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C&C
High Tatras, Slovakia

Workshop

Cycles and Colourings 2014

7th – 12th September 2014 **Nový Smokovec**

Abstracts of the 23rd Workshop Cycles and Colourings
7–12 September 2014, Nový Smokovec, High Tatras, Slovakia

Editors: Igor Fabrici, František Kardoš

Dear Participant,

welcome to the Twenty-third Workshop Cycles and Colourings. Except for the first workshop in the Slovak Paradise (Čingov 1992), the remaining twenty one workshops took place in the High Tatras (Nový Smokovec 1993, Stará Lesná 1994–2003, Tatranská Štrba 2004–2010, Nový Smokovec 2011–2013).

The series of C&C workshops is organised by combinatorial groups of Košice and Ilmenau. Apart of dozens of excellent invited lectures and hundreds of contributed talks the scientific outcome of our meetings is represented also by special issues of journals Tatra Mountains Mathematical Publications and Discrete Mathematics (TMMP 1994, 1997, DM 1999, 2001, 2003, 2006, 2008, 2013).

The scientific programme of the workshop consists of 50 minute lectures of invited speakers and of 20 minute contributed talks. This booklet contains abstracts as were sent to us by the authors.

Invited speakers:

Zdeněk Dvořák	Charles University, Prague, Czech Republic
Gregory Z. Gutin	Royal Holloway, University of London, UK
Ken-ichi Kawarabayashi	National Institute of Informatics, Tokyo, Japan
Dirk Meierling	Univesität Ulm, Germany
Mickaël Montassier	Université Montpellier, France
Shakhar Smorodinsky	Ben-Gurion University, Be'er Sheva, Israel
Alexander Soifer	University of Colorado at Colorado Springs, USA

Have a pleasant and successful stay in Nový Smokovec.

Organising Committee:

Igor Fabrici
František Kardoš
Tomáš Madaras
Roman Soták

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Sufficient conditions for the existence of proper hamiltonian cycles in edge-colored multigraphs

Raquel Águeda

(joint work with Raquel Díaz, Valentin Borozan, Yannis Manoussakis, and Leandro Montero)

In this paper we give sufficient conditions that guarantee the existence of a proper Hamiltonian cycle in an edge-colored multigraph: we give bounds for the number of edges so that there exists a proper Hamiltonian cycle. Furthermore, analogue bounds are also obtained for multigraphs under the condition of the rainbow degree being maximum. Naturally, these latter bounds are smaller than the former ones.

Number decompositions and their relations with Fibonacci numbers

Urszula Bednarz

(joint work with Małgorzata Wołowiec-Musiał)

In the talk we present new kinds of (2,k)-distance Fibonacci numbers being generalizations of Fibonacci numbers in the distance sense. We show that they are closely related to special number decompositions. We also give some interesting relations between them.

References

[1] U. Bednarz, D. Bród, I. Włoch, M. Wołowiec-Musiał, On three types of $(2, k)$-distance Fibonacci numbers and number decompositions, Ars Combin., in press.

[2] M. Kwaśnik, I. Włoch, The total number of generalized stable sets and kernels in graphs, Ars Combin. 55 (2000), 139–146.

[3] I. Włoch, U. Bednarz, D. Bród, A. Włoch, M. Wołowiec-Musiał, On a new type of distance Fibonacci numbers, Discrete Appl. Math. 161 (2013), 2695–2701.

Irregular vertex colorings of cartesian products of paths and cycles

Jens-P. Bode

(joint work with Arnfried Kemnitz and Zuzana Moravcová)

The color code of a vertex $v \in V$ with respect to a proper vertex k-coloring c is the ordered $(k+1)$-tuple $\text{code}(v) = (a_0, a_1, \ldots, a_k)$ where $a_0 = c(v)$ and a_i, $i = 1, \ldots, k$, is the number of vertices of color i that are adjacent to v. The coloring c is called irregular if the codes of all vertices are pairwise different. The irregular chromatic number $\chi_{ir}(G)$ of G is the minimum k such that G has an irregular k-coloring. We consider the irregular chromatic number of cartesian products of paths and cycles.

Non-repetitive colourings

Barbora Candráková

(joint work with Robert Lukoťka and Xuding Zhu)

A vertex-colouring c of a graph G is *non-repetitive* if for any path $P = v_1v_2\ldots v_{2k}$ in the graph, $(c(v_1), c(v_2), \ldots, c(v_k)) \neq (c(v_{k+1}), c(v_{k+2}), \ldots, c(v_{2k}))$. The Thue chromatic number of a graph G is the least integer d such that G has a non-repetitive colouring using d colours. A supergraph H of G is called *non-repetitive supergraph* of G if $V(G) = V(H)$ and for each path $P = v_1v_2\ldots v_{2k}$ of G, at least one of the edges v_1v_{k+1}, v_2v_{k+2}, $\ldots v_kv_{2k}$ is an edge of H. We show that the Thue chromatic number of G equals the minimum chromatic number over all non-repetitive supergraphs of G. The result extends to both fractional, circular, and list version of non-repetitive colourings. Our result implies that both the fractional Thue chromatic number and the circular Thue chromatic number is attained and rational, and that the fractional Thue chromatic number equals the fractional Thue choice number. We prove that the fractional Thue number of Petersen graph is 5 and we determine the fractional Thue chromatic number of circuits of length 10, 14, and 17, which were the only circuits with unknown fractional Thue chromatic numbers.

3-coloring triangle-free planar graphs with a precolored 9-cycle

Ilkyoo Choi

(joint work with Jan Ekstein, Přemek Holub, and Bernard Lidický)

We are interested in characterizing when a 3-coloring of a cycle in a 3-colorable planar graph does not extend to a 3-coloring of the entire graph. Grötzsch's Theorem states that a triangle-free planar graph G is 3-colorable. Moreover, every 3-coloring of a 4-cycle or a 5-cycle in G extends to all of G. Given a 3-coloring of a cycle C of length at most 8 in a triangle-free planar graph, it is characterized when the 3-coloring extends to the entire graph. We extend this result to cycles of length 9. Namely, we characterize all situations where a 3-coloring of a cycle of length 9 in a triangle-free planar graph does not extend to a 3-coloring of the whole graph.

