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Abstract
A proper edge colouring of a graph $G$ without isolated edges is neighbour-distinguishing if any two adjacent vertices have distinct sets consisting of colours of their incident edges. The neighbour-distinguishing index of $G$ is the minimum number $\text{ndi}(G)$ of colours in a neighbour-distinguishing edge colouring of $G$. According to a conjecture by Zhang, Liu and Wang (2002), $\text{ndi}(G) \leq \Delta(G) + 2$ provided that $G$ is a connected graph of order at least 6. The conjecture is proved for planar graphs $G$ with $\Delta(G) \geq 12$.

Keywords: colour set, neighbour-distinguishing edge colouring, neighbour-distinguishing index, planar graph

MSC 2010: 05C15

1 Introduction
Let $G$ be a finite simple graph (loops and multiple edges are not allowed), $C$ a finite set of colours and let $\varphi : E(G) \to C$ be a proper edge colouring of $G$. The colour set (with respect to $\varphi$) of a vertex $x \in V(G)$ is the set $S_\varphi(x) := \{ \varphi(xy) : xy \in E(G) \}$. The colouring $\varphi$ is neighbour-distinguishing (an nd-colouring for short) provided that $S_\varphi(x) \neq S_\varphi(y)$ for every $xy \in E(G)$. Clearly, to have an nd-colouring, $G$ must not have an isolated edge (it is normal). Since $|S_\varphi(x)|$ is equal to the degree of $x$, the condition $S_\varphi(x) \neq S_\varphi(y)$ is trivially fulfilled for any edge $xy$ joining vertices of distinct degrees. The neighbour-distinguishing index of a normal graph $G$ is the minimum number of colours in an nd-colouring of $G$. Since any nd-colouring is proper, $\text{ndi}(G)$ is bounded from below by the chromatic index $\chi'(G)$ of $G$. Thus, by Vizing’s Theorem, $\Delta(G) \leq \chi'(G) \leq \text{ndi}(G)$.

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If a normal graph $G$ has connected components $G_1, \ldots, G_k$, then evidently $\text{ndi}(G) = \max(\text{ndi}(G_i) : i = 1, \ldots, k)$. Therefore, when analysing the neighbour-distinguishing index, we can restrict our attention to connected graphs.

The neighbour-distinguishing index was introduced by Zhang, Liu and Wang in [8]. The authors determined there the invariant (also) for cycles, complete bipartite graphs and trees. Their results motivated them to formulate

**Conjecture (Neighbour-Distinguishing Conjecture, NDC)** If $G$ is a connected graph with $|V(G)| \geq 6$, then $\text{ndi}(G) \leq \Delta(G) + 2$.

It should be noted that $\text{ndi}(C_5) = 5 = \Delta(C_5) + 3$. NDC was proved by Balister et al. in [1] for bipartite graphs and for graphs with maximum degree 3. Edwards, Horňák and Woźniak showed in [3] that $\text{ndi}(G) \leq \Delta(G) + 1$ if $G$ is a bipartite planar graph with $\Delta(G) \geq 12$. It is known that NDC is true for almost all 4-regular graphs. Namely, Greenhill and Ruciński in [5] proved that asymptotically almost surely the neighbour-distinguishing index of $d$-regular graphs is at most $\left\lceil \frac{3d^2}{2} \right\rceil$ for $d \geq 4$. (If $G_{n,d}$ is the space of all uniformly random $d$-regular graphs with vertex set $\{1, \ldots, n\}$, then the probability of the set of all $G \in G_{n,d}$ satisfying $\text{ndi}(G) \leq \left\lceil \frac{3d^2}{2} \right\rceil$ tends to 1 as $n$ tends to infinity.) Baril, Kheddouci and Togni in [2] succeeded to show that $\text{ndi}(Q_d) = d + 1$ for the $d$-dimensional cube $Q_d$ with $d \geq 3$. Wang and Wang brought into play the maximum average degree of a graph $G$, $\text{mad}(G) := \max\left(\frac{2|E(H)|}{|V(H)|} : H \subseteq G, V(H) \neq \emptyset\right)$; they proved NDC in [7] for graphs $G$ with $\text{mad}(G) < 3$ and $\Delta(G) \geq 3$. The best general result is due to Hatami [6] who bounded (by a probabilistic method) $\text{ndi}(G)$ from above by $\Delta(G) + 300$ provided that $\Delta(G) > 10^{20}$.

The aim of the present paper is to prove NDC for planar graphs $G$ with $\Delta(G) \geq 12$. More precisely, our result is a little bit stronger:

**Theorem 1** If $G$ is a normal connected planar graph, then $\text{ndi}(G) \leq \max(14, \Delta(G) + 2)$.

A normal connected planar graph $G$ is said to be $\text{ndi-minimal}$ if $\text{ndi}(G) > \max(\Delta(G) + 2, 14)$ and $\text{ndi}(H) \leq \max(\Delta(H) + 2, 14)$ for any normal planar graph $H$ such that $(|V(H)|, |E(H)|) \prec (|V(G)|, |E(G)|)$, where “$\prec$” stands for the binary relation “to be lexicographically smaller than”.

If $H$ is a normal planar graph with $|E(H)| \leq 14$, then $\text{ndi}(H) \leq |E(H)| \leq \max(\Delta(H) + 2, 14)$. Therefore, if Theorem 1 is not true, there exists an ndi-minimal planar graph $G$. To obtain a contradiction with this assumption we use the Discharging Method. Roughly speaking, we shall show that an ndi-minimal planar graph must have so many properties that it cannot exist.

