allows to study these objects of complex analysis using techniques of time-scale analysis.

The relationship among the introduced spaces $A^{(k)}$ of wavelet transforms of $H_2(\mathbb{R})$-functions, and the unitary operators of continuous wavelet transform W_k and the Fourier transform $\mathscr{F}$ is schematically described on Figure 1. For each $k \in \mathbb{Z}_+$ the spaces $A^{(k)}$ are the reproducing kernel Hilbert spaces. Explicit formulas for their reproducing kernels

$$K_\zeta^{(k)}(\eta) = \Big\langle \pi(\eta)\psi^{(k)}, \pi(\zeta)\psi^{(k)} \Big\rangle$$

and orthogonal projections $P^{(k)} : L_2(\mathbb{G}, \mathrm{d}\nu_L) \to A^{(k)}$ are described in [11].

$$\begin{tikzcd}
L_2(\mathbb{R}) \arrow[d, "P_{\mathbb{R}}"] & L_2(\mathbb{R}) \arrow[l, "\mathscr{F}^{-1}"'] \arrow[d, "\chi_+ I"] & L_2(\mathbb{G}, \mathrm{d}\nu_L) \arrow[l, "W_k^{-1}"'] \arrow[d, "P^{(k)}"] \\
H_2(\mathbb{R}) \arrow[r, "\mathscr{F}"] & L_2(\mathbb{R}_+) \arrow[r, "W_k"] & A^{(k)}
\end{tikzcd}$$

Figure 1: Relationship among the introduced spaces and operators

Structural results. In accordance with the representation of $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ as tensor product in the form

$$L_2(\mathbb{G}, \mathrm{d}\nu_L(\zeta)) = L_2(\mathbb{R}_+, u^{-2}\mathrm{d}u) \otimes L_2(\mathbb{R}, \mathrm{d}v)$$

with $\zeta = (u, v) \in \mathbb{G}$, we consider the unitary operator

$$U_1 = (I \otimes \mathscr{F}) : L_2(\mathbb{R}, u^{-2}\mathrm{d}u) \otimes L_2(\mathbb{R}, \mathrm{d}v) \longrightarrow L_2(\mathbb{R}_+, u^{-2}\mathrm{d}u) \otimes L_2(\mathbb{R}, \mathrm{d}\omega).$$

For the purpose to "linearize" the hyperbolic measure $\mathrm{d}\nu_L$ onto the usual Lebesgue plane measure we introduce the unitary operator

$$U_2 : L_2(\mathbb{R}_+,\, u^{-2}\mathrm{d}u) \otimes L_2(\mathbb{R},\, \mathrm{d}\omega) \longrightarrow L_2(\mathbb{R}_+,\, \mathrm{d}x) \otimes L_2(\mathbb{R},\, \mathrm{d}y)$$

given by

$$U_2 : F(u, \omega) \longmapsto \frac{\sqrt{2|y|}}{x} F\left(\frac{x}{2|y|}, y\right).$$

Immediately we get the following theorem describing the structure of $A^{(k)}$ inside $L_2(\mathbb{G}, \mathrm{d}\nu_L)$.

Theorem 2.2 ([11], Theorem 2.1) *The unitary operator $U = U_2 U_1$ gives an isometrical isomorphism of the space $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ onto $L_2(\mathbb{R}_+,\, \mathrm{d}x) \otimes L_2(\mathbb{R},\, \mathrm{d}y)$ under which*

(i) the wavelet subspace $A^{(k)}$ is mapped onto $L_k \otimes L_2(\mathbb{R}_+)$, where L_k is the rank-one space generated by Laguerre function $\ell_k(x)$;

(ii) the orthogonal projection $P^{(k)} : L_2(\mathbb{G}, \mathrm{d}\nu_L) \to A^{(k)}$ is unitarily equivalent to the following one $UP^{(k)}U^{-1} = P_0^{(k)} \otimes \chi_+ I$, where $P_0^{(k)}$ is the one-dimensional projection of $L_2(\mathbb{R}_+, \mathrm{d}x)$ onto L_k.

Remark 2.3 Let us mention that connection between certain spaces of wavelet transforms and Bergman spaces is already well-known, see e.g. [15]: the Bergman transform $[B^\alpha F](u, v) = u^{-\alpha-1/2}F(u, v)$ with $\alpha > 0$ gives an isometrical isomorphism of the space $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ onto $L_2(\mathbb{G}, u^{2\alpha-1}\mathrm{d}u\mathrm{d}v)$ under which the space of continuous wavelet transforms of $H_2(\mathbb{R})$-functions with respect to the Bergman wavelet given by $\hat{\psi}^\alpha(\xi) = \chi_+(\xi) c_\alpha \xi^\alpha \mathrm{e}^{-2\pi\xi}$ is mapped onto the weighted Bergman space $\mathscr{A}_{2\alpha-1}(\mathbb{G})$. Here, c_α is a certain normalization constant.

However, we may say more about the connection between the wavelet subspaces and Hardy spaces which reads as follows. Let us mention that the orthogonal projection $P_{\mathbb{R}}$ of $L_2(\mathbb{R})$ onto $H_2(\mathbb{R})$ is called the *Szegö projection.*

Theorem 2.4 *The unitary operator $V = (I \otimes \mathscr{F}^{-1})U_2(I \otimes \mathscr{F})$ gives an isometrical isomorphism of the space $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ onto $L_2(\mathbb{R}_+,\, \mathrm{d}x) \otimes L_2(\mathbb{R},\, \mathrm{d}y)$ under which*

(i) $A^{(k)}$ and $H_2(\mathbb{R})$ are connected by the formula $V\left(A^{(k)}\right) = L_k \otimes H_2(\mathbb{R})$;

(ii) $P^{(k)}$ and $P_{\mathbb{R}}$ are connected by the formula $VP^{(k)}V^{-1} = P_0^{(k)} \otimes P_{\mathbb{R}}$.

$$
\begin{tikzcd}
L_2(\mathbb{G}, \mathrm{d}\nu_L) \arrow[d, "P^{(k)}"] & L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}) \arrow[l, "U^{-1}"'] \arrow[d, "P_0^{(k)} \otimes \chi_+ I"] & L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}) \arrow[l, "U_1"'] \arrow[d, "P_0^{(k)} \otimes P_{\mathbb{R}}"] \\
A^{(k)} \arrow[r, "U"] & L_k \otimes L_2(\mathbb{R}_+) \arrow[r, "U_1^{-1}"] & L_k \otimes H_2(\mathbb{R})
\end{tikzcd}
$$

Figure 2: Visualizing the results of Theorem 2.2 and Theorem 2.4

The diagram on Figure 2 schematically describes all the relations among the constructed operators and spaces appearing in the above two theorems. A detailed analysis of the construction and origin of unitary operators describing the structure of wavelet subspaces from the viewpoint of induced representations of $\mathbb{G}$ is done in [8]. It was shown that these unitary maps have the following properties related to group representations:

(i) they intertwine respective representations of the affine group $\mathbb{G}$;

(ii) they provide a spatial separation of the irreducible components of the affine group's representations.

Indeed, these properties make the unitary maps useful for characterization of $A^{(k)}$ inside the space $L_2(\mathbb{G}, \mathrm{d}\nu_L)$.