An introduction to clique-width of graphs

Konrad Dąbrowski

(joint work with Shenwei Huang and Daniël Paulusma)

The *clique-width* of a graph G, is the minimum number of labels needed to construct G using the following four operations:

- creating a new graph consisting of a single vertex v with label i;
- taking the disjoint union of two labelled graphs G_1 and G_2;
- joining each vertex with label i to each vertex with label j $(i \neq j)$;
- renaming label i to j.

Clique-width is of great theoretical interest because many computational problems that are hard to solve in general can be solved efficiently on graphs of bounded clique-width. In particular if a class of graphs has bounded clique-width then for graphs in this class we can find an optimal *colouring* in polynomial time. It is therefore important to know which classes of graphs have bounded clique-width.

I will give a short introduction to clique-width and some of the simple tricks that can be used when dealing with it. I will also give a short summary of our recent results on classifying which classes of graphs have bounded clique-width. (This talk will not contain any algorithms and will be accessible to anyone who likes playing with graphs.)

Fractional chromatic number of triangle-free graphs

Zdeněk Dvořák

Fractional coloring is a natural generalization of the ordinary coloring, where vertices are colored by sets of colors instead of individual colors. Similarly to the ordinary coloring, fractional chromatic number is unbounded even for triangle-free graphs. However, forbidding triangles can improve bounds for the fractional chromatic number in particular graph classes. We survey the results in this area, including a recent proof [1] that triangle-free subcubic graphs have fractional chromatic number at most 14/5.

References

[1] Z. Dvořák, J.-S. Sereni, J. Volec, Subcubic triangle-free graphs have fractional chromatic number at most 14/5, J. Lond. Math. Soc., accepted.

New argument on universality of homomorphism orders

Jiří Fiala

(joint work with Jan Hubička and Yangjing Long)

We give a new and significantly easier proof of the universality of homomorphism order that can be carried into several other more specific cases, e.g. on classes of line graphs of grapgs of bounded degree or to locally constrained homomorphisms. We show that even the class of oriented graphs with indegree and outdegree one ordered by the existence of homomorphism is universal. By a more systematic approach we simplify several other proofs of earlier results in this area.

In particular, our argument leads to a simpler construction of universality of line graphs of subcubic graphs without the need of complex gadgets (Blanuša snarks) used by Šámal [1].

References

[1] R. Šámal, Cycle-continuous mappings — order structure, in: J. Nešetřil, M. Pellegrini (eds.), The Seventh European Conference on Combinatorics, Graph Theory and Applications, CRM Series 16 (2013), 513–520.

Threshold coloring of prisms et al.

Gašper Fijavž
(joint work with Matthias Kriesell)

Let G be a graph and $\mathcal{L} = (E_F, E_N)$ a bipartition of its edges. We say that G is (r,t)-*threshold colorable with respect to* $\mathcal{L}$ if there exists a mapping $c : V(G) \to \{1,\ldots,r\}$ so that for every edge $uv \in E_F$ we have $|c(u) - c(v)| > t$ and for every edge $uv \in E_N$ we have $|c(u) - c(v)| \leq t$.

A graph G is *total threshold colorable* if there exist parameters r, t, so that G admits a (r,t)-threshold coloring with respect to *every edge bipartition.*

Threshold colorings were introduced in [1] as an approach for realizing several graph classes as contact graphs of unit cubes.

For a fixed graph G it is apparently difficult to decide whether G is total threshold colorable, and examples of graphs which are not total threshold colorable include K_4 and the sun graph.

We have managed to show that prisms are total threshold colorable (which is the first family of total threshold colorable 3-connected graphs) and so is the class of triangle-free series-parallel graphs. To put these results in context we show that none of the Möbius ladders (prisms with a twist) is total threshold colorable. We have also managed to show that Petersen graph is total threshold colorable.

In the direction of optimizing the complexity of the color space we have established that there does not exist a uniform choice of parameters r, t, so that every triangle-free series-parallel graph is (r,t)-threshold colorable.

References

[1] Md.J. Alam, S. Chaplick, G. Fijavž, M. Kaufmann, S. Kobourov, S. Pupyrev, Threshold coloring and unit-cube contact representaiton of graphs, in: A. Brandstädt et al. (eds.), WG 2013, LNCS 8165 (2013), 26–37.

Parameterized rural postman problem

Gregory Z. Gutin
(joint work with Magnus Wahlstrom and Anders Yeo)

The Directed Rural Postman Problem (DRPP) can be formulated as follows: given a strongly connected directed multigraph $D = (V, A)$ with nonnegative integral weights on the arcs, a subset R of A and a nonnegative integer ℓ, decide whether D has a closed directed walk containing every arc of R and of total weight at most ℓ. Let k be the number of weakly connected components in the the subgraph of D induced by R. M. Sorge, R. van Bevern, R. Niedermeier and

M. Weller (2012) ask whether the DRPP is fixed-parameter tractable (FPT) when parameterized by k, i.e., whether there is an algorithm of running time $O^*(f(k))$ where f is a function of k only and the O^* notation suppresses polynomial factors.

Sorge et al. (2012) note that this question is of significant practical relevance and has been open for more than thirty years. Using an algebraic approach, we prove that DRPP has a randomized algorithm of running time $O^*(2^k)$ when ℓ is bounded by a polynomial in the number of vertices in D. We also show that the same result holds for the undirected version of DRPP, where D is a connected undirected multigraph.

Colorings and the total domination game

Michael A. Henning

(joint work with Sandi Klavžar and Douglas Rall)

In this talk, we study the total domination game played in graphs. This game is played on a graph G by two players, named Dominator and Staller. They alternately take turns choosing vertices of G such that each chosen vertex totally dominates at least one vertex not totally dominated by the vertices previously chosen. Dominator's goal is to totally dominate the graph as fast as possible, and Staller wishes to delay the process as much as possible. The game total domination number of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The Staller-start game total domination number of G is the number of vertices chosen when Staller starts the game and both players play optimally. We present a key lemma, known as the Total Continuation Principle, to compare the Dominator-start total domination game and the Staller-start total domination game. Using a vertex colouring strategy, we obtain upper bounds for both games played on any graph that has no isolated vertices.