For $p, q \in \mathbb{Z}$ let $[p, q]$ denote the integer interval bounded by $p$ and $q$, i.e., the set $\{z \in \mathbb{Z} : p \leq z \leq q\}$. If $X$ is a finite set and $k$ a positive integer, then $\binom{X}{k}$ is the set of all $k$-element subsets of $X$. 
Graphs we are dealing with are undirected. However, it will be useful to consider an edge \(xy\) as a pair of oriented edges \((x, y)\) and \((y, x)\). Let \(G = (V, E, F)\) be a plane graph (a spherical embedding of a planar graph) and let \(\sigma\) be one of two possible orientations of the sphere carrying \(G\). Let \(N_G(x)\) be the set of all neighbours of a vertex \(x \in V\). The degree of \(x\) (in the graph \(G\)) is the number \(\deg_G(x) := |N_G(x)|\). For a face \(f \in F\) let \(\vec{E}(f)\) denote the set of all oriented edges \((x, y)\) such that \(y\) is the successor of \(x\) in an ordering of vertices incident with \(f\) that is obtained if the boundary of \(f\) is traversed in the sense of the orientation \(\sigma\). The degree of the face \(f\) (in the graph \(G\)) is the number \(\deg_G(f) := |\vec{E}(f)|\). If \(x, y\) are adjacent vertices, the (unique) face \(f\) such that \((x, y) \in \vec{E}(f)\), will be denoted by \(\text{face}(x, y)\). A trail \((x_1, \ldots, x_k) \in V^k\) is facial if there is a (unique) face \(f \in F\) such that \((x_i, x_{i+1}) \in \vec{E}(f)\) for every \(i \in [1, k - 1]\). A \((d_1, \ldots, d_k)\)-path is a path \((x_1, \ldots, x_k) \in V^k\) in which \(\deg_G(x_i) = d_i\) for every \(i \in [1, k]\). We shall write \([x_1, x_2, x_3]\) instead of a cycle \((x_1, x_2, x_3, x_1)\) (of length 3). A \((d_1, d_2, d_3)\)-cycle is a cycle \([x_1, x_2, x_3]\) with \(\deg_G(x_i) = d_i, i = 1, 2, 3\). A \(d\)-vertex is a vertex \(x \in V\) with \(\deg_G(x) = d\); similarly is defined a \(d\)-face and a \(d\)-neighbour (of a vertex). Let \(N_d(x)\) denote the set of all \(d\)-neighbours of the vertex \(x\), let \(N_{d-}(x) := \bigcup_{k=1}^{d} N_k(x)\) and let \(n_d(x) := |N_d(x)|, n_{d-}(x) := |N_{d-}(x)|\). An edge \(xy \in E\) is special provided that \(\deg_G(x) = \deg_G(y) \leq 5\). The set of all special edges in \(G\) will be denoted \(E_s(G)\).

2. The structure of an ndi-minimal graph

Lemma 2 Let \(G\) be an ndi-minimal graph, \(C\) a set of \(\max(\Delta(G) + 2, 14)\) colours and let \((x_1, x_2)\) be a \((d_1, d_2)\)-path in \(G\).

1. If \(d_1 + d_2 \leq 8\), there is no nd-colouring \(\varphi : E(G - x_1x_2) \to C\) such that \(S_\varphi(x_1) \neq S_\varphi(x_2)\).

2. If \(d_1 \neq d_2\), then \(\sum_{i=1}^{2}(d_i + n_d(x_i)) \geq 16\).

Proof. 1. Suppose the opposite and consider a proper extension \(\psi : E(G) \to C\) of \(\varphi\) which clearly exists, because we have at least 14 colours in \(C\) and at most \(\bigcup_{i=1}^{2} S_\varphi(x_i)\) \(\leq \sum_{i=1}^{2}(d_i - 1) \leq 6\) out of them are forbidden for \(\psi(x_1x_2)\).

Thus, \(S_\psi(x_1) = S_\varphi(x_1) \cup \{\psi(x_1x_2)\} \neq S_\varphi(x_2) \cup \{\psi(x_1x_2)\} = S_\psi(x_2)\), i.e., \(\psi\) distinguishes \(x_1\) from \(x_2\). Moreover, \(\psi\) does not distinguish \(x_i\) from its neighbour \(y \neq x_{3-i}\) if and only if \(\deg(y) = \deg(x_i)\) and \(\psi(y) - \psi(x_i) = \{\psi(x_1x_2)\}, i = 1, 2\).

Thus, the number of colours that can be used as \(\psi(x_1x_2)\) so that \(\psi\) is neighbour-distinguishing, is at least \(|C| - (|S_\varphi(x_1)| + |S_\varphi(x_2)|) + \bigcup_{i=1}^{2}(N_\psi(x_i) - \{x_{3-i}\})| \geq 14 - 2 \sum_{i=1}^{2}(d_i - 1) = 18 - 2(d_1 + d_2) \geq 18 - 2 \cdot 8 > 0\) and ndi(G) \(\leq \max(\Delta(G) + 2, 14)\), a contradiction.

2. Assume that \(\sum_{i=1}^{2}(d_i + n_d(x_i)) \leq 15\). As \(G\) is ndi-minimal, there is an nd-colouring \(\varphi : E(G - x_1x_2) \to C\). Analogously as above, the number of colours available as \(\psi(x_1x_2)\) in an nd-extension \(\psi\) of \(\varphi\) is at least \(|C| - (|S_\varphi(x_1)| + |S_\varphi(x_2)|) + \bigcup_{i=1}^{2}(N_\psi(x_i) - \{x_{3-i}\})| \geq 14 - 2 \sum_{i=1}^{2}(d_i - 1) = 18 - 2(d_1 + d_2) \geq 18 - 2 \cdot 8 > 0\) and ndi(G) \(\leq \max(\Delta(G) + 2, 14)\), a contradiction.
Lemma 4

If $d_1 \neq d_2$ it follows that $x_1$ is distinguished from $x_2$ “automatically” provided that $\psi$ is proper.

Lemma 3

If $m, n \in \mathbb{Z}$, $1 \leq m \leq n$ and $X, Y$ are finite sets with $|X| \geq m$ and $|Y| \geq n$, then $|\{(x, y) : x \in X, y \in Y\}| \geq \frac{m(2n-m-1)}{2}$.

Proof. The set $Z := \{(x, y) : x \in X, y \in Y\}$ has a decomposition $\{Z_1, Z_2, Z_3, Z_4\}$, where $Z_1 := \{(x, y) : x \in X - Y, y \in Y - X\}$, $Z_2 := \{(x, y) : x \in X - Y, y \in X \cap Y\}$, $Z_3 := \{(x, y) : x \in X \cap Y, y \in Y - X\}$ and $Z_4 := \{(x, y) : x, y \in X \cap Y, x \neq y\}$. For $l := |X \cap Y|$ then $|Z| = \sum_{i=1}^{4} |Z_i| = (|X| - l)(|Y| - l) + (|X| - l)l + l(|Y| - l) + \binom{l}{2} = |X||Y| - \binom{l+1}{2}$.

If $l \leq m$, then $|Z| \geq mn - \binom{m+1}{2} = \frac{m(2n-m-1)}{2}$.

If $l \geq m + 1$, then $|X| \geq l$. Provided that $n = m$ we have $|Y| \geq l$ and $|Z| \geq l^2 - \binom{l+1}{2} = \binom{l}{2} \geq \binom{n+1}{2} = \frac{m(2n-m+1)}{2} > \frac{m(2n-m-1)}{2}$.

Suppose that $n \geq m+1$. If $l \in [m+1, n-1]$, then $|Z| \geq ln - \binom{l+1}{2} = \frac{l(2n-l-1)}{2} \geq \frac{m(2n-m-1)}{2}$, since the function $\frac{x(2n-x)}{2}$ is nondecreasing for $x \in \langle m, \frac{2n-1}{2} \rangle$. Finally, if $l \geq n$, then $|Z| \geq l^2 - \binom{l+1}{2} = \binom{l}{2} \geq \binom{n}{2} = \frac{m(2n-m-1)}{2} + \frac{(n-m)(n-m-1)}{2} \geq \frac{m(2n-m-1)}{2}$.