Remark 2.5 The suggested time-scale (or, more general time-frequency) point of view was recently successfully used in Abreu's paper [1] to obtain a complete characterization of all lattice sampling and interpolating sequences in the Segal-Bargmann-Fock space of poly-analytic functions or, equivalently, of all lattice vector-valued Gabor frames and vector-valued Gabor Riesz sequences with Hermite functions for $L_2(\mathbb{R}, \mathbb{C}^n)$. This again underlines a new, and perhaps unexpected, connection between poly-analytic functions and time-frequency analysis having a great potential in various applications.

3 Toeplitz operators on poly-analytic spaces

Toeplitz operators form one of the most significant classes of concrete operators because of their importance both in pure and applied mathematics and in many other sciences. In the context of true-poly-analytic Bergman spaces (or, equivalently, wavelet subspaces) for a given bounded function a on $\mathbb{G}$ define the *Toeplitz operator* $T_a^{(k)} : A^{(k)} \to A^{(k)}$ with symbol a usually as $T_a^{(k)} := P^{(k)} M_a$, where M_a is the operator of pointwise multiplication by a on $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ and $P^{(k)}$ is the orthogonal projection from $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ onto $A^{(k)}$. In fact, this provides a mapping between the same wavelet subspaces. It is worth noting that in the case of many wavelet subspaces (parameterized by k) other Toeplitz- and Hankel-type operators may be defined, e.g.

$$
\begin{aligned}
T_a^{(k,l)} &:= P^{(k)} M_a P^{(l)}, \\
h_a^{(k,l)} &:= \bar{P}^{(k)} M_a P^{(l)}, \\
H_a^{(k,l)} &:= \left(I - \sum_{j=0}^{k} P^{(j)} \right) M_a P^{(l)}.
\end{aligned}
$$

In what follows we restrict our attention only to the case $k = l$. The mapping $a \mapsto T_a^{(k)}$ is then interpreted as the quantization rule "on the level k".

3.1 Unitarily equivalent images of Toeplitz operators for symbols depending on $\Im\zeta$

It was observed in several cases, cf. [18], that Toeplitz operators can be transformed into pseudo-differential operators by means of certain unitary maps constructed as an exact analog of the Bargmann transform mapping the Segal-Bargmann-Fock space $F_2(\mathbb{C}^n)$ of Gaussian square-integrable entire functions on complex n-space onto $L_2(\mathbb{R}^n)$. Via this mapping Toeplitz operator $T_a^{(k)} : A^{(k)} \to A^{(k)}$ can be identified with certain pseudo-differential operator $\mathfrak{T}_a^{(k)} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ which provides an analog of the Berezin reducing of Toeplitz operators with anti-Wick symbols on the Fock space $F_2(\mathbb{C}^n)$ to Weyl pseudo-differential operators on $L_2(\mathbb{R}^n)$. This construction may be better seen if we apply this procedure to operator symbols $a(u, v)$ that are only depending on individual variables. Indeed, for the case when $a = a(u)$ depends only on the first spatial variable of $\mathbb{G}$ the operator $T_a^{(k)}$ is simply a multiplication operator with explicitly computable symbol.

Theorem 3.1 ([13], Theorem 3.2) *Let $(u, v) \in \mathbb{G}$. If a measurable symbol $a = a(u)$ does not depend on v, then $T_a^{(k)}$ acting on $A^{(k)}$ is unitarily equivalent to the multiplication operator $\mathfrak{A}_a^{(k)} = \gamma_{a,k} I$ acting on $L_2(\mathbb{R}_+)$, where the function $\gamma_{a,k} : \mathbb{R}_+ \to \mathbb{C}$ is given by*

$$\gamma_{a,k}(\xi) = \int_{\mathbb{R}_+} a\left(\frac{u}{2\xi}\right) \ell_k^2(u)\,\mathrm{d}u, \quad \xi \in \mathbb{R}_+. \tag{2}$$

Example 3.2 Given a point $\lambda_0 \in \mathbb{R}_+$ we have

$$\gamma_{\chi_{[0,\lambda_0]},k}(\xi) = \chi_+(\xi) \int_{\mathbb{R}_+} \chi_{[0,\lambda_0]}\left(\frac{u}{2\xi}\right) \ell_k^2(u)\,\mathrm{d}u = \chi_+(\xi) \int_0^{2\lambda_0\xi} \ell_k^2(u)\,\mathrm{d}u.$$

Immediately, for $\lambda_0 = \frac{1}{2}$ and $k = 0$ we have $\gamma_{\chi_{[0,1/2]},0}(\xi) = 1 - \mathrm{e}^{-\xi}$, $\xi \in \mathbb{R}_+$.

The function $\gamma_{a,k}$ is obtained by integrating a dilation of a symbol $a = a(u)$ of $T_a^{(k)}$ against a Laguerre function of order k. This result extends the result of Vasilevski for the classical Toeplitz operators acting on the Bergman space (i.e., the case $k = 0$ in our notation) in very interesting way which differs from the case of Toeplitz operators acting on weighted Bergman spaces summarized in Vasilevski book [18]. Moreover, the function $\gamma_{a,k}$ sheds a new light upon the investigation of main properties of the corresponding Toeplitz operator $T_a^{(k)}$ with a symbol $a = a(u)$ (measurable and unbounded, in general), such as boundedness, spectrum, invariant subspaces, norm value, etc. Furthermore, the use unitarily equivalent images as model, or local representatives permits us to study Toeplitz operators with much more general symbols.

Since the function $\gamma_{a,k} : \mathbb{R}_+ \to \mathbb{C}$ is responsible for many interesting features of the corresponding Toeplitz operator $T_a^{(k)}$, we present here certain interesting and useful properties of $\gamma_{a,k}$ in what follows.

Theorem 3.3 *Let $(u, v) \in \mathbb{G}$. If $a = a(u) \in L_1(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+)$ such that $\gamma_{a,k}(\xi) \in L_\infty(\mathbb{R}_+)$, then for each $n = 1, 2, \ldots$*

$$\lim_{\xi\to+\infty} \frac{\mathrm{d}^n \gamma_{a,k}(\xi)}{\mathrm{d}\xi^n} = 0$$

holds for each $k \in \mathbb{Z}_+$. Moreover, if $a = a(u) \in C_b^\infty(\mathbb{R}_+)$ such that for each $n \in \mathbb{N}$ holds

$$\lim_{u\to+\infty} u^n\, a^{(n)}(u) = 0,$$

then also

$$\lim_{\xi\to 0} \xi^n \frac{\mathrm{d}^n \gamma_{a,k}(\xi)}{\mathrm{d}\xi^n} = 0$$

for each $n \in \mathbb{N}$ and each $k \in \mathbb{Z}_+$.

The result of above theorem states that the behavior of derivatives of $\gamma_{a,k}$ *does not depend on parameter* k. In fact, it depends only on behavior of the corresponding symbol a (or, its derivatives), but not on the particularly chosen Laguerre functions. This is quite surprising because, as we have already stated, the wavelet transforms with Laguerre functions of order k live, up to a multiplier isomorphism, in the true-poly-analytic Bergman space of order k, which is rather different from the classical Bergman space of analytic functions. Thus, we have the remarkable observation that, asymptotically, all the true-poly-analytic Bergman spaces have "the same behavior". This result has some important consequences in quantum physics, signal analysis and in the asymptotic theory of random matrices, which are not yet completely understood.

Remark 3.4 The special case $n = 1$ yields the following result: if $a \in C_b^1(\mathbb{R}_+)$ with $\lim\limits_{u\to+\infty} ua'(u) = 0$, then for each $k \in \mathbb{Z}_+$ the function $\gamma_{a,k}$ is *slowly varying at infinity* (in the additive sense) and *slowly varying at zero* (in the multiplicative sense).