On path-kipas Ramsey numbers

Přemek Holub

(joint work with Binlong Li and Halina Bielak)

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer r such that, for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2. We use P_n to denote the path on n vertices, and $\widehat{K}_m$ the kipas on $m+1$ vertices, i.e., the graph obtained by joining K_1 and P_m.

In 1967, Gerencsér and Gyárfás [4] determined all the path-path Ramsey numbers. After that, Ramsey numbers for paths versus other graphs have been investigated in several papers, e.g., the path-cycle Ramsey numbers by Faudree et al. in [2], the path-star Ramsey numbers by Parsons in [6], the path-wheel Ramsey numbers by Chen et al. in [1], by Zhang in [8] and by Li and Ning in [5], and the path-tree Ramsey numbers by Faudree et al. in [3]. In [7], Salman and Broersma studied the path-kipas Ramsey numbers and they derived $R(P_n, \widehat{K}_n)$ for some values of m, n.

In this talk, we complete the discussion on the path-kipas Ramsey number for all the values of m, n.

References

[1] Y. Chen, Y. Zhang, K. Zhang, The Ramsey numbers of paths versus wheels, Discrete Math. 290 (2005), 85–87.

[2] R.J. Faudree, S.L. Lawrence, T.D. Parsons and R.H. Schelp, Path-cycle Ramsey numbers, Discrete Math. 10 (1974), 269–277.

[3] R.J. Faudree, R.H. Schelp, M. Simonovits, On some Ramsey type problems connected with paths, cycles and trees, Ars Combin. 29 (1990), 97–106.

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[6] T.D. Parsons, Path-star Ramsey numbers, J. Combin. Theory Ser. B 17 (1974), 51–58.

[7] A.N.M. Salman, H.J. Broersma, Path-kipas Ramsey numbers, Discrete Appl. Math. 155:14 (2007), 1878–1884.

[8] Y. Zhang, On Ramsey numbers of short paths versus large wheels, Ars Combin. 89 (2008), 11–20.

Light graphs in a family of 4-critical planar graphs

Peter Hudák

(joint work with Tomáš Madaras)

A graph G is k-critical if $\chi(G) = k$ and $\chi(H) < \chi(G)$ for every proper subgraph H of G. It is known that k-critical planar graphs possess specific structural properties. In this talk, we will focus on 4-critical planar graph. Koester [1] proved that every 4-critical planar graph contains a vertex of degree at most 4 and this bound is best possible. This result improves the general upper bound 5 valid for minimum degree of planar graphs. We will present an improvement of Kotzig result for existence of a light edge in the family of 4-critical planar graphs and characterize all light graphs in this family.

References

[1] G. Koester, On 4-critical planar graphs with high edge density, Discrete Math. 98 (1991), 147–151.

A few open problems concerning edge colourings of plane graphs

Stanislav Jendrol'

Let $\mathbb{A} = \{a, b, c, \dots\}$ be a finite alphabet, whose element are called letters (digits, colours, symbols, ...). The *word* of length n over $\mathbb{A}$ is an expression $w = a_1a_2 \dots a_n$, where $a_i \in \mathbb{A}$ for all $i = 1, 2, \dots, n$. *Subword* $\bar{w}$ of the word w is an expression $\bar{w} = a_ia_{i+1} \dots a_j$ with $1 \leq i \leq j \leq n$. The *cyclic word* of length n is an expression $w = a_1a_2 \dots a_n$, $n \geq 2$ (consider the cyclic word as a sequence of consecutive labells on the edges of a cycle of length n). A subword of a cyclic word is its arbitrary part.

A word is *proper* if in it no two consecutive letters are the same. The word $a_1a_2 \dots a_n$, $n \geq 1$, is *rainbow* if $a_i \neq a_j$ for $i \neq j$. The word of the form $a_1a_2 \dots a_{2k}$ with property that $a_i = a_{i+k}$ for all $i = 1, 2, \dots, k$ is called the *repetition*. A word is called *nonrepetitive* is none of its subwords is a repetition. A *palindrom* is any word which can be read in the same way from the front and from the back. The word is *palindromfree* if no its subword is a palindrom. A word is an *odd* one if at least one letter in it appears there an odd number of times. A word is a *strong odd* one if each used letter in it is used an odd numbers of times. A word is a *unique maximum* one if the "largest" letter in it appear exactly once.

Consider a 2-connected plane graph. All its faces are bounded by cycles, called the *facial cycles*. If we label all the edges of a 2-connected plane graph G with letters from an alphabet $\mathbb{A}$, then any face $\alpha = [e_1, e_2, \dots, e_k]$ determined by the edges $e_1, e_2, \dots, e_k$ can be associated with a cyclic word $a_1a_2 \dots a_k$, where k is size (degree) of the face α and a_i is a labell of the edge e_i. The word $a_1a_2 \dots a_k$ is called the *facial word* of the face α of G.

In our talk we will consider the following problem:

Problem. What is the minimum number of letters in an alphabet $\mathbb{A}$ that allow to label the edges of a given 2-connected plane graph G in such a way that all the facial words of G over $\mathbb{A}$ have a given property $\mathcal{P}$?

We will give a survey on results and open questions concerning this problem for several properties of words. See, e.g. [1], [2], [3].

References

[1] I. Fabrici, S. Jendroľ, M. Vrbjarová, Unique-maximum edge-colourings with respect to faces of plane graphs, submitted.

[2] F. Havet, S. Jendroľ, R. Soták, E. Škrabuľáková, Facial non-repetitive edge-colouring of plane graphs, J. Graph Theory 66 (2011), 38–48.

[3] J. Czap, S. Jendroľ, F. Kardoš, R. Soták, Facial parity edge colouring of plane pseudographs, Discrete Math. 312 (2012), 2735–2740.