Let us define $g(i, j)$ as the entry in the $i$th row and the $j$th column of the (symmetric) matrix

$$
\begin{pmatrix}
11 & 11 & 10 & 10 & 9 \\
11 & 10 & 10 & 9 & 9 \\
10 & 10 & 8 & 8 & 8 \\
10 & 9 & 8 & 7 & 7 \\
9 & 9 & 8 & 7 & 7 \\
\end{pmatrix}
$$

and let $h(3) := 10, h(4) := 10, h(5) := 9$.

Lemma 4

An ndi-minimal graph $G$ does not contain

1. a $(1, d)$-path with $d \in [1, 7]$;
2. a $(2, 2)$-path;
3. an induced $(d, d)$-path with $d \in [3, 4]$;
4. a $(d, d, d)$-cycle with $d \in [2, 5]$;
5. a $(d_1, d_2)$-path with $d_1 \in [2, 4]$ and $d_2 \in [d_1 + 1, 9 - d_1]$;
6. a 5-vertex $x$ with $n_5(x) \in [2, 4]$;
7. an induced $(5, 5, 5)$-path;
8. a $(d, d, d_3)$-cycle with $d \in [3, 5]$ and $d_3 \in [2, h(d)]$;
9. a $(d_1, d_2, d_3)$-path with $d_1, d_3 \in [1, 5]$ and $d_2 \in [2, g(d_1, d_3)]$;
10. a $(4, d_2, 6)$-path with $d_2 \in [2, 5]$;
11. a $(4, 6)$-path;
12. a $(5, 6, 6)$-cycle;
13. a $(1, d_2, d_3, 1)$-path with $d_2, d_3 \in [2, \Delta(G)]$. 
Proof. An edge colouring of $G$ using each colour exactly once is clearly neighbour-distinguishing. Therefore, from ndi$(G) > \max(\Delta(G) + 2, 14)$ it follows that $|E(G)| \geq 15$. Pick a set $C$ of max$(\Delta(G) + 2, 14)$ colours and suppose $G$ contains a configuration of Lemma 4, i., $i \in [1, 13]$. We modify $G$ to a planar graph $H$ “close” to $G$ that has no isolated edges and satisfies the conditions $([V(H)], |E(H)|) \prec (|V(G)|, |E(G)|)$ as well as $\Delta(H) \leq \Delta(G)$. Since $G$ is ndi-minimal, we have ndi$(H) \leq \max(\Delta(H) + 2, 14) \leq \max(\Delta(G) + 2, 14)$ and there is an nd-colouring $\varphi : E(H) \rightarrow C$.

$i = 1$: Let $(x_1, x_2)$ be a $(1, d)$-path in $G$. If $d = 1$, then $G$ has an isolated edge, a contradiction. If $d \geq 2$, put $H := G - x_1x_2$. Since deg$(x_1) + \text{deg}(x_2) \leq 8$, we have $S_\varphi(x_1) = \emptyset \neq S_\varphi(x_2)$ in contradiction to Lemma 2.1.

$i = 2$: Suppose $(x_1, x_2)$ is a $(2, 2)$-path in $G$ and let $N(x_j) - \{x_{j-1}\} = \{y_j\}$, $j = 1, 2$. If $y_1 = y_2$ and $H := G - x_1x_2$, then $S_\varphi(x_1) \neq S_\varphi(x_2)$ (because $\varphi$ is proper in contradiction to Lemma 2 (note that $\text{deg}(x_1) + \text{deg}(x_2) = 4 \leq 8$). If $y_1 \neq y_2$, and $H$ is a planar graph with $V(H) = V(G) - \{x_1, x_2\} \cup \{x\}$, $x \notin V(G)$ and $E(H) = E(G) - \{y_1x_1, x_1x_2, x_2y_2\} \cup \{y_1x, xy_2\}$, then $\varphi(y_1x) \neq \varphi(xy_2)$. Let $\tau : E(G - x_1x_2) \rightarrow C$ be defined by $\tau(c) := \varphi(c)$ for $c \in E(H) \cap E(G)$ and $\tau(y_jx) = \varphi(y_jx)$, $j = 1, 2$. Clearly, $\tau$ is neighbour-distinguishing and $S_\tau(x_1) = \{\varphi(y_1x)\} \neq \{\varphi(xy_2)\} = S_\tau(x_2)$ in contradiction to Lemma 2.1 again.

In the remaining cases we modify $\varphi$ in order to obtain an nd-colouring $\psi : E(G) \rightarrow C$ (i.e., a contradiction to the assumption ndi$(G) > \max(\Delta(G) + 2, 14)$).

$i = 3$: Let $(x_1, x_2, x_3)$ be an induced $(d, d, d)$-path in $G$ and let $H := G - \{x_1x_2, x_2x_3\}$. For $i \in \{1, 3\}$ let $C(x_i) \subseteq \text{set of all colours that can be used as } \tau_i(x_i) \text{ in an extension } \tau_i : E(H + x_ix_2) \rightarrow C \text{ of } \varphi \text{ that is proper, distinguishes } x_i \text{ from all its neighbours and satisfies } S_{\tau_i}(x_2) \neq S_{\tau_i}(x_{4-i})$. Thus, a colour $c \in C$ does not belong to $C(x_i)$ if either $c \in S_\varphi(x_i) \cup S_\varphi(x_2)$ or there is a $d$-neighbour $y$ of $x_i$ with $S_\varphi(y) - S_\varphi(x_i) = \{c\}$. If $S_\varphi(x_{4-i}) - S_\varphi(x_2) = \{c\}$, then $C(x_i) \geq \{S_\varphi(x_i) \cup S_\varphi(x_2)\} + |\text{Neig}(x_i) - \{x_2\}| + 1 \geq 14 - [(2d-3)+(d-1)+1] = 17-3d \geq 5$, $i = 1, 3$. Any extension $\psi : E(G) \rightarrow C \text{ of } \varphi \text{ with } \psi(x_1x_2) = c_i \in C(x_i)$, $i = 1, 3$, and $c_1 \neq c_3$, is proper and distinguishes adjacent vertices of $G$ with a possible exception of $x_2$ and its $d$-neighbours other than $x_1$ and $x_3$. (Notice that $S_\psi(x_2) = S_\tau(x_2) \cup \{c_{4-i}\} \neq S_\tau(x_{4-i}) \cup \{c_{4-i}\} = S_\psi(x_{4-i})$, $i = 1, 3$.) According to Lemma 3 a 2-element set $\{c_1, c_3\}$ with $c_i \in C(x_i)$ may be chosen in at least $25 \cdot 5 \cdot 9 = 10$ ways. As $|\text{Neig}(x_2) - \{x_1, x_3\}| \leq 2 < 10$, we can choose $\{c_1, c_3\}$ so that $S_\psi(x_2) = S_\varphi(x_2) \cup \{c_1, c_3\} \neq S_\psi(y)$ for each $y \in \text{Neig}(x_2) - \{x_1, x_3\}$ and the obtained colouring $\psi$ is neighbour-distinguishing.