Easily, for each $k \in \mathbb{Z}_+$ and each $a = a(u) \in L_\infty(\mathbb{R}_+)$ we have

$$\sup_{\xi\in\mathbb{R}_+} |\gamma_{a,k}(\xi)| \le \sup_{u\in\mathbb{R}_+} |a(u)| \int_{\mathbb{R}_+} \ell_k^2(u)\,\mathrm{d}u < +\infty,$$

i.e., $\gamma_{a,k}$ is bounded on $\mathbb{R}_+$ for each $k \in \mathbb{Z}_+$. Moreover, in such a case of bounded symbol a the function $\gamma_{a,k}(\xi)$ is also continuous in each finite point $\xi \in \mathbb{R}_+$, and thus $\gamma_{a,k} \in C_b(\mathbb{R}_+)$. However, as the following examples show $\gamma_{a,k}$ may be bounded even for unbounded symbols.

Example 3.5 (i) For unbounded symbol

$$a(u) = \frac{1}{\sqrt{u}} \sin\frac{1}{u}, \quad u \in \mathbb{R}_+,$$

we have

$$\gamma_{a,1}(\xi) = \frac{\sqrt{2\pi}}{4}\,\mathrm{e}^{-2\sqrt{\xi}} \left[\left(2\sqrt{\xi} - 8\xi\right)\frac{\cos 2\sqrt{\xi}}{2\sqrt{\xi}} + \left(3 - 2\sqrt{\xi}\right)\frac{\sin 2\sqrt{\xi}}{2\sqrt{\xi}}\right]$$

for $\xi \in \mathbb{R}_+$, which is a bounded function on $\mathbb{R}_+$. However, due to computational limitations we can not say anything about the boundedness of $\gamma_{a,k}(\xi)$ for arbitrary k.

(ii) For oscillating symbol $a(u) = \mathrm{e}^{2u\mathrm{i}}$ we have again the bounded function

$$\gamma_{a,k}(\xi) = \frac{(-1)^k}{(\xi - \mathrm{i})^{2k+1}} \sum_{j=0}^{k} (-1)^j \left[\binom{k}{j}\right]^2 \xi^{2j+1}, \quad \xi \in \overline{\mathbb{R}}_+.$$

Moreover, $\gamma_{a,k}(\xi) \in C[0, +\infty]$ for each $k \in \mathbb{Z}_+$.

These examples motivate to study this interesting feature in more detail considering unbounded symbols to have a sufficiently large class of them common to all admissible k. For this purpose denote by $L_1(\mathbb{R}_+, 0)$ the class of functions $a = a(u)$ such that $a(u)\,\mathrm{e}^{-\varepsilon u} \in L_1(\mathbb{R}_+)$ for any $\varepsilon > 0$. For any such $L_1(\mathbb{R}_+, 0)$-symbol $a(u)$ define the following averaging functions

$$C_a^{(1)}(u) = \int_0^u a(t)\,\mathrm{d}t, \quad C_a^{(m)}(u) = \int_0^u C_a^{(m-1)}(t)\,\mathrm{d}t, \quad m = 2, 3, \ldots$$

The functions $C_a^{(m)}$ constitute a "sequence of iterated integrals" of symbol a.

Theorem 3.6 *Let $a = a(u) \in L_1(\mathbb{R}_+, 0)$.*

(i) If for any $m \in \mathbb{N}$ the function $C_a^{(m)}$ has the following asymptotic behavior

$$C_a^{(m)}(u) = O(u^m), \quad \text{as } u \to 0 \text{ as well as } u \to +\infty, \tag{3}$$

then for each $k \in \mathbb{Z}_+$ we have $\sup_{\xi\in\mathbb{R}_+} |\gamma_{a,k}(\xi)| < +\infty$.

(ii) If for any $m, n \in \mathbb{N}$, any $\lambda_1 \in \mathbb{R}_+$ and any $\lambda_2 \in (0, n+1)$ holds

$$C_a^{(m)}(u) = O\left(u^{m+\lambda_1}\right), \quad as \;\; u \to 0, \tag{4}$$

and

$$C_a^{(n)}(u) = O\left(u^{n-\lambda_2}\right), \quad as \;\; u \to +\infty,$$

then for each $k \in \mathbb{Z}_+$ we have $\lim\limits_{\xi \to +\infty} \gamma_{a,k}(\xi) = 0 = \lim\limits_{\xi \to 0} \gamma_{a,k}(\xi)$.

Remark 3.7 The condition (3) guarantees the boundedness of the function $\gamma_{a,k}(\xi)$ at a neighborhood of $\xi = +\infty$, as well as at a neighborhood of $\xi = 0$. Observe that if the both conditions in (3) hold for some $m = m_0$, then they hold also for $m = m_0 + 1$. Indeed,

$$|C_a^{(m_0+1)}(u)| \le \int_0^u |C_a^{(m_0)}(t)| \, \mathrm{d}t \le \mathrm{const} \int_0^u t^{m_0} \, \mathrm{d}t \le \mathrm{const}\, u^{m_0+1}.$$

The main advantage of Theorem 3.6 is that we need not have an explicit form of the corresponding function $\gamma_{a,k}$ for an unbounded symbol $a = a(u)$ to decide about its boundedness. Also, it gives the condition on the behavior of $L_1(\mathbb{R}_+, 0)$-symbols such that the function $\gamma_{a,k}(\xi) \in C[0, +\infty]$.

Example 3.8 For $\alpha > 0$ and $\beta \in (0, 1)$ consider the unbounded symbol

$$a(u) = u^{-\beta} \sin u^{-\alpha}, \quad u \in \mathbb{R}_+.$$

However, the function $a(u)$ is continuous at $u = +\infty$ for all admissible values of parameters, and therefore $\gamma_{a,k}(0) = a(+\infty) = 0$. On the other side, it is difficult to verify the behavior of function $\gamma_{a,k}(\xi)$ at the endpoint $+\infty$ by a direct computation. Since

$$C_a^{(1)}(u) = \frac{u^{\alpha-\beta+1}}{\alpha} \cos u^{-\alpha} + O(u^{2\alpha-\beta+1}), \quad \text{as } u \to 0,$$

then for $\alpha > \beta$ the first condition in (4) holds for $m = 1$ and $\lambda_1 = \alpha - \beta$. By Theorem 3.6 the function $\gamma_{a,k}(\xi)$ is bounded.
If $\alpha \le \beta$, then

$$C_a^{(m)}(u) = O(u^{m\alpha-\beta+m}), \quad \text{as } u \to 0.$$

Thus, for each $\alpha \le \beta$ there exists $m_0 \in \mathbb{N}$ such that $m_0\alpha > \beta$, and therefore the first condition in (4) holds for $m = m_0$ and $\lambda_1 = m_0\alpha - \beta$, which guarantees that $\gamma_{a,k}(\xi)$ is continuous at $\xi = 0$. Thus, for all parameters $\alpha > 0$ and $\beta \in (0, 1)$ the function $\gamma_{a,k}(\xi) \in C[0, +\infty]$ for each $k \in \mathbb{Z}_+$.