Barnette was right: (not only) fullerene graphs are hamiltonian

František Kardoš

Fullerene graphs, i.e., 3-connected planar cubic graphs with pentagonal and hexagonal faces, are conjectured to be hamiltonian. This is a special case of a conjecture of Barnette, dating back to the 60s, stating that 3-connected planar graphs with faces of size at most 6 are hamiltonian. We present a computer-assisted proof of the conjecture.

Towards the grid minor theorem for directed graphs

Ken-ichi Kawarabayashi
(joint work with Stephan Kreutzer)

The seminal graph minor theory is one of the deepest results in all of mathematics, but it only works for undirected graphs. What about directed graphs?

Researchers come to know that there is a big difference. So the first step would be to show "the excluded grid theorem" for digraphs. For undirected graphs, this was originally proved by Robertson and Seymour in Graph Minors V, and is one of the most central results in the study of graph minors. It has found numerous applications in algorithmic graph structure theory.

Reed, and Johnson, Robertson, Seymour and Thomas (in 1997) conjectured an analogous theorem for directed graphs, i.e. the existence of a function $f(k)$ such that every digraph of directed tree-width at least $f(k)$ contains a directed grid of order k. In an unpublished manuscript from 2001, Johnson, Robertson, Seymour and Thomas give a proof of this conjecture for planar digraphs.

In this talk, we shall report recent progress. We shall discuss the followings

- The excluded grid theorem for directed graphs with no H-minor.
- The half-integral grid theorem.
- Some applications to the disjoint paths problem.

Total colorings of cartesian products of graphs

Arnfried Kemnitz
(joint work with Massimiliano Marangio)

A *total coloring* of a finite and simple graph G is an assignment of colors to the elements (vertices and edges) of G such that neighbored elements (two adjacent vertices or two adjaccent edges or a vertex and an incident edge) are colored differently. The *total chromatic number* $\chi''(G)$ of G is defined to be the minimum numbers of colors in a total coloring of G. Obviously, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G, and Behzad and Vizing independently conjectured that $\chi''(G) \leq \Delta(G) + 2$ for every graph G. The truth of this so-called *total coloring conjecture* would imply that $\chi''(G)$ attains one of two possible values for every graph G. The total coloring conjecture is proved so far for some specific classes of graphs, e. g., for complete graphs, for bipartite graphs, for complete multipartite graphs, for graphs G with $\Delta(G) \geq \frac{3}{4}|V(G)|$ or $\Delta(G) \leq 5$, and for planar graphs G with $\Delta(G) \neq 6$.

We determine the total chromatic number of cartesian products $K_n \square K_m$ of complete graphs, $C_n \square C_m$ of cycles, and $K_n \square H$ as well as $C_n \square H$ where H is a bipartite graph.

Wiener index in the iterated line graphs of trees

Martin Knor

(joint work with Riste Škrekovski, Primož Potočnik, and Martin Mačaj)

Let G be a graph. Its Wiener index, $W(G)$, is the sum of distances between all pairs of vertices. Line graph of G, $L(G)$, is obtained by taking all edges of G as vertices of $L(G)$, and connecting the vertices of $L(G)$ whenever the corresponding edges in G share a vertex. For $i \geq 1$ we recursively define $L^i(G) = L(L^{i-1}(G))$, where $L^0(G) = G$.

Dobrynin, Entringer and Gutman posed a problem if there is $i \geq 3$ and a nontrivial tree T, such that $W(L^i(T)) = W(T)$. In a series of six papers we solved this problem completely. That is, we characterize all $i \geq 3$ and all T's, such that the above equation holds.

Maximally ambiguously k-colorable graphs

Matthias Kriesell

A graph is *ambiguously k-colorable* if its vertex set admits two distinct partitions each into at most k anticliques. We give a full characterization of the maximally ambiguously k-colorable graphs in terms of $k \times k$-matrices. As an application, we calculate the maximum number of edges an ambiguously k-colorable graph can have, and characterize the extremal ones.

Weakly convex domination subdivision number of a graph

Magdalena Lemańska

(joint work with Magda Dettlaff, S. Kosary, and Seyed Mahmoud Sheikholeslami)

A set X is *weakly convex* in G if for any two vertices $a, b \in X$ there exists an ab–geodesic such that all of its vertices belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* if X is weakly convex and dominating. The *weakly convex domination number* $\gamma_{\text{wcon}}(G)$ of a graph G equals the minimum cardinality of a weakly convex dominating set in G. *The weakly convex domination subdivision number* $\text{sd}_{\gamma_{\text{wcon}}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the weakly convex domination number. We initiate the study of weakly convex domination subdivision number and we establish upper bounds for it.

A proof of the Tuza-Vestergaard conjecture

Christian Löwenstein

(joint work with Anders Yeo)

The transversal number, denoted $\tau(H)$, of a hypergraph H is the minimum number of vertices that intersect every edge. A hypergraph is k-uniform if every edge has size k. Tuza and Vestergaard [1] conjectured that every 3-regular 6-uniform hypergraph of order n has a transversal of size at most $n/4$. We will prove this conjecture.

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2-factors in cubic graph without many 6-circuits

Robert Lukoťka

(joint work with Barbora Candráková)

Let G be a bridgeless cubic graph of girth 6 and order n. We show that G has a 2-factor that contains at most $n/9$ circuits of length 6. We use a similar approach to show that G has a travelling salesman tour of length at most $1 + 7/24 \cdot n - 2 \approx 1.292 \cdot n$. We sketch how to prove similar results without girth requirement.

On injective colorings of graphs

Borut Lužar

An *injective coloring* of a graph is a (not necessarily proper) coloring of vertices such that for every vertex v, the neighbors of v receive distinct colors. In other words, every two vertices having a common neighbor are colored distinctly.

Injective colorings were defined by Hahn et al. [1] in 2002 and since then a number of results was published. In the talk a short survey on this topic will be presented.

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Describing 3-paths in plane graphs of girth at least 4

Mária Maceková
(joint work with Stanislav Jendroľ and Roman Soták)

The *girth* of a graph is the length of a shortest cycle in the graph. A path on three vertices u, v and w in this order is an (i, j, k)-*path* if $\deg(u) \leq i, \deg(v) \leq j$, and $\deg(w) \leq k$.