$i = 4$: For $d = 2$ the statement follows from the fact that $G$ is connected and $|E(G)| \geq 15$. Suppose $d \in [3, 5]$, let $[x_1, x_2, x_3]$ be a $(d, d, d)$-cycle in $G$ and let $H := G - \{x_1x_2, x_2x_3, x_3x_1\}$. A pair of colours $\{c_1, c_2\} \in \binom{c}{2}$ is said to be dangerous for $x_j$, $j \in [1, 3]$, if there is $y \in \text{Neig}(x_j) - \{x_1, x_2, x_3\}$ with $S_\varphi(y) - S_\varphi(x_j) = \{c_1, c_2\}$ and dangerous if it is dangerous for some $x_j$. Clearly, the number of all exemplars of colours belonging to a dangerous pair is at most
2|\bigcup_{j=1}^{3}(N_d(x_j) - \{x_1, x_2, x_3\})| \leq 6(d - 2) = 6d - 12. Let c ∈ C - \bigcup_{j=1}^{3}S_\varphi(x_j) be a colour having the minimum frequency f(c) in dangerous pairs. Since |C - \bigcup_{j=1}^{3}S_\varphi(x_j)| \geq 14 - 3(d - 2) = 20 - 3d, we have f(c) ≤ \left\lceil \frac{6d - 12}{20 - 3d} \right\rceil ≤ \left\lceil \frac{6 \times 5 - 12}{20 - 3 \times 5} \right\rceil = 3.

Let f(c) = \sum_{j=1}^{3}f_j(c) where f_j(c) is the frequency with which c appears in pairs dangerous for x_j, j = 1, 2, 3. We may suppose without loss of generality that the notation is chosen so that f_1(c) ≤ f_2(c) ≤ f_3(c), hence f_2(c) ≤ 1. Consider the extension ρ: E(H + x_1x_2) → C of ϕ with ρ(x_1x_2) := c. As c /∈ S_\varphi(x_3), any proper extension ψ: E(G) → C of ρ distinguishes x_j from x_3, j = 1, 2, 3. Therefore, the number of colours that are available as ψ(x_jx_3), j ∈ [1, 2], if ψ: E(G) → C has to be a proper extension of ρ that distinguishes x_j from all vertices in N_d(x_j) - \{x_3 - j\}, is at least 14 - [2(d - 2) + 1 + f_2(c)] = 17 - f_2(c) - 2d ≥ 16 - 2d. (If ψ does not distinguish x_j from a vertex y ∈ N_d(x_j) - \{x_3 - j\}, then \{c, ψ(x_jx_3)\} is a pair that is dangerous for x_j.) Further, ψ does not distinguish x_1 from x_2 if and only if \{ψ(x_jx_3) = S_\varphi(x_3 - j) - S_\varphi(x_j), j = 1, 2, 3\}. So, distinguishing x_1 from x_2 forbids at most one pair of colours from being used as \{ψ(x_1x_3), ψ(x_2x_3)\}. On the other hand, distinguishing x_3 from its neighbours forbids at most |N_d(x_3) - \{x_1, x_2\}| ≤ d - 2 pairs. By Lemma 3, the number of 2-element colour sets appropriate as \{ψ(x_1x_3), ψ(x_2x_3)\} is at least \frac{1}{2}(16 - 2d)[2(16 - 2d) - (16 - 2d) - 1] = (8 - d)(15 - 2d). Because d ∈ [3, 5] implies (8 - d)(15 - 2d) > (d - 2) + 1 ≥ |N_d(x_3) - \{x_1, x_2\}| + 1, an extension ψ can be found as necessary.

i = 5: Let (x_1, x_2) be a (d_1, d_2)-path in G and let H := G - x_1x_2. By Lemma 4.1 G does not contain a (1, 2)-path, and so H is without isolated edges. Since d_1 ∈ [2, 4], from Lemma 4.2, 3, 4 it follows that n_d(x_1) ≤ d_1 - 2. Obviously, we have n_d(x_2) ≤ d_2 - 1. Therefore, \sum_{i=1}^{2}(d_i + n_d(x_i)) ≤ 2d_1 - 2 + 2d_2 - 1 = 2(d_1 + d_2) - 3 ≤ 15 in contradiction to Lemma 2.2.

i = 6: Consider a 5-vertex x ∈ V(G) with n_5(x) ∈ [2, 4], pick x_1, x_2 ∈ N_5(x), x_1 ≠ x_2, and let H := G - \{xx_1, xx_2\}. By Lemma 4.4 we know that x_1x_2 ∉ E(G).

Suppose there is j ∈ [1, 2] with S_\varphi(x) ⊆ S_\varphi(x_j). Let C(xx_k), k ∈ [1, 2], be the set of all colours in C that can be used as ϕ_k(xx_k) in a proper extension ϕ_k: E(H + xx_k) → C of ϕ that distinguishes x_k from vertices in N_5(x_k) - \{x\} and satisfies S_\varphi(x) ≠ S_\varphi(x_3 - k). If c ∈ C(xx_k), then necessarily c /∈ S_\varphi(x) ∪ S_\varphi(x_k), c /∈ S_\varphi(y) - S_\varphi(x_k) provided that y ∈ N_5(x_k) - \{x\} and |S_\varphi(y) - S_\varphi(x_k)| = 1, and, finally, c /∈ S_\varphi(x_3 - k) - S_\varphi(x) provided that |S_\varphi(x_3 - k) - S_\varphi(x)| = 1. We have |S_\varphi(x) ∪ S_\varphi(x_j)| = |S_\varphi(x_j)| = 4. Therefore, |C(xx_k)| ≥ 14 - (4 + 4 + 1) = 5 and |C(xx_3 - j)| ≥ 14 - [(4 + 3) + 4 + 1] = 2. Because of Lemma 3 the number of 2-element sets \{c_1, c_2\} with c_k ∈ C(xx_k), k = 1, 2, is at least \frac{2(5 - 2 - 1)}{2} = 7. Since 7 > |N_5(x) - \{x_1, x_2\}|, we can choose \{c_1, c_2\} so that taking ψ(xx_k) := c_k, k = 1, 2, yields an nd-extension ψ: E(G) → C of ϕ.

So, we can now assume that S_\varphi(x) /∈ S_\varphi(x_k), k = 1, 2. In such a case any proper extension ψ: E(G) → C of ϕ distinguishes x from x_k, k = 1, 2. Therefore, if C(xx_k) is defined as above, we have |C(xx_k)| ≥ 14 - (4 + 3 + 4) = 3, k = 1, 2,
and the number of 2-element sets \{c_1, c_2\} with \(c_k \in C(xx_k), k = 1, 2\), is at least 
\[
\frac{3(3-3-1)}{2} = 3 \geq |N_5(x) - \{x_1, x_2\}|
\]
and \(\psi\) can be found again.