We also mention an alternative way to the properties of $\gamma_{a,k}$: using the explicit form of Laguerre polynomial we may write

$$\gamma_{a,k}(\xi) = 2\xi \int_{\mathbb{R}_+} a(u)\, \ell_k^2(2u\xi) \, \mathrm{d}u = \sum_{i=0}^{k} \sum_{j=0}^{k} \kappa(k, i, j)\, \widetilde{\gamma}_{a,i+j}(\xi), \tag{5}$$

where

$$\widetilde{\gamma}_{a,\lambda}(\xi) := (2\xi)^{\lambda+1} \int_{\mathbb{R}_+} a(u)\, u^\lambda \, \mathrm{e}^{-2u\xi} \, \mathrm{d}u = \xi^{\lambda+1} \int_{\mathbb{R}_+} a(u/2)\, u^\lambda \, \mathrm{e}^{-u\xi} \, \mathrm{d}u,$$

and

$$\kappa(k, i, j) := \frac{(-1)^{i+j}}{i!j!} \binom{k}{i} \binom{k}{j}.$$

It is immediate that the boundedness of each function $\widetilde{\gamma}_{a,\lambda}$ for $\lambda \in \{0, 1, \ldots, 2k\}$ implies the boundedness of function $\gamma_{a,k}$. Therefore, given $\lambda \in \mathbb{Z}_+$ and a locally sumable function $a = a(u)$ we now introduce the weighted means of symbol a as follows

$$D_{a,\lambda}^{(1)}(u) = \int_0^u a(t/2) t^\lambda \, \mathrm{d}t, \quad D_{a,\lambda}^{(m)}(u) = \int_0^u D_{a,\lambda}^{(m-1)}(t) \, \mathrm{d}t, \quad m = 2, 3, \ldots$$

It is obvious that $D_{a,0}^{(m)}(u) = 2^m C_a^{(m)}(u/2)$ for each $m \in \mathbb{N}$. In this setting the result of Theorem 3.6 reads as follows: if $a = a(u) \in L_1(\mathbb{R}_+, 0)$ and for any $m \in \mathbb{N}$ the function $D_{a,0}^{(m)}$ has the asymptotic behavior

$$D_{a,0}^{(m)}(u) = O(u^m), \quad \text{as} \quad u \to 0 \ \text{ as well as } \ u \to +\infty,$$

then each function $\widetilde{\gamma}_{a,\lambda}$ is bounded on $\mathbb{R}_+$ for every $\lambda \in \mathbb{Z}_+$, which means that $\gamma_{a,k}$ is bounded on $\mathbb{R}_+$ for each $k \in \mathbb{Z}_+$. Moreover, we may extend the above observation about asymptotic behavior using any (positive) weight $\lambda_0 \in \mathbb{Z}_+$ appearing in weighted means $D_{a,\lambda_0}^{(m)}$.

Theorem 3.9 *Let $a = a(u) \in L_1(\mathbb{R}_+, 0)$. If for any $\lambda_0 \in \mathbb{Z}_+$ and for any $m \in \mathbb{N}$ the function $D_{a,\lambda_0}^{(m)}$ has the following asymptotic behavior*

$$D_{a,\lambda_0}^{(m)}(u) = O(u^{\lambda_0+m}), \quad \textit{as} \quad u \to 0 \ \textit{ as well as } \ u \to +\infty,$$

then $\gamma_{a,k}$ is bounded for each $k \in \mathbb{Z}_+$.

Furthermore, if $a = a(u)$ is a bounded symbol on $\mathbb{G}$, then the operator $T_a^{(k)}$ is clearly bounded on each $A^{(k)}$, and for its operator norm holds $\|T_a^{(k)}\| \leq \text{ess-sup}\,|a(u)|$. Thus, all spaces $A^{(k)}$, $k \in \mathbb{Z}_+$, are naturally appropriate for Toeplitz operators with bounded symbols. However, we may observe that the result of Theorem 3.1 suggests considering not only $L_\infty(\mathbb{G}, \mathrm{d}\nu_L)$-symbols, but also *unbounded* ones. In this case we obviously have

Corollary 3.10 *The operator $T_a^{(k)}$ with a measurable symbol $a = a(u)$, $u \in \mathbb{R}_+$, is bounded on $A^{(k)}$ if and only if the corresponding function $\gamma_{a,k}(\xi)$ is bounded on $\mathbb{R}_+$, and*

$$\|T_a^{(k)}\| = \sup_{\xi \in \mathbb{R}_+} |\gamma_{a,k}(\xi)|.$$

From this result we immediately have that *all the obtained results for boundedness of $\gamma_{a,k}$ in terms of iterated integrals are, in fact, sufficient conditions for boundedness of the corresponding Toeplitz operator $T_a^{(k)}$ on each $A^{(k)}$.* The following result provides another criteria for simultaneous boundedness of Toeplitz operators on each wavelet subspaces.

Theorem 3.11 *(i) Let $a = a(u) \in L_1(\mathbb{R}_+, 0)$ be non-negative almost everywhere. If $T_a^{(0)}$ is bounded on $A^{(0)}$, then the operator $T_a^{(k)}$ is bounded on $A^{(k)}$ for each $k \in \mathbb{Z}_+$.*

(ii) Let $C_a^{(m)}$ be non-negative almost everywhere for a certain $m = m_0$. If $T_a^{(0)}$ is bounded on $A^{(0)}$, then the operator $T_a^{(k)}$ is bounded on $A^{(k)}$ for each $k \in \mathbb{Z}_+$.

Theorem states that under the assumption of non-negativity of a symbol $a \in L_1(\mathbb{R}_+, 0)$, or its mean $C_a^{(m)}$ for certain $m \in \mathbb{N}$, the boundedness of Toeplitz operator $T_a = T_a^{(0)}$ on the Bergman space $\mathscr{A}_2(\Pi) = A^{(0)}$ implies the boundedness of Toeplitz operator $T_a^{(k)}$ acting on $A^{(k)}$ for each $k \in \mathbb{Z}_+$. However, if $a \in L_1(\mathbb{R}_+, 0)$, the question whether the boundedness of $T_a^{(k_0)}$ on $A^{(k_0)}$ for certain $k_0 \in \mathbb{N}$ implies the boundedness of $T_a^{(k)}$ acting on $A^{(k)}$ for each $k \in \mathbb{Z}_+$ (smaller, or greater than k_0) is still open. This is related to question whether the boundedness of $T_a^{(k)}$ may happen only simultaneously for all $k \in \mathbb{Z}_+$.

It is immediate that an unbounded symbol must have a sufficiently sophisticated oscillating behavior at neighborhoods of the points 0 and $+\infty$ to generate a bounded Toeplitz operator. In what follows we show that infinitely growing positive symbols cannot generate bounded Toeplitz operators in general. For this purpose for a non-negative function $a = a(u)$ put

$$\theta_a(u) = \inf_{t \in (0,u)} a(t) \ \text{ and } \ \Theta_a(u) = \inf_{t \in (u/2,u)} a(t).$$

Theorem 3.12 *For a given non-negative symbol $a = a(u)$ if*

$$\text{either} \quad \lim_{u\to 0} \theta_a(u) = +\infty, \quad \text{or} \quad \lim_{u\to +\infty} \Theta_a(u) = +\infty, \tag{6}$$

then $T_a^{(k)}$ is unbounded on each $A^{(k)}$, $k \in \mathbb{Z}_+$.

Example 3.13 For the family of non-negative symbols on $\mathbb{R}_+$ in the form

$$a(u) = u^{-\beta} \ln^2 u^{-\alpha}, \quad \beta \in [0,1],\ \alpha > 0,$$

we have that for all admissible parameters holds $\lim_{u\to 0} \theta_a(u) = +\infty$, and thus $T_a^{(k)}$ is unbounded on $A^{(k)}$ for each $k \in \mathbb{Z}_+$.