The motivation for our research has come from the following results. Already in 1922 Franklin proved that every normal plane map G of minimum degree five contains a $(6, 5, 6)$-path. In 1993 Ando, Iwasaki, and Kaneko showed that every 3-polytope contains a 3-path such that the sum of degrees of vertices of this path is at most 21. Jendroľ extended this result and described the types of 3-paths contained in each 3-polytope. In 2013 Borodin described the 3-paths in plane maps with minimum degree three and with girth at least three.

In this talk we consider simple plane graphs with minimum degree at least two and girth at least four. We describe the structure of the 3-paths in such graphs.

Improvements on the Erdős-Pósa function for long cycles

Dirk Meierling

A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property* if there is a function $f_{\mathcal{F}} \colon \mathbb{N} \to \mathbb{N}$ such that for every graph G and every $k \in \mathbb{N}$, either G contains k vertex-disjoint subgraphs that belong to $\mathcal{F}$ or there is a set X of at most $f_{\mathcal{F}}(k)$

vertices of G such that $G - X$ has no subgraph that belongs to $\mathcal{F}$. The origin of this notion is [3] where Erdős and Pósa prove that the family of all cycles has this property.

Let ℓ be an integer at least 3 and let $\mathcal{F}_\ell$ denote the family of cycles of length at least ℓ. In [4], Fiorini and Herinckx show that $\mathcal{F}_\ell$ has the Erdős-Pósa property with $f_\ell(k) := f_{\mathcal{F}_\ell}(k) = \mathcal{O}(\ell k \log k)$ which improves an earlier bound obtained by Birmelé, Bondy, and Reed [2]. The main contribution of Birmelé, Bondy, and Reed [2] is to prove a bound for two cycles, that is, to show

$$f_\ell(2) \leq 2\ell + 3, \tag{1}$$

For $k \geq 3$, inductive arguments allow to deduce general bounds from (1).

Birmelé, Bondy, and Reed [2] conjecture that ℓ vertices always suffice.

Conjecture. $f_\ell(2) \leq \ell$.

In other words, for every graph G containing no two vertex-disjoint cycles of length at least ℓ, there is a set X of at most ℓ vertices such that $G - X$ has no cycle of length at least ℓ. In view of the complete graph of order $2\ell - 1$, this bound would be best possible. The first open case of the conjecture is the case $\ell = 6$. (The case $\ell = 3$ was shown by Lovász [5] and the cases $\ell = 4$ and $\ell = 5$ where shown by Birmelé [1].)

Recently, Meierling, Rautenbach, and Sasse [6] improved (1) to $f_\ell(2) \leq 5\ell/3 + 29/2$. Our contribution is the following result.

Theorem. $f_\ell(2) \leq \frac{3}{2}\ell + \frac{25}{4}$.

Moreover, we show that if G contains no cycles of lengths between ℓ and $7\ell/2$, then $f_\ell(2) \leq \ell + 4$ which is essentially the bound suggested by Birmelé, Bondy, and Reed.

In [1], Birmelé confirmed the conjecture for planar graphs. Hence, Birmelé, Bondy, and Reed [1, 2] also posed the question whether their conjecture is true for particular classes of graphs, e.g. cubic, bipartite, or Hamiltonian graphs. We show that the conjecture holds for (sub-)cubic graphs.

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On coloring of double disk graphs

Martina Mockovčiaková

(joint work with Jaka Kranjc, Borut Lužar, and Roman Soták)

The coloring of disk graphs is motivated by the frequency assignment problem. In 1998, Malesińska et al. [2] introduced double disk graphs as a generalization of disk graphs. They showed that the chromatic number of a double disk graph G is at most $33\omega(G) - 35$, where $\omega(G)$ denotes the size of a maximum clique in G. In 2004, Du et al. [1] improved the upper bound to $31\omega(G) - 1$.

In this talk we establish a new upper bound, namely we show that the chromatic number of G is at most $15\omega(G) - 14$. We also discuss a tightness of this bound.

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Entropy compression method and graph coloring problems

Mickaël Montassier

(joint work with Daniel Gonçalves and Alexandre Pinlou)

Based on the algorithmic proof of Lovász local lemma due to Moser and Tardos [4], and the works of Dujmović et al. [1], Esperet and Parreau [2] developed a framework to prove upper bounds for several chromatic numbers (in particular acyclic chromatic index, star chromatic number and Thue chromatic number) using the so-called *entropy compression method*.

Inspired by these works, we propose a more general framework with a better analysis. This leads to improved upper bounds on chromatic numbers and indices. In particular, every graph with maximum degree Δ has an acyclic chromatic number at most $\frac{3}{2}\Delta^{\frac{4}{3}} + O(\Delta)$.

The aim of this talk is to present this framework, starting from an example on acyclic vertex-coloring and going to a more general context and applications. All the details can be found here [3].

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Hamiltonian-connectedness of 5-connected graphs on the torus

Kenta Ozeki
(joint work with Ken-ichi Kawarabayashi)

Tutte [3] showed that every 4-connected planar graph is Hamiltonian, and Thomassen [2] extended it, showing that every 4-connected planar graph is Hamiltonian-connected, i.e., there is a Hamiltonian path connecting any two prescribed vertices. For graphs on the torus, Thomas and Yu [1] proved that every 5-connected graph on the torus has a Hamiltonian cycle. In this talk, we improve it; every 5-connected graph on the torus is Hamiltonian-connected. Since there exists a 4-connected graph on the torus that is not Hamiltonian-connected, the connectivity condition is best possible.

References

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Edges on a common circuit

Ondřej Pangrác
(joint work with Karel Tesař)

It is a well known result that in any k-connected graph every k vertices are contained in a common circuit. We study the edge variant of the problem. The goal is to find bounds on the connectivity of graph to ensure that every k disjoint edges belong to a common circuit. We present some partial results on the problem together with open questions.

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What is the maximum order of a planar signed clique?