\(i = 7\): Let \((x_1, x_2, x_3)\) be a \((5, 5, 5)\)-path in \(G\) and let \(G_5\) be the maximum
connected subgraph of \(G\) with \(V(G_5) \supseteq \{x_1, x_2, x_3\}\) induced on 5-vertices.
By Lemma 4.6 then \(\text{deg}_{G_5}(x) \in \{1, 5\}\) for every \(x \in V(G_5)\).
Further, by Lemma 4.4, no plane embedding of \(G_5\) has a 3-face. As a consequence, there are \(y_1, y, y_2 \in V(G_5)\) with \(\text{deg}_{G_5}(y_1) = 1\) and \(\text{deg}_{G_5}(y) = 5\). (It is well known that any plane
graph has either a 3-face or a vertex of degree at most 3.) Then, however,
repeating the proof of Lemma 4.6 with \((y_1, y, y_2)\) instead of \((x_1, x_2)\) leads to a
desired contradiction. If there is \(j \in [1, 2]\) with \(S_{\varphi}(y) \subseteq S_{\varphi}(y_j)\), the reasoning
is the same as above. In the case \(S_{\varphi}(y) \not\subseteq S_{\varphi}(y_k), k = 1, 2\), we have \(|C(yy_k)| \geq
14 - (4 + 3 + 0) = 7\) and \(|C(yy_k)| \geq 3\) so that the number of 2-element sets \{c_1, c_2\}
with \(c_k \in C(yy_k), k = 1, 2\), is at least 
\[
\frac{3(27-3-1)}{2} = 15 \geq |N_5(y) - \{y_1, y_2\}|
\]

\(i = 8\): Let \([x_1, x_2, x_3]\) be a \((d, d, d)\)-cycle in \(G\) and let \(H := G - x_1x_2\).
By Lemma 4.3,4,5,7 we know that \(d_3 \geq d + 1\) and \(N_d(x_j) = \{x_3, \ldots, x\}\), \(j = 1, 2\).
If \(d \in [3, 4]\), Lemma 2 yields \(S_{\varphi}(x_1) = S_{\varphi}(x_2)\). The equality holds true for \(d = 5\),
too: otherwise there are at least 14 (4 + 4) colours for a proper extension of \(\varphi\)
and each such extension is neighbour-distinguishing. Let \(c_j := \varphi(x_jx_3), j = 1, 2, \) and
let \(\tau: E(H - \{x_1x_3, x_2x_3\}) \rightarrow C\) be the restriction of \(\varphi\). Any pair \(\{\tilde{c}_1, \tilde{c}_2\} \in \tilde{C}\)
with \(\tilde{C} := C - (S_{\varphi}(x_1) \cup S_{\varphi}(x_3))\) can be used for a proper extension \(\xi: E(H) \rightarrow C\)
of \(\tau\) with \(\xi(x_jx_3) := \tilde{c}_j, j = 1, 2\). Another possibility how to define \(\xi\) is to take
\(\xi(x_kx_3) := c_k\) and \(\xi(x_3-kx) \in \tilde{C}\) with \(k \in [1, 2]\). All mentioned possibilities lead
to \(S_{\xi}(x_1) \not\subseteq S_{\xi}(x_2)\). Since \(|\tilde{C}| \geq 14 - [(d - 1) + (d_3 - 1)] = 17 - d - d_3\), we
have altogether at least \((17-d-d_3) + 2(17-d-d_3) = \frac{17}{2}(17 - d - d_3)(20 - d - d_3)\)
proper extensions \(\xi\) of \(\tau\) satisfying \(S_{\xi}(x_1) \not\subseteq S_{\xi}(x_2)\). A sufficient condition for
finding an appropriate extension \(\psi: E(G) \rightarrow C\) of \(\xi\) is then \(\frac{1}{2}(17 - d - d_3)(20 - d - d_3) > d_3 - 2\) \((\geq n_d(x_3))\). Therefore,
to obtain a requested contradiction it suffices that \(d_3\) is smaller than the minimum root of the quadratic equation
\[
x^2 - x(39 - 2d) + d^2 - 37d + 344 = 0, \text{ i.e., } d_3 < \frac{1}{2}(39 - 2d - \sqrt{145 - 8d})\]
and, finally, \(d_3 \leq h(d)\).

\(i = 9\): Let \((x_1, x_2, x_3)\) be a \((d_1, d_2, d_3)\)-path in \(G\) and let \(H := G - \{x_1x_2, x_2x_3\}\).
Because of \(g(d_3, d_1) = g(d_1, d_3)\) we may suppose without loss of generality that
\(d_1 \leq d_3\). Further, from Lemma 4.2,8 and the inequality \(g(d, d) \leq h(d), d = 3, 4, 5, \) it
follows that in the case \(d_1 = d_3 = d\) we may suppose that \((x_1, x_2, x_3)\) is an
induced path. Also, due to Lemma 4.3,5,7 we may suppose that \(d_2 \geq 6\). So,
there will be no need to take care about distinguishing \(x_j\) from \(x_k\) for \(j, k \in [1, 3]\),
\(j \neq k\).

Let \(n(j)\) be an upper bound for \(n_d(x_j), j = 1, 3\). From Lemma 4.1–4,7 we see
that we can take \(n(1) = n(2) = 0\) and \(n(3) = n(4) = n(5) = 1\). Let \(C(x_jx_2)\)
be the set of all colours in \(C\) that can be used as \(\varphi_j(x_jx_2)\) in a proper extension
\(\varphi_j: E(H + x_jx_2) \rightarrow C\) of \(\varphi\) that distinguishes \(x_j\) from all vertices in \(N_d(x_j)\).
Thus \(c \in C(x_jx_2)\) implies \(c \not\in S_{\varphi}(x_j) \cup S_{\varphi}(x_2)\) as well as \(c \not\in S_{\varphi}(y) - S_{\varphi}(x_j)\)
under the assumptions \( y \in N_d(x_j) \) and \( |S_\varphi(y) - S_\varphi(x_j)| = 1 \). As a consequence, 
\[ |C(x, x_2)| \geq 14 - ([d_1 - 1] + (d_2 - 2) + n(d_j)) = 17 - d_j - n(d_j) - d_2, \ j = 1, 2. \]