In the following we provide an interesting example of symbols a, b for which $T_a^{(k)}, T_b^{(k)}$ are bounded, but $T_{ab}^{(k)}$ is not on the whole scale of parameters k.

Example 3.14 Let us consider two symbols on $\mathbb{R}_+$ in the form

$$a(u) = u^{-\beta} \sin u^{-\alpha}, \ \ \beta \in (0,1), \ \alpha \geq \beta, \quad \text{and} \quad b(u) = u^{\tau} \sin u^{-\alpha}, \ \ \tau \in (0,\beta).$$

Then $T_a^{(k)}$ is bounded for each $k \in \mathbb{Z}_+$, and since $b(u) \in C[0,+\infty]$, then $T_b^{(k)}$ is bounded for each $k \in \mathbb{Z}_+$ as well. Put

$$c(u) = a(u)b(u) = \frac{u^{-\delta}}{2} - \frac{u^{-\delta}}{2} \cos 2u^{-\alpha} = c_1(u) + c_2(u),$$

where $\delta = \beta - \tau \in (0,1)$. Clearly, $c(u)$ is an unbounded symbol. However, $T_{c_2}^{(k)}$ is bounded for each $k \in \mathbb{Z}_+$. Since

$$\theta_{c_1}(u) = \inf_{t\in(0,u)} \frac{1}{2t^{\delta}} = \frac{1}{2u^{\delta}} \to +\infty, \quad \text{as} \quad u \to 0,$$

then the operator $T_{c_1}^{(k)}$ is unbounded for each $k \in \mathbb{Z}_+$. Thus, the Toeplitz operator $T_{ab}^{(k)}$ is unbounded on $A^{(k)}$ for each $k \in \mathbb{Z}_+$ showing that the semi-commutator $\left[T_a^{(k)}, T_b^{(k)}\right)$ *is not compact.*

Perhaps the most surprising feature of behavior of Toeplitz operators on $A^{(k)}$ with symbols depending only on vertical coordinate in the upper half-plane is appearance of certain commutative algebras of Toeplitz operators on these spaces which are practically unknown in the literature. Therefore, denote by $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ the C^*-subalgebra of $L_\infty(\mathbb{R}_+)$ which consists of all functions having limits at the points 0 and $+\infty$. For $k \in \mathbb{Z}_+$ denote by $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ the C^*-algebra generated by all operators $T_a^{(k)}$ acting on $A^{(k)}$ with symbols $a \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$.

Theorem 3.15 *For $a \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ the corresponding functions $\gamma_{a,k}(\xi)$, $k \in \mathbb{Z}_+$, possess the following properties*

(i) $\gamma_{a,k}(\xi) \in C[0,+\infty]$;

(ii) $\gamma_{a,k}(+\infty) = \lim_{\xi\to+\infty} \gamma_{a,k}(\xi) = \lim_{u\to 0} a(u) = a(0)$;

(iii) $\gamma_{a,k}(0) = \lim_{\xi\to 0} \gamma_{a,k}(\xi) = \lim_{u\to+\infty} a(u) = a(+\infty)$.

Indeed, the behavior of a bounded function $a(u)$ near the point 0, or $+\infty$ determines the behavior of function $\gamma_{a,k}(\xi)$ near the point $+\infty$, or 0, respectively. An interesting observation is that the limits at infinity and at zero of the function $\gamma_{a,k}$ *are completely independent of parameter* k, which is rather surprising in this case and it again may be useful in various contexts. The existence of limits of $a(u)$ at these endpoints guarantees the continuity of $\gamma_{a,k}(\xi)$ on $[0,+\infty]$, however this condition is not necessary even for bounded symbols as the following example shows.

Example 3.16 For $a(u) = \sin u$, $u \in \mathbb{R}_+$, we have

$$\gamma_{a,1}(\xi) = 2\xi \int_{\mathbb{R}_+} \sin u \, \mathrm{e}^{-2u\xi}(1-2u\xi)^2 \, \mathrm{d}u = \frac{2\xi\,(1-16\xi^2+48\xi^4)}{(1+4\xi^2)^3}, \quad \xi \in \mathbb{R}_+,$$

which yields

$$\lim_{\xi\to+\infty} \gamma_{a,1}(\xi) = \lim_{\xi\to 0} \gamma_{a,1}(\xi) = 0.$$

On the other hand, Example 3.5(i) provides an example of unbounded symbol $a(u)$ such that the corresponding function $\gamma_{a,k}$ *is continuous* on $[0,+\infty]$ for each $k \in \mathbb{Z}_+$.

Corollary 3.17 *Each C^*-algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$, $k \in \mathbb{Z}_+$, is isometric and isomorphic to $C[0,+\infty]$. The corresponding isomorphism is generated by the following mapping $\tau^{(k)}$: $T_a^{(k)} \longmapsto \gamma_{a,k}(\xi)$.*

According to this result the symbol algebra $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ is an example of algebra such that the operator algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ generated by Toeplitz operators $T_a^{(k)}$ with symbol $a \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ *is commutative for each* $k \in \mathbb{Z}_+$. An interesting question may be a characterization of all such algebras $\mathscr{A}$ of symbols for which the operator algebra $\mathscr{T}_k(\mathscr{A})$ is commutative for each k. It seems to be a challenging problem.

Easily, continuity of function $\gamma_{a,k}$ on the whole $\overline{\mathbb{R}}_+$ guarantees its boundedness, and therefore the boundedness of the corresponding Toeplitz operator $T_a^{(k)}$ on $A^{(k)}$. Moreover, $T_a^{(k)}$ with a symbol $a = a(u)$ belongs to the algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ if and only if the corresponding function $\gamma_{a,k}(\xi)$ belongs to $C[0,+\infty]$. This means that the algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ also contains (bounded) Toeplitz operators whose (generally unbounded) symbols $a(u)$ need not have limits at the endpoints 0 and $+\infty$. For instance, Example 3.5(ii) provides oscillating symbols $a = a(u)$ for which Toeplitz operator $T_a^{(k)}$ belongs to the algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ for the whole range of parameters k. Now we give an example of a bounded oscillating symbol such that the bounded operator $T_a^{(k)}$ does not belong to the algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$.

Example 3.18 The function $a(u) = u^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln u}$, $u \in \mathbb{R}_+$, is oscillating near the endpoints 0 and $+\infty$, but it is bounded on $\mathbb{R}_+$, and therefore $T_a^{(k)}$ is bounded for each $k \in \mathbb{R}_+$. Changing the variable $x = 2u\xi$ yields

$$\gamma_{a,k}(\xi) = 2\xi \int_{\mathbb{R}_+} u^{\mathrm{i}} \ell_k^2(2u\xi)\,\mathrm{d}u = (2\xi)^{-\mathrm{i}} \int_{\mathbb{R}_+} x^{\mathrm{i}} \ell_k^2(x)\,\mathrm{d}x.$$

Since the last integral is a constant depending on k, the function $\gamma_{a,k}(\xi)$ oscillates and has no limit when $\xi \to 0$ as well as when $\xi \to +\infty$. Thus, the bounded Toeplitz operator $T_a^{(k)}$ *does not belong* to the algebra $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$. Hence not all oscillating symbols (even bounded and continuous) generate an operator from $\mathscr{T}_k\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$.