Edita Rollová

(joint work with Reza Naserasr and Éric Sopena)

A homomorphism of G to H is a mapping from $V(G)$ to $V(H)$ such that an edge of G is mapped to an edge of H. Homomorphisms are related to (proper) vertex colourings since a vertex v of G can be "coloured" by the vertex of H that is a homomorphic image of v. In this way the chromatic number of G is simply the smallest order of a homomomorphic image of G.

In this talk we will consider signed graphs, graphs where each edge is either positive or negative. We will define vertex colouring of signed graphs using signed graph homomorphisms. We will focus on signed cliques, which are signed graphs on n vertices whose chromatic number equals to n, and we will answer the question of the title in general. For all-positive signed graphs, which correspond to graphs, the answer is given by the Four-Colour-Theorem and by the existence of a planar clique on four vertices, and therefore is 4.

Palettes in block-colourings of designs

Alexander Rosa

(joint work with Curt Lindner and Mariusz Meszka)

For any proper block-colouring of a Steiner system, a *palette* of an element is the set of colours on blocks incident with it. We obtain bounds on the minimum possible number of distinct palettes in proper block colourings of Steiner triple systems and of Steiner systems $S(2, 4, v)$.

Forbidden subgraphs for rainbow connection

Zdeněk Ryjáček
(joint work with Jan Brousek, Přemek Holub, and Petr Vrána)

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors, and the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors such that G is rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In the talk, we give a complete answer for any finite family $\mathcal{F}$.

On the duality between matchings and vertex covers in balanced hypergraphs

Robert Scheidweiler
(joint work with Eberhard Triesch)

We present a new min-max theorem for an optimization problem closely connected to matchings and vertex covers in balanced hypergraphs. The result generalizes Kőnig's Theorem (cf. [2] and [3]) and Hall's Theorem (cf. [1]) for balanced hypergraphs.

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Rainbow connection and size of graphs

Ingo Schiermeyer

An edge-coloured connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a path whose edges have distinct colours. The *rainbow connection number* of G, denoted by $rc(G)$, is the minimum number of colours such that G is rainbow-connected. In [2] the following problem was introduced.

Problem 1 For all integers n and k with $1 \leq k \leq n-1$ compute and minimize the function $f(n,k)$ with the following porperty: If $|V(G)| = n$ and $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$.

In [2] the following lower bound for $f(n,k)$ has been shown.

Proposition For n and k with $1 \leq k \leq n-1$ it holds that $f(n,k) \geq \binom{n-k+1}{2} + k - 1$.

This lower bound is tight.

Problem 2 Determine all values of n and k such that

$$f(n,k) = \binom{n-k+1}{2} + k - 1. \quad (2)$$

It has been shown that $f(n,k) = \binom{n-k+1}{2} + k - 1$ for $k = 1, 2, n-2$, and $n-1$ in [2], for $k = 3$ and 4 in [3], and for $n - 6 \leq k \leq n - 3$ in [1].

In this talk we will report about these results and show some further recent progress obtained for this problem. Moreover, we will consider the following characterization Problem 3:

Problem 3 Let G be a connected graph of order n with $diam(G) = 2$ and $\binom{n-2}{2} + 2 \leq m \leq \binom{n-1}{2}$. Which graphs satisfy $rc(G) = 2$ and which graphs satisfy $rc(G) = 3$?

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Counting K_4-subdivisions

Jens Schmidt

A fundamental theorem of Isbell states that every 3-connected graph contains a K_4-subdivision. A generalization of this statement asks for the minimum number of K_4-subdivisions that are contained in every 3-connected graph on n vertices. We show that there are at least $\Theta(n^3)$ K_4-subdivisions and ask some open questions.

Minimal graphs with respect to geometric distance realizability

Pavol Široczki
(joint work with Tomáš Madaras)

A unit distance drawing of a graph G is a drawing of G in the Euclidean plane so that all edges are represented by line segments of unit length. A graph that admits a unit distance drawing is called a unit distance graph. In [1] the authors state an unsolved problem, which can be formulated (in the weaker version) as follows: Characterize all graphs G such that G is not a unit distance graph, but every proper subgraph of G is a unit distance graph. We call graphs satisfying the previous property non-unit distance minimal graphs. In our talk we give some examples of such graphs together with arguments why their complete characterization seems highly unlikely.

Furthermore, we define analogous concepts for odd distance drawings, which are graph drawings in the plane where all edges are represented by line segments of odd integer length. We also present known examples of non-odd distance graphs and investigate the connection between non-unit distance minimal and non-odd distance minimal graphs.

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The Thue chromatic number vs. the Thue choice number of graphs

Erika Škrabuľáková

A finite sequence $R = r_1 r_2 \dots r_{2n}$ of symbols is called a *repetition* if $r_i = r_{n+i}$ for all $i = 1, 2, \dots, n$. Let G be a simple graph and let φ be a proper colouring of its vertices, $\varphi : V(G) \to \{1, \dots, k\}$. We say that φ is *non-repetitive* if for any simple path on vertices $v_1 \dots v_{2n}$ in G the associated sequence of colours $\varphi(v_1) \dots \varphi(v_{2n})$ is not a repetition. The minimum number of colours in a non-repetitive colouring of a graph G is the *Thue chromatic number* $\pi(G)$. For the case of list-colourings let the *Thue choice number* $\pi_{ch}(G)$ of a graph G denote the smallest integer k such that for every list assignment $L : V(G) \to 2^{\mathbb{N}}$ with minimum list length at least k, there is a colouring of the vertices of G from the assigned lists such that the sequence of vertex colours of no path in G forms a repetition.

Recently it was proved (see [1]) that the Thue chromatic number and the Thue choice number of the same graph may have an arbitrary large difference. The most interesting open problem from this area is whether the Thue chromatic number of a path equals its Thue choice number (see [2]). Here we give some overview of the known results where we compare these two parameters for several families of graphs and we also give a list of some open problems.

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Conflict-free colorings

Shakhar Smorodinsky

There are few generalizations of the classical graph coloring notion to arbitrary hypergraphs. One such generalization is the conflict-free coloring notion. This notion originated recently in the context of frequency assignment in cellular networks. It attracted a lot of attention both from mathematicians and computer scientists.