Let \( p \) be the number of all proper extensions \( \psi : E(G) \to C \) of \( \varphi \) with 
\( \psi(x, x_2) \in C(x, x_2) \). From the inequality \( n_{d_2}(x_2) \leq d_2 - 2 \) it follows that \( p > d_2 - 2 \) is a sufficient condition for obtaining a required contradiction. Since \( d_1 \leq d_3 \) and 
\( n(d_1) \leq n(d_3) \), we obtain \( 17 - d_1 - n(d_1) - d_2 \geq 17 - d_3 - n(d_3) - d_2 \). Thus,
using Lemma 3, a contradiction follows from the inequality 
\[ \frac{1}{2}(17 - d_3 - n(d_3) - d_2) \geq |2(17 - d_1 - n(d_1) - d_2) - (17 - d_3 - n(d_3) - d_2) - 1| > d_2 - 2, \]
or, equivalently, \( (17 - d_3 - n(d_3) - d_2)(16 + d_3 + n(d_3) - 2d_1 - 2n(d_1) - d_2) > 2(d_2 - 2) \). The quadratic equation
\[ (17 - d_3 - n(d_3) - x)(16 + d_3 + n(d_3) - 2d_1 - 2n(d_1) - x) = 2(x - 2), \]
i.e.,
\[ x^2 - x(35 - 2d_1 - 2n(d_1)) + 276 + d_3 + n(d_3) - 34d_1 - 34n(d_1) - d_3^2 \]
\[- (n(d_3))^2 + 2d_1d_3 - 2d_3n(d_3) + 2d_3n(d_1) + 2d_1n(d_3) + 2n(d_1)n(d_3) = 0, \]
has roots
\[ \frac{1}{2} \left( 35 - 2d_1 - 2n(d_1) \pm \sqrt{D(d_1, d_3)} \right), \]
where \( D(d_1, d_3) := 4(d_3 - d_1 + n(d_3) - n(d_1))^2 + 121 - 4d_1 - 4d_3 - 4n(d_1) - 4n(d_3) \geq 121 - 20 - 20 - 4 - 4 = 73. \) As a consequence, the inequality \( d_2 < \frac{1}{2} \left( 35 - 2d_1 - 2n(d_1) - \sqrt{D(d_1, d_3)} \right) \) is sufficient to obtain a contradiction. The number \( g(d_1, d_3) \) is just the maximum integer smaller than the above upper bound for \( d_2 \).

\( i = 10 \): Suppose \((x_1, x_2, x_3)\) is a \((4, d_2, 6)\)-path in \( G \) and let \( H := G - \{x_1, x_2, x_2x_3\} \). From Lemma 4.5 it follows that \( d_2 = 4 \). Then, by Lemma 4.3.4, \( x_3 - j \) is the unique 4-neighbour of \( x_j, \ j = 1, 2 \). Let \( C(x_1x_2) \) be the set of all colours available as \( \varphi_1(x_1x_2) \) in a proper extension \( \varphi_1 : E(H + x_1x_2) \to C \) of \( \varphi \) and let \( C(x_2x_3) \) be the set of all colours available as \( \varphi_2(x_2x_3) \) in a proper extension \( \varphi_2 : E(H + x_2x_3) \to C \) of \( \varphi \) that distinguishes \( x_3 \) from all its 6-neighbours and satisfies \( S_\varphi(x_2) \neq S_\varphi(x_1) (= S_\varphi(x_1)) \). We have \( |C(x_1x_2)| \geq 14 - (3 + 2) = 9 \) and \( |C(x_2x_3)| \geq 14 - (5 + 2 + 5 + 1) = 1 \). A proper extension \( \psi : E(G) \to C \) with 
\( \psi(x_jx_{j+1}) \in C(x_jx_{j+1}), \ j = 1, 2, \) is neighbour-distinguishing, a contradiction.

\( i = 11 \): Suppose \((x_1, x_2)\) is a \((4, 6)\)-path in \( G \) and let \( H := G - x_1x_2 \).
By Lemma 4.10, \( x_1 \) has no 4-neighbour. Thus, the number of colours that are available as \( \psi(x_1x_2) \) in an nd-extension \( \psi : E(G) \to C \) of \( \varphi \), is at least \( 14 - (5 + 3 + 5) = 1 \).

\( i = 12 \): Let \([x_1, x_2, x_3]\) be a \((5, 6, 6)\)-cycle in \( G \) and let \( H \) be the graph 
\( G - \{x_1x_2, x_2x_3, x_3x_1\} \). Similarly as in the proof of Lemma 4.4, the number of exemplars of colours belonging to a pair that is dangerous for \( x_2 \) or \( x_3 \) is at most \( 2 \cdot 2 \cdot (6 - 2) = 16 \). Let \( c \in C - \bigcup_{j=1}^{3} S_\varphi(x_j) \) be a colour having
the minimum frequency \( f(c) \) in pairs that are dangerous for \( x_2 \) or \( x_3 \). Since \(|C - \bigcup_{j=1}^{3} S_{\phi}(x_j)| \geq 14 - (3 + 4 + 4) = 3\), we have \( f(c) \leq \lfloor \frac{16}{3} \rfloor = 5 \). Then \( f(c) = f_2(c) + f_3(c) \), where \( f_j(c) \) is the frequency with which \( c \) appears in a pair dangerous for \( x_j \), \( j = 2, 3 \). We may suppose without loss of generality that the notation is chosen so that \( f_2(c) \leq f_3(c) \). For \( \hat{H} := H + x_2x_3 \) consider the extension \( \tau : E(\hat{H}) \rightarrow C \) of \( \phi \) with \( \tau(x_2x_3) := c \). Let \( C(x_1x_j), j \in [2,3] \), be the set of colours that are available as \( \psi_j(x_1x_j) \) if \( \psi_j : E(\hat{H} + x_1x_j) \rightarrow C \) has to be a proper extension of \( \tau \) that distinguishes \( x_j \) from all vertices in \( N_6(x_j) - \{x_{5-j}\} \); then \(|C(x_1x_j)| \geq 14 - (3 + 4 + 1 + f_j(c)) = 6 - f_j(c) \). Note that \( f_3(c) \leq 4 \) implies \( f_2(c) \leq 1\), hence \(|C(x_1x_3)| \geq 2\) and \(|C(x_1x_2))| \geq 5\). Analogously, with \( f_3(c) \leq 3 \) we have \( f_2(c) \leq 2 \) so that \(|C(x_1x_3)| \geq 3\) and \(|C(x_1x_2))| \geq 4\). By Lemma 3, the number of 2-element sets \( c_1, c_2 \) with \( c_j \in C(x_1x_j), j = 1, 2\), is at least \( \min(2^{\frac{2(25^2 - 2\cdot 1)}{2}}, 3^{\frac{2(24 - 3 - 1)}{2}}) = 6 \). From the inequality \( 6 > 4 \geq n_5(x_1) + 1 \) (where the summand 1 “guarantees” distinguishing \( x_2 \) from \( x_3 \)) it follows that one can find an nd-extension \( \psi : E(G) \rightarrow C \) of \( \tau \).

\[ i = 13: \text{Let } (x_1, x_2, x_3, x_4) \text{ be a } (1, d_2, d_3, 1)-\text{path in } G. \text{ Pick a vertex } x \notin V(G) \text{ and consider the planar graph } H \text{ with } V(H) := V(G) - \{x_1, x_2\} \cup \{x\} \text{ and } E(H) := E(G) - \{x_1x_2, x_3x_4\} \cup \{xx_2, xx_3\}. \text{ The colouring } \psi : E(G) \rightarrow C, \text{ that is determined by } \psi(x_1x_2) := \varphi(xx_2), \psi(x_3x_4) := \varphi(xx_3) \text{ and } \psi(e) := \varphi(e) \text{ for } e \in E(G) \cap E(H), \text{ is neighbour-distinguishing.} \]

### 3 The discharging procedure

Consider an ndi-minimal planar graph \( G \). Let \( \tilde{G} = (V, E, F) \) be a standard embedding of \( G \), i.e., one with the following property: If \( N_1(v) \neq \emptyset \) for a vertex \( v \in V \), there is a face \( f \in F \) that is incident with all vertices of \( N_1(v) \), and, moreover, \( f \) is of maximum possible degree. For simplicity we shall not distinguish vertices in \( V(G) \) and \( V \) as well as edges in \( E(G) \) and \( E \).