Introduce now the C^*-subalgebra $\mathscr{T}\left(T_{a_+}^{(0)}\right)$ of the algebra $\mathscr{T}_0\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right)$ which is generated by identity and the Toeplitz operator $T_{a_+}^{(0)}$ with symbol $a_+ = \chi_{[0,1/2]}$. By Example 3.2 the corresponding function $\gamma_{a_+,0} \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ is continuous on $[0,+\infty]$, therefore $T_{a_+}^{(0)}$ is self-adjoint and $\operatorname{sp} T_{a_+}^{(0)} = \operatorname{Range} \gamma_{a_+,0} = [0,1]$. Also, the function $\gamma_{a_+,0}$ is strictly increasing real-valued function which separates the point of $[0,+\infty]$. Thus, the algebra $\mathscr{T}\left(T_{a_+}^{(0)}\right)$ consists of all operators of the form $h\left(T_{a_+}^{(0)}\right)$ with $h \in C[0,1]$ by functional calculus for C^*-algebras. Summarizing, we have

Theorem 3.19 *The algebra* $\mathscr{T}_0\left(L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)\right) = \mathscr{T}\left(T_{a_+}^{(0)}\right)$, *and*

(i) is isomorphic and isometric to $C[0,+\infty]$*;*

(ii) is generated by identity and the single Toeplitz operator $T_{a_+}^{(0)}$*;*

(iii) consists of all operators of the form $h\left(T_{a_+}^{(0)}\right)$ *with* $h \in C[0,1]$.

We have chosen Toeplitz operator $T_{a_+}^{(0)}$ as the starting operator because in this specific case the equation $x = \gamma_{a_+,0}(\xi) = 1 - \mathrm{e}^{-\xi}$ admits an explicit solution. But we can start from any operator $T_{\chi_{[0,\lambda]}}^{(0)}$ with symbol $a(u) = \chi_{[0,\lambda]}(u)$, $\lambda \in \mathbb{R}_+$. Indeed, the function $\gamma_{\chi_{[0,\lambda]},0}(\xi)$ is strictly increasing which implies that the function

$$\Delta_\lambda(x) = 1 - (1-x)^{2\lambda}, \quad x \in [0,1],$$

is strictly increasing as well, and thus the function Δ_λ^{-1} is well defined and continuous on $[0,1]$. Clearly, for $\lambda_1, \lambda_2 \in \mathbb{R}_+$ we have

$$\left(\Delta_{\lambda_2} \circ \Delta_{\lambda_1}^{-1}\right)\left(T_{\chi_{[0,\lambda_1]}}^{(0)}\right) = T_{\chi_{[0,\lambda_2]}}^{(0)},$$

where $\circ$ is the usual composition of real functions. This means that for any symbol $a = a(u) = \chi_{[0,\lambda]}(u)$ the operator $T_a^{(0)}$ belong to the algebra $\mathscr{T}\left(T_{a_+}^{(0)}\right)$, and is the function of $T_{a_+}^{(0)}$, i.e., $T_{\chi_{[0,\lambda]}}^{(0)} = \Delta_\lambda\left(T_{a_+}^{(0)}\right)$. In fact, the (localization) operator $T_{a_+}^{(0)}$ gives a reconstruction of a signal on the segment $\Omega_{1/2} = \mathbb{R} \times (0,1/2]$ and level 0. Then we may obtain any operator $T_{\chi_{[0,\lambda]}}^{(0)}$ (giving a reconstruction of a signal on the segment $\Omega_\lambda = \mathbb{R} \times (0,\lambda]$ and level 0) from this operator $T_{a_+}^{(0)}$, i.e., from the reconstruction of a signal on the segment $\Omega_{1/2}$ and level 0 we may obtain a reconstruction of the same signal on the same level on an arbitrary segment Ω_λ using the function Δ_λ which is easy to compute.

In particular and quite surprisingly, each Toeplitz operator $T_a^{(k)}$ with symbols $a \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ is a certain continuous function of the initial operator and this function can be figured out.

Theorem 3.20 *For each* $a = a(u) \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ *the Toeplitz operator* $T_a^{(k)}$ *belongs to the algebra* $\mathscr{T}\left(T_{a_+}^{(0)}\right)$, *and is the following function of the operator* $T_{a_+}^{(0)}$

$$\left(\nabla_{a,\lambda}^{(k)} \circ \Delta_\lambda\right)\left(T_{a_+}^{(0)}\right) = T_a^{(k)},$$

where

$$\nabla_{a,\lambda}^{(k)}(x) = -\frac{1}{\lambda}\ln(1-x)\int_{\mathbb{R}_+} a(u)(1-x)^{u/\lambda} L_k^2\left(-\frac{u}{\lambda}\ln(1-x)\right)\,\mathrm{d}u$$

with $\lambda \in \mathbb{R}_+$ *and* $x \in [0,1]$.

Theorem 3.20 states that if we know the reconstruction of a signal on a segment Ω_λ and level 0, we might get an arbitrary reconstruction of the signal (as its filtered version using a real bounded function a of scale having limits in critical points of boundary of $\mathbb{R}_+$ such that the corresponding function $\gamma_\cdot$ separates the points of $\overline{\mathbb{R}}_+$) on an arbitrary level k using the function $\nabla_{a,\lambda}^{(k)}$. Theoretically, for the purpose to study localization of a signal in the time-scale plane the result of Theorem 3.20 suggests to consider certain "nice" symbols on the first level 0 (indeed, Toeplitz operators on $\mathscr{A}_2(\Pi)$ with symbols as characteristic functions of some interval in $\mathbb{R}_+$) instead of possibly complicated $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$-symbols with respect to "different microscope" represented by the level k. On the other hand, to compute the corresponding function $\nabla_{a,\lambda}^{(k)}$ need not be always easy.

3.2 Unitarily equivalent images of Toeplitz operators for symbols depending on $\Re\zeta$

For general symbols $a = a(u, v)$ the operator $T_a^{(k)}$ is no longer unitarily equivalent to a multiplication operator $\mathfrak{A}_a^{(k)}$. For symbols depending on the second (vertical) variable in the upper half-plane Π a certain class of pseudo-differential operaors appears. In what follows $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$.

Theorem 3.21 ([13], Theorem 3.3) *Let $(u, v) \in \mathbb{G}$. If a measurable function $b = b(v)$ does not depend on u, then $T_b^{(k)}$ acting on $A^{(k)}$ is unitarily equivalent to the operator $\mathfrak{B}_b^{(k)}$ acting on $L_2(\mathbb{R}_+)$ given by*

$$\left[\mathfrak{B}_b^{(k)} f\right](\xi) = \int_{\mathbb{R}_+} B_k(\xi, t)\, \hat{b}(\xi - t) f(t)\, \mathrm{d}t, \quad \xi \in \mathbb{R}_+,$$

where the function $B_k : \mathbb{R}_+^2 \to \mathbb{C}$ has the form

$$B_k(\xi, t) = \frac{2\sqrt{t\xi}}{t + \xi} P_k\left(\frac{8t\xi}{(t + \xi)^2} - 1\right) \tag{7}$$

with

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$$

being the Legendre polynomial of degree $n \in \mathbb{Z}_+$ for $x \in [-1, 1]$.