Let $H = (V, \mathcal{E})$ be a hypergraph. A k-coloring, for some $k \in \mathbb{N}$, of (the vertices of) H is a function $\varphi : V \to \{1, \dots, k\}$. A k-coloring φ of H is called *proper* or *non-monochromatic* if every hyperedge $e \in \mathcal{E}$ with $|e| \geq 2$ is non-monochromatic. That is, there exists at least two vertices $x, y \in e$ such that $\varphi(x) \neq \varphi(y)$. Let $\chi(H)$ denote the least integer k for which H admits a proper coloring with k colors.

In this talk, we focus on the following colorings which are more restrictive than proper coloring:

Conflict-Free and Unique-Maximum Colorings: Let $H = (V, \mathcal{E})$ be a hypergraph and let $C : V \to \{1, \dots, k\}$ be some coloring of H. We say that C is a *conflict-free* coloring (*CF-coloring* for short) if every hyperedge $e \in \mathcal{E}$ contains at least one uniquely colored vertex. More formally, for every hyperedge $e \in \mathcal{E}$ there is a vertex $x \in e$ such that $\forall y \in e, y \neq x \Rightarrow C(y) \neq C(x)$. We say that C is a *unique-maximum* coloring (*UM-coloring* for short) if the maximum color in every hyperedge is unique. That is, for every hyperedge $e \in \mathcal{E}$, $|e \cap C^{-1}(\max_{v \in e} C(v))| = 1$.

Let $\chi_{\mathrm{cf}}(H)$ (respectively, $\chi_{\mathrm{um}}(H)$) denote the least integer k for which H admits a CF-coloring (respectively, a UM-coloring) with k colors. Obviously, every UM-coloring of a hypergraph H is also a CF-coloring of H, and every CF-coloring of H is also a proper coloring of H. Hence, we have the followng inequalities:

$$\chi(H) \leq \chi_{\mathrm{cf}}(H) \leq \chi_{\mathrm{um}}(H)$$

Notice that for simple graphs, the three notions of coloring (non-monochromatic, CF and UM) coincide. Also, for 3-uniform hypergraphs (i.e., every hyperedge has cardinality 3), the two first notions (non-monochromatic and CF) coincide. The curious reader might want to play with finding an example of a 3-uniform hypergraph H such that, say $\chi_{\mathrm{cf}}(H) = 2$ and $\chi_{\mathrm{um}}(H) > 10^{100}$.

These notions of CF-coloring and UM-coloring are related to various other classical notions in combinatorics such as discrepancy, VC-dimension, epsilon-nets, vertex ranking of graphs, etc.

In this talk we give a survey of these notions and related results. The talk is based mainly on the papers [1, 2]. We refer the curious reader for the recent survey [3] for more on CF-colorings.

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Geometric graphs and mathematics of coloring

Alexander Soifer

Graph Theory was born out of our neglecting geometric considerations of shape and size, preserving only adjacency. Surprisingly, the past century has witnessed a renewed interest in graphs, where geometrical considerations such as distance define the adjacency. The wealth of material related to geometric graphs is vast. I have chosen here to address a small but colorful area of geometric graphs: the problem of finding the chromatic number of the plane and related problems. My monograph *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators*, Springer, 2009, contains much material on this topic.

We can create a graph G from the Euclidean plane $\mathbb{E}^2$ by taking all of its points as vertices, and joining two vertices by an edge if and only if they are at distance 1 apart. More generally, we call a graph *unit-distance* when any two vertices are adjacent if and only if they are at distance 1 apart. We get the main open problem:

Problem (Nelson, 1958). Find the chromatic number of the above graph G.

This number is called the *chromatic number of the plane* (CNP) and is denoted by $\chi(\mathbb{E}^2)$. This problem was created in October-November 1950 by the 18-year old Edward Nelson, who determined a lower bound; his 20-year old friend John Isbell found an upper bound: $4 \le \chi(\mathbb{E}^2) \le 7$. After sixty-four years of intensive work, using tools from geometry, graph theory, abstract algebra, topology and measure theory, we have been unable to improve on the above bounds for $\chi(\mathbb{E}^2)$ in the general case. Ronald L. Graham believes that the chromatic number of the plane is 5 or 6. He cites a theorem of Paul O'Donnell, showing the existence of 4-chromatic unit-distance graphs of arbitrarily large girth as "perhaps, the evidence that χ is at least 5". Paul Erdős believed that the chromatic number of the plane was 5, 6, or 7. I expect the answer to be 4 or 7. What are the consequences of $\chi(\mathbb{E}^2) = 4$? The chromatic number of the plane may depend upon the system of axioms we adopt for set theory.

Let G be a graph and let $\boldsymbol{A}$ be a system of axioms for set theory. The *set of chromatic cardinalities* $\chi^{\boldsymbol{A}}(G)$ of G is the set of all cardinal numbers $\tau \le |V(G)|$ for which there is a proper coloring of the vertices of G in τ colors, and τ is minimum with respect to this property. As can be seen, the set of chromatic cardinalities needs not have just one element as with $\boldsymbol{A} = \mathbf{ZFC}$. It can also be empty. The advantage of this definition is its simplicity. Best of all, we can use inequalities on sets of chromatic cardinalities as follows. Let τ be a cardinal number. The inequality $\chi^{\boldsymbol{A}}(G) > \tau$ means that, for every $\sigma \in \chi^{\boldsymbol{A}}(G)$, $\sigma > \tau$; the inequalities $<$, $\le$ and $\ge$ are defined analogously. We also agree that the empty set is greater than or equal to any other set of cardinal numbers. Finally, if τ is a cardinal number and $\chi^{\boldsymbol{A}}(G) = \{\tau\}$ is a one-element set of chromatic cardinalities (as is the case with the chromatic number when $\boldsymbol{A} = \mathbf{ZFC}$), then we will simplify our notation by omitting parentheses and writing $\chi^{\boldsymbol{A}}(G) = \tau$.