Consider vertices \( v, w \in V \) and a face \( f \in F \). We have \( \deg_{\tilde{G}}(f) = |\tilde{E}(f)| \) and \( (v, w) \in \tilde{E}(f) \iff (w \in N(v) \land f = \text{face}(v, w)) \). Therefore, by Euler’s formula and handshaking lemma,

\[
2 = |V| - |E| + |F| = \sum_{v \in V} 1 - \frac{1}{2} \sum_{v \in V} \deg_{\tilde{G}}(v) + \sum_{f \in F} \sum_{(v,w) \in E(f)} \frac{1}{\deg_{\tilde{G}}(f)}
\]

\[
= \sum_{v \in V} \left(1 - \frac{1}{2} \deg_{\tilde{G}}(v) \right) + \sum_{v \in V} \sum_{w \in N(v)} \frac{1}{\deg_{\tilde{G}}(\text{face}(v, w))} = \sum_{v \in V} c_0(v),
\]

where

\[
c_0(v) := 1 - \frac{1}{2} \deg_{\tilde{G}}(v) + \sum_{w \in N(v)} \frac{1}{\deg_{\tilde{G}}(\text{face}(v, w))}.
\]
is the initial charge of the vertex $v_0$. Since

$$\sum_{w \in N(v)} \frac{1}{\deg_G(\text{face}(v, w))} = \sum_{w \in N(v)} \frac{1}{\deg_G(\text{face}(w, v))}$$

(for any $v \in V$), it follows that

$$c_0(v) = 1 - \frac{1}{2} \deg_G(v) + \sum_{w \in N(v)} \left[ \frac{1}{2 \deg_G(\text{face}(v, w))} + \frac{1}{2 \deg_G(\text{face}(w, v))} \right]$$

$$= \sum_{w \in N(v)} s(v, w),$$

where

$$s(v, w) := \frac{1}{\deg_G(v)} - \frac{1}{2} + \frac{1}{2 \deg_G(\text{face}(v, w))} + \frac{1}{2 \deg_G(\text{face}(w, v))}$$

is the contribution of the oriented edge $(v, w)$ to $c_0(v)$.

We shall transform the mapping $c_0 : V \rightarrow \mathbb{Q}$ to a mapping $c_1 : V \rightarrow \mathbb{Q}$ and then $c_1$ to a mapping $c_2 : V \rightarrow \mathbb{Q}$. Both transformations will be simply charge redistributions (using several redistribution rules) which means that the total charge is constant, i.e., $\sum_{v \in V} c_i(v) = 2$, $i = 0, 1, 2$. On the other hand, we will be able to show that $c_2(v) \leq 0$ for every $v \in V$ so that $\sum_{v \in V} c_2(v) \leq 0$. The obtained contradiction will prove our Theorem by showing that there is no ndi-minimal planar graph.

The redistribution rules are as follows (a Rule $i.j$ applies when defining $c_i$ from $c_{i-1}$, $i = 1, 2$):

**Rule 1.1** If $s(v, w) > 0$ and $\deg_G(v) < \deg_G(w)$, the vertex $v$ sends to the vertex $w$ the amount $s(v, w)$.

**Rule 1.2** Let $v_1v_2$ be a special edge with $s(v_1, v_2) > 0$. If $T(v_1, v_2) := \{w \in V : v_1v_2w \in F\} \neq \emptyset$, the vertex $v_i$ sends to each vertex $w \in T(v_i, v_2)$ the amount $s(v_i, v_2)/|T(v_i, v_2)|$, $i = 1, 2$.

**Rule 1.3** Let $v_1v_2$ be a special edge with $s(v_1, v_2) > 0$ that is not incident with a 3-face. If $(w_1^j, v_1, v_2, w_1^j)$ and $(w_2^j, v_2, v_1, w_2^j)$ are facial trails in $\tilde{G}$, the vertex $v_i$ sends to the vertex $w_i^j$ the amount $s(v_i, v_2)/2$, $i, j = 1, 2$.

**Rule 2.1** Let $w \in V$ be an 8-vertex with $c_1(w) < 0$. If $N_8^+(w) := \{v \in N_8(w) : c_1(v) > 0\} \neq \emptyset$, the vertex $w$ sends to each vertex $v \in N_8^+(w)$ the amount $c_1(w)/|N_8^+(w)|$.

## 4 Some useful claims

A vertex $v \in V$ is said to be low if $\deg_G(v) \leq 5$ and high if $\deg_G(v) \geq 7$.

**Claim 1** If $v \in V$ and $c_0(v) > 0$, then $v$ is a low vertex.
Proof. If \( \deg_G(v) \geq 6 \) and \( w \in N(v) \), then \( s(v, w) \leq \frac{1}{6} - \frac{1}{2} + \frac{1}{6} + \frac{1}{6} = 0 \), and so \( c_0(v) = \sum_{w \in N(v)} s(v, w) \leq 0 \).

Claim 2 If a vertex \( w \) receives an amount by one of redistribution rules, it is a high vertex.

Proof. From Lemma 4.1,5,11 it follows that if \( G \) contains a \((d_1, d_2)\)-path with \( d_1 \in [1, 5] \) and \( d_2 \in [d_1 + 1, 6] \), then \((d_1, d_2) = (5, 6)\). Suppose \((v, w)\) is a \((5, 6)\)-path. If the edge \( vw \) is incident with a face of degree at least 4, then \( s(v, w) \leq \frac{1}{5} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = -\frac{1}{120} \). If \( vw \) is incident with 3-faces only, the third vertex of each such face is, by Lemma 4.5,8,12, high; thus, \( n_6(w) \leq 3 \). By Lemma 4.6, \( n_5(v) \leq 1 \). As a consequence, \( 5 + n_5(v) + 6 + n_6(w) \leq 15 \) in contradiction to Lemma 2.2. So, we are done for Rule 1.1.

For Rule 1.2 use Lemma 4.8.