Immediately, for each $k \in \mathbb{Z}_+$ the function $B_k : \mathbb{R}_+^2 \to \mathbb{R}$ has the following remarkable properties:

(i) B_k is continuous and bounded on $\mathbb{R}_+^2$;

(ii) B_k is a symmetric function, i.e., $B_k(\xi, t) = B_k(t, \xi)$ for each $(\xi, t) \in \mathbb{R}_+^2$;

(iii) $B_k(\xi, t) \in C^\infty(\mathbb{R}_+^2)$;

(iv) B_k is homogeneous (of order 0), i.e., for each $\alpha > 0$ holds $B_k(\alpha\xi, \alpha t) = B_k(\xi, t)$ for each $(\xi, t) \in \mathbb{R}_+^2$;

(v) $B_k(\xi, \xi) = 1$ for all $\xi \in \mathbb{R}_+$.

Again, the above result includes the well-known result for classical Toeplitz operators on the Bergman space $\mathscr{A}_2(\Pi)$ as a special case. In fact, for $k = 0$ Toeplitz operator T_b with a symbol $b = b(\Re\zeta) = b(v)$ acting on the Bergman space $\mathscr{A}_2(\Pi)$ is unitarily equivalent to the following integral operator

$$[\mathfrak{B}_b f](x) = \int_{\mathbb{R}_+} \frac{2\sqrt{xy}}{x + y} K(y - x) f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}_+,$$

where K is the Fourier transform of the function $b(-v)$. As we have already mentioned, our case of Toeplitz operators $T_a^{(k)}$ depending on a "discrete" weight parameter $k \in \mathbb{Z}_+$ is different from the case of Toeplitz operators $T_a^{(\lambda)}$ depending on a "continuous" weight parameter $\lambda \in (-1, +\infty)$ studied in [18] for weighted Bergman spaces.

In what follows we deal with the Fredholm theory for Toeplitz operator algebras on poly-analytic spaces. Thus, consider the class of integral operators on $\mathbb{R}_+$ of the form

$$(Hf)(x) = \int_{\mathbb{R}_+} h(x, y) K(x - y) f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}_+, \tag{8}$$

where

(H1) for the Fourier transform $\hat{K}$ of the function K holds $|\hat{K}^{(j)}(\omega)| \le \frac{C_j}{(1+\omega^2)^{j/2}}$ for each $j \in \mathbb{Z}_+$;

(H2) there exists limits $\hat{K}_\pm = \lim\limits_{\omega\to\pm\infty} \hat{K}(\omega)$, and the Fourier transform of a function $K_0 \in L_1(\mathbb{R})$ may be written in the form

$$\hat{K}_0(\omega) = \hat{K}(\omega) - \hat{K}_+\chi_+(\omega) - \hat{K}_-\chi_-(\omega);$$

(H3) $h(x,y) \in C^\infty(\mathbb{R}^2_+)$, and for all $\alpha > 0$ holds $h(\alpha x, \alpha y) = h(x,y)$;

(H4)

$$\int_{\mathbb{R}_+} \frac{|h(1,t)|}{1+t}\,\frac{\mathrm{d}t}{\sqrt{t}} < +\infty.$$

With the operator H we associate the function q_H on $\mathbb{R}$ as follows

$$q_H(\lambda) = \frac{1}{\mathrm{i}\pi}\int_{\mathbb{R}_+} t^{-\mathrm{i}\lambda}\,\frac{h(1,t)}{1-t}\,\frac{\mathrm{d}t}{\sqrt{t}}, \quad \lambda \in \mathbb{R}.$$

Denote by $SV^\infty(\mathbb{R}_+)$ the class of functions $f(x) \in C_b^\infty(\mathbb{R}_+)$ which are slowly varying at infinity (in the additive sense) and slowly varying at zero (in the multiplicative sense), see Remark 3.4, i.e., $\lim_{x\to+\infty} f'(x) = 0$ and $\lim_{x\to 0} xf'(x) = 0$. Let $SV(\mathbb{R}_+)$ be the closure of $SV^\infty(\mathbb{R}_+)$ in $C_b(\mathbb{R}_+)$. Further, denote by $\mathscr{H}$ the C^*-algebra generated by the integral operators H of the form (8) with the property

$$\sup_{\lambda\in\mathbb{R}} |q_H(\lambda)| < +\infty, \tag{9}$$

and the multiplication operators $f(x)I$ with $f \in SV^\infty(\mathbb{R}_+)$. In such a case the commutator $[f(x)I, H] = f(x)H - Hf(x)I$ is compact on $L_2(\mathbb{R}_+)$.

For each $k \in \mathbb{Z}_+$ the operator $\mathfrak{B}_b^{(k)}$ with $b(v) \in L_\infty(\mathbb{R})$ is of the form (8) with $\hat{K}(\omega) = b(-\omega)$ and is clearly bounded on $L_2(\mathbb{R}_+)$. The associated function $q_{\mathfrak{B}_b^{(k)}}$ has the form

$$q_{\mathfrak{B}_b^{(k)}}(\lambda) = \frac{2}{\mathrm{i}\pi}\int_{\mathbb{R}_+} \frac{t^{-\mathrm{i}\lambda}}{(1+t)(1-t)} P_k\left(\frac{8t}{(t+1)^2} - 1\right)\mathrm{d}t, \quad \lambda \in \mathbb{R},$$

and it is possible to prove, see [14], that for each $k \in \mathbb{Z}_+$

$$\sup_{\lambda\in\mathbb{R}} \left| q_{\mathfrak{B}_b^{(k)}}(\lambda)\right| < +\infty,$$

which implies that $\mathfrak{B}_b^{(k)} \in \mathscr{H}$ for each $k \in \mathbb{Z}_+$ whenever $b = b(v) \in C(\overline{\mathbb{R}})$. According to Theorem 3.3 $\gamma_{a,k}(x) \in SV(\mathbb{R}_+)$ whenever $a = a(u) \in C_b^1(\mathbb{R}_+)$ such that $\lim\limits_{u\to+\infty} ua'(u) = 0$, and therefore $\gamma_{a,k}(x)I \in \mathscr{H}$. Thus, denote by $\widetilde{SV}(\mathbb{R}_+)$ the C^*-algebra generated by functions $a = a(u) \in C_b^1(\mathbb{R}_+)$ with $\lim\limits_{u\to+\infty} ua'(u) = 0$. For each $k \in \mathbb{Z}_+$ consider the C^*-algebra

$$\mathscr{T}_k = \mathscr{T}_k\left(C(\overline{\mathbb{R}}), \widetilde{SV}(\mathbb{R}_+)\right)$$

generated by all Toeplitz operators $T_b^{(k)}$ with $b = b(v) \in C(\overline{\mathbb{R}})$, and $T_a^{(k)}$ with $a = a(u) \in \widetilde{SV}(\mathbb{R}_+)$ acting on $A^{(k)}$, where $\zeta = (u,v) \in \mathbb{G}$. It is immediate that $\mathscr{T}_k$ provides a parameterized family of operator algebras with compact commutator property and non-compact semi-commutator property.