An infinite cardinal $\aleph_\alpha$ is regular if cf $\omega_\alpha = \omega_\alpha$, and κ is a *strong limit* cardinal if, for every cardinal λ, $\lambda < \kappa$ implies that $2^\lambda < \kappa$. A cardinal κ is called *inaccessible* if $\kappa >$ cardinal. Assuming the existence of an inaccessible cardinal, and using Paul Cohen's forcing, Robert Solovay constructed in 1964 and published in 1970 a model that proved a remarkable theorem. In his honor the author introduced the following definitions.

The standard Zermelo-Fraenkel-Choice system of axioms for set theory will be denoted by **ZFC**; the countable axiom of choice by $\mathbf{AC}_{\aleph_0}$, the principle of dependent choices by **DC**. We will use one further axiom, **LM**: every set of real numbers is Lebesgue measurable. *The Zermelo-Fraenkel-Solovay system of axioms* **ZFS** for set theory is defined by $\mathbf{ZFS} = \mathbf{ZF} + \mathbf{AC}_{\aleph_0} + \mathbf{LM}$; and **ZFS+** stands for $\mathbf{ZF} + \mathbf{DC} + \mathbf{LM}$.

Solovay's Theorem. ZFS+ is consistent.

Consequences Theorem (Shelah and Soifer, 2003). Assume that any finite unit-distance plane graph has chromatic number not exceeding 4.
Then $\chi^{\mathbf{ZFC}}(\mathbb{E}^2) = 4$, but $\chi^{\mathbf{ZFS+}}(\mathbb{E}^2) \ge 5$.

And finally, let me share three conjectures:

Chromatic number of the plane conjecture (Soifer). $\chi(\mathbb{E}^2) = 7$.

Chromatic number of the 3-space conjecture (Soifer). $\chi(\mathbb{E}^3) = 15$.

Main conjecture (Soifer). For the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 1$, $\chi(\mathbb{E}^n) = 2^{n+1} - 1$.

Spanning k-forests with large components in $K_{1,k+1}$-free graphs

Takeshi Sugiyama
(joint work with Kenta Ozeki)

Let k be an integer with $k \geq 2$. In this talk, we consider a spanning forest with maximum degree at most k, which is called a *spanning k-forest*. First we introduce the following well-known proposition. For example, we obtain Proposition 1 combining Lemma 2.2 (ii) and Theorem 3.1 (i) in [1].

Proposition 1. *For an integer k with $k \geq 1$, every connected $K_{1,k+1}$-free graph contains a spanning $(k+1)$-tree.*

For a positive integer r, recall that a graph having no $K_{1,r}$ as an induced subgraph is said to be *$K_{1,r}$-free*.

Moreover, it is known that this condition is best possible, that is, there exist infinitely many connected $K_{1,k+1}$-free graphs containing no spanning k-tree. Therefore, instead of a spanning k-tree, in a connected $K_{1,k+1}$-free graph, we are interested in a spanning k-forest with only large components. This is the main purpose of this talk and we show the following two results.

Theorem 2. *Let G be a connected $K_{1,4}$-free graph. Then G contains a spanning 3-tree or a spanning 3-forest F such that for any component C of F, we have*

$$|V(C)| \geq \begin{cases} \sigma_3(G) + 1 & \text{if } \Delta(C) = 3, \\ \sigma_2(G) + 1 & \text{if } \Delta(C) = 2. \end{cases}$$

Theorem 3. *Let k be a positive integer with $k \geq 4$ and let G be a connected $K_{1,k+1}$-free graph. Then G contains a spanning k-tree or a spanning k-forest F such that for any component C of F, we have $\Delta(C) \geq k-1$ and*

$$|V(C)| \geq \begin{cases} \sigma_{2k-3}(G) - 1 & \text{if } \Delta(C) = k, \\ \sigma_{k-1}(G) + 1 & \text{if } \Delta(C) = k-1. \end{cases}$$

For a graph G, let $\delta(G)$ and $\alpha(G)$ be the minimum degree and the independence number of G, respectively. For a positive integer k, if $\alpha(G) \geq k$, then let

$$\sigma_k(G) = \min\left\{ \sum_{x \in X} d_G(x) \colon X \text{ is an independent set of } G \text{ with } |X| = k \right\};$$

otherwise let $\sigma_k(G) = +\infty$.

Note that the lower bounds on the orders of each component of a spanning k-forest obtained in Theorems 2 and 3 are almost best possible.

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Distance-constrained labeling of complete trees

Zsolt Tuza
(joint work with Veronika Halász)

We study vertex labelings $\varphi : V \to \{0, 1, 2, \dots\}$ of a graph $G = (V, E)$, which assign nonnegative integers to the vertices and the restrictions depend on the distances in G. Fixing a positive integer d, the requirement is that if vertices u and v are at distance i apart (where $1 \leq i \leq d$), then $|\varphi(u) - \varphi(v)| > d - i$ must hold. The goal is to make $\max_{v \in V} \varphi(v)$ as small as possible. We present two methods to construct optimal or nearly optimal labelings on a class of trees. A corollary of the main result is an exact formula for the minimum for trees whose internal vertices all have the same degree and all leaves are at distance $d/2$ from the central vertex (for d even) or at distance $(d-1)/2$ from the central edge (for d odd). The case of even diameter extends the main theorem of Li, Mak, and Zhou [1] on complete rooted trees with fixed down-degree and height.

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On types of distance Fibonacci numbers generated by number decompositions

Andrzej Włoch
(joint work with Anetta Szynal-Liana and Iwona Włoch)

In the talk we consider three types of the distance Fibonacci numbers $Fd^{(i)}(k, n)$ for $i = 1, 2, 3$ defined by the kth order linear recurrence relation on the form $Fd^{(i)}(k, n) = Fd^{(i)}(k, n - k + 1) + Fd^{(i)}(k, n - k)$ with special initial conditions for $i = 1, 2, 3$.

We give an interpretations of the numbers $Fd^{(i)}(k,n)$, $i = 1,2,3$ with respect to special number decompositions (i.e. ordered number partition) and using these interpretations we will show relations between all three types of distance Fibonacci numbers. We also give matrix generators for distance Fibonacci numbers $Fd^{(i)}(k,n)$ and their direct formulas.

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