Now, introduce two ideals of the algebra $SV(\mathbb{R}_+)$,

$$C_0^0(\mathbb{R}_+) = \left\{a(u) \in SV(\mathbb{R}_+);\ \lim_{u\to 0} a(u) = 0\right\};$$
$$C_0^{+\infty}(\mathbb{R}_+) = \left\{a(u) \in SV(\mathbb{R}_+);\ \lim_{u\to+\infty} a(u) = 0\right\},$$

and let $\operatorname{Sym}\mathscr{H} = \mathscr{H}/\mathscr{K} = \widehat{\mathscr{H}}$ be the Fredholm symbol algebra of the algebra $\mathscr{H}$, where $\mathscr{K}$ is the ideal of all compact operators on $L_2(\mathbb{R}_+)$. For each $\xi_0 \in \overline{\mathbb{R}}_+$ the local algebra is defined as $\mathscr{H}(\xi_0) = \widehat{\mathscr{H}}/J(\xi_0)$, where $J(\xi_0)$ is the closed two-sided ideal of the algebra $\widehat{\mathscr{H}}$ generated by the maximal ideal of $C(\overline{\mathbb{R}}_+)$ corresponding to the point ξ_0. Clearly, $C(\overline{\mathbb{R}}_+)$ is a central commutative subalgebra of $\widehat{\mathscr{H}}$. Denote by $\mathfrak{S}$ the C^*-algebra of all vector-valued functions σ continuous on $\overline{\mathbb{R}}_+$, where $\sigma(\xi) \in \mathscr{H}(\xi)$ for each $\xi \in \overline{\mathbb{R}}_+$, with point-wise operations, and the norm $\|\sigma\| = \sup_{\xi\in\overline{\mathbb{R}}_+} \|\sigma(\xi)\|$.

Theorem 3.22 *For each $k \in \mathbb{Z}_+$ the Fredholm symbol algebra $\operatorname{Sym}\mathscr{T}_k = \mathscr{T}_k/\mathscr{K}$ of Toeplitz operator algebra $\mathscr{T}_k$ is isomorphic and isometric to the algebra $\mathfrak{S}$. The symbol homomorphism*

$$\operatorname{sym}_k : \mathscr{T}_k \longrightarrow \operatorname{Sym}\mathscr{T}_k = \mathfrak{S}$$

is generated by the following mapping of the generators of the algebra $\mathscr{H}$

$$\operatorname{sym}_k : T_a^{(k)} \longmapsto \begin{cases} \gamma_{a,k}(\xi) + C_0^0(\mathbb{R}_+), & \xi = 0 \\ (\gamma_{a,k}(\xi), \gamma_{a,k}(\xi)), & \xi \in \mathbb{R}_+; \\ \gamma_{a,k}(\xi) + C_0^{+\infty}(\mathbb{R}_+), & \xi = +\infty \end{cases}$$

$$\operatorname{sym}_k : T_b^{(k)} \longmapsto \begin{cases} \frac{1}{2}\left[\Big(b(-\infty) + b(+\infty)\Big) + \Big(b(-\infty) - b(+\infty)\Big) q_{\mathfrak{B}_b^{(k)}}(\lambda)\right], & \xi = 0 \\ (b(-\infty), b(+\infty)), & \xi \in \mathbb{R}_+; \\ b(-\omega), & \xi = +\infty \end{cases}$$

where $a = a(u) \in \widetilde{SV}(\mathbb{R}_+)$ and $b = b(v) \in C(\overline{\mathbb{R}})$ with $\zeta = (u, v) \in \mathbb{G}$.

4 Conclusion

In this paper we review our recent results on Toeplitz operators on wavelet subspaces with respect to a special parameterized family of wavelets from Laguerre functions. As it was shown in [2] such spaces are in fact the true-poly-analytic Bergman spaces over the upper half-plane Π providing thus a useful and interesting tool for investigating basic properties of Toeplitz operators acting on them and their algebras via time-scale approach. We list here some questions and possible directions which could be interesting (from our point of view) for further study.

(i) We suppose that considering an appropriate family of admissible wavelets $\psi^{(k,\alpha)}$ related to generalized Laguerre functions

$$\ell_k^{(\alpha)}(x) := \left[\frac{k!}{\Gamma(k+\alpha+1)}\right]^{1/2} x^{\alpha/2}\, \mathrm{e}^{-x/2} L_k^{(\alpha)}(x), \quad x \in \mathbb{R}_+,$$

we may obtain analogous structural results and representations of weighted (true)-poly-analytic Bergman spaces. *What are the corresponding Toeplitz (Hankel, etc.) operators acting on these spaces and which properties do they have*?

(ii) In connection with the results of this paper we may ask *what happens to properties of Toeplitz operators $T_a^{(k)}$* acting on true-poly-analytic Bergman spaces (wavelet subspaces $A^{(k)}$) *when the weight parameter $k \in \mathbb{Z}_+$ varies*?

(iii) In particular, *to study the spectral properties of a Toeplitz operator $T_a^{(k)}$* and the related asymptotic properties of function $\gamma_{a,k}$ *in dependence on k*, and compare their limit behavior under $k \to +\infty$ with corresponding properties of the initial symbol a. Also, the similar questions may be stated for the above mentioned Toeplitz operator acting on weighted poly-analytic spaces, but here the question of a varying parameter is not clear, because in this case at least two weighted parameters will appear.

(iv) Recently, an interesting result has been obtained in [10] for the classical Bergman space of analytic functions on the upper half-plane. As it is already known, the C^*-algebra generated

by Toeplitz operators with bounded symbols (depending on vertical coordinate $u = \mathrm{Im}(\zeta)$ only, the so-called vertical Toeplitz operators in terminology of [10]) is isometrically isomorphic to the C^*-algebra generated by the set $\Gamma_0 := \{\gamma_{a,0};\, a \in L_\infty(\mathbb{R}_+)\}$. In [10] authors showed that Γ_0 is dense in the space $\mathrm{VSO}(\mathbb{R}_+)$ of all very slowly oscillating functions on the positive half-line. Thus, we conjecture the following more general result: *for each $k \in \mathbb{Z}_+$ the set*

$$\Gamma_k := \{\gamma_{a,k};\, a \in L_\infty(\mathbb{R}_+)\}$$

is dense in $\mathrm{VSO}(\mathbb{R}_+)$.

(v) In practical applications certain algebraic operations with symbols and operators naturally appear. In signal analysis, the problem of finding a filter that has the same effect as two filters arranged in series amounts to the computation of the product of two localization operators. Thus, *what is the product of two Toeplitz operators (in exact, or at least approximate formulas)*? The answer does not seem to be so simple and seems to depend on the availability of a useful formula for Toeplitz operator. We think that some technical information in this direction obtained in this paper studying the particular cases of symbols on the affine group $\mathbb{G}$ (e.g., if a depends only on horizontal variable $u \in \mathbb{R}_+$ in the upper half-plane, then $T_a^{(k)}$ is a Fourier multiplier) may be helpful and crucial.

(vi) The continuous wavelet transform in the one-dimensional case can be obtained in two ways: one from the theory of square-integrable group representation, and the other from the Calderón representation formula. Also, it is known that in the one-dimensional case these two different ways can induce the same results. However, in the higher dimensional case these two ways will induce two different results. One is the Calderón representation formula, which induces a decomposition of $L_2(\mathbb{R}_+ \times \mathbb{R}^n, u^{-n-1}\mathrm{d}u\mathrm{d}v)$, and the other is the wavelet transform associated with the square-integrable group representation. Also, there exist other ways how to extend wavelet analysis to higher dimensions, cf. [4]. Each such a case generates its own (possibly different) class of localization operators. *What is the "natural" extension of our results for Toeplitz operators to higher dimensions*?

Immediately, there are many other questions dealing with various contexts, e.g., in quantization problems, (discrete and continuous) frame theory, engineering applications, etc. We hope this paper will stimulate a further interest and development in this topic of intersection of poly-analytic function theory and time-scale (or, more generally, time-frequency) analysis.

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