Let \( v_1v_2 \) be a special edge not incident with a 3-face. Then \( s(v_1, v_2) \leq \frac{\frac{1}{5}}{\deg_G(v_1)} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = \frac{1}{\deg_G(v_1)} - \frac{1}{4} \) so that \( s(v_1, v_2) > 0 \) implies \( v_1, v_2 \) are 3-vertices. If a vertex \( w \) receives an amount either from \( v_1 \) or from \( v_2 \) (by Rule 1.3), it is high by Lemma 4.5.

In the case of Rule 2.1 there is nothing to prove.

Claim 3 If \( k \in [3, 5] \) and \( x \in V(G) \) is a \( k \)-vertex, then \( n_k(x) \leq 1 \).

Proof. See Lemma 4.3,4,7.

In what follows \( C \) will denote a set of colours with \( |C| = \max(\Delta(G) + 2, 14) \).

Claim 4 If \( \emptyset \neq E' = \{x_{2i-1}x_{2i} : i \in [1, l]\} \subseteq E_s(G) \), there is no nd-colouring \( \varphi : E(G - E') \to C \) satisfying \( S_\varphi(x_{2i+1}) \neq S_\varphi(x_{2i}) \) for every \( i \in [1, l] \).

Proof. Suppose the opposite. If \( i \in [1, l] \) and \( d_i := \deg_G(x_{2i-1}) \), then, by Claim 3, \( N_{d_i}(x_{2i-1}) = \{x_{2i}\} \) and \( N_{d_i}(x_{2i}) = \{x_{2i-1}\} \). Clearly, one can choose an extension \( \psi : E(G) \to C \) of \( \varphi \) so that it is proper: for every \( i \in [1, l] \) there are at least \( 14 - [\deg_G(x_{2i-1}) - 1] + \deg_G(x_{2i}) \geq 6 \) colours available as \( \psi(x_{2i-1}x_{2i}) \). Since \( S_\psi(x_{2i-1}) = S_\varphi(x_{2i-1}) \cup \{\psi(x_{2i-1}x_{2i})\} \neq S_\varphi(x_{2i}) \cup \{\psi(x_{2i-1}x_{2i})\} = S_\varphi(x_{2i}) \), \( \psi \) is neighbour-distinguishing and we have obtained a contradiction to the fact that \( G \) is ndi-minimal.

Claim 5 If \( x_1x_2 \in E_s(G) \) and \( \varphi : E(G - x_1x_2) \to C \) is an nd-colouring, then \( S_\varphi(x_1) = S_\varphi(x_2) \).

Proof. Otherwise we have a contradiction to Claim 4.

Claim 6 If \( k, n \in \mathbb{Z}, k \geq 2 \) and \( n \geq 2 \), then \( \binom{n+k}{k} - \binom{n+k-2}{k-2} \geq 2k + 1 \).
Proof. From the assumptions it follows that \((\binom{n+k}{k} - \binom{n+k-1}{k-2}) + [\binom{n+k-1}{k-1} - \binom{n+k-2}{k-2}] = (\binom{n+k-1}{k} + \binom{n+k-2}{k-1}) = (\binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}) \geq \binom{k+1}{k} = 2k+1.

The rest of the paper will be devoted to showing that \(c_2(x) \leq 0\) for every \(x \in V\). Suppose first that \(x\) is a low vertex and let \(S^+(x) := \{y \in N(x) : s(x, y) > 0\}\). Then, by Rules 1.1, 1.2, 1.3 and by Claim 2, \(c_2(x) = c_1(x) = \sum_{y \in N(x) - S^+(x)} s(x, y) \leq 0\).

If \(x\) is a 6-vertex, then, by Claims 1 and 2, \(c_2(x) = c_1(x) = c_0(x) \leq 0\).

In the sequel (including all remaining claims) we suppose that \(x\) is a high vertex of degree \(d\) and we use a simplified notation \(N_i\) instead of \(N_i(x), n_i\) instead of \(n_i(x)\) and so on. Also, \(\Delta\) is written instead of \(\Delta(G)\).

Claim 7 If \(e_1, e_2 \in E_\alpha(G)\), \(e_1 \neq e_2\), then \(e_1, e_2\) are at the distance at least 2 apart.

Proof. Use Lemma 4.3.4.5.7.

Let \(n_{k,k}\) denote the number of special edges \(y_1y_2\) such that \(y_1, y_2 \in N_k\).

Claim 8 If \(n_{k,k} \geq 1\) for some \(k \in \{3, 5\}\), then \(n_1 = 0, n_2 \leq k - 3\) and \(n_d \geq \binom{7 - k + n_2 + \Delta - d}{2} - 1\).

Proof. Let \([x, x_1, x_2]\) be a cycle in \(G\) with \(\deg_G(x_1) = \deg_G(x_2) = k\) and let \(\varphi : E(G - x_1x_2) \to C\) be an nd-colouring. By Claim 5 then \(S := S_\varphi(x_1) = S_\varphi(x_2)\).

Let \(C_{1,2} := \{\varphi(xx_1), \varphi(xx_2)\}\).

If \(x_0 \in N_1\), then exchanging the colours (under \(\varphi\)) on the edges \(xx_0\) and \(xx_1\) leads to an nd-colouring \(\hat{\varphi} : E(G - x_1x_2) \to C\) with \(S_{\hat{\varphi}}(x_1) \neq S_{\hat{\varphi}}(x_2)\) in contradiction to Claim 5. (Note that, by Claim 3, \(x_i\) does not have a \(k\)-neighbour in \(G - x_1x_2, i = 1, 2\).)

Next, assume that \(n_2 \geq k - 2\). Since \(|S - C_{1,2}| = k - 3\), there is \(x_3 \in N_2\) such that \(\varphi(xx_3) \notin S - C_{1,2}\). Moreover, there is \(j \in \{1, 2\}\) such that \(\varphi(xx_j) \notin S_\varphi(x_3)\).

As \(N_2(x_3) = \emptyset\) (Lemma 4.2), it is sufficient to exchange the colours on the edges \(xx_j\) and \(xx_3\) to obtain a contradiction to Claim 5.

Finally, suppose that \(n_d < \binom{7 - k + n_2 + \Delta - d}{2} - 1\). If \(y \in N_2\), then \(\varphi(xy) \in S - C_{1,2}\), for otherwise we could proceed as above with \(y\) in the role of \(x_3\). Thus, \(|S_\varphi(x) \cap S| \geq n_2 + 2\). As \(|(S_\varphi(x) \cup S) - C_{1,2}| = (|S_\varphi(x)| + |S| - |S_\varphi(x) \cap S|) - 2 \leq |d + k - 1 - (n_2 + 2)| - 2 = d + k - n_2 - 5\), with \(\hat{C} := C - ((S_\varphi(x) \cup S) - C_{1,2})\) we have \(|\hat{C}| \geq \Delta + 2 - (d + k - n_2 - 5) = 7 - k + n_2 + \Delta - d\). Because of the assumption on \(n_d\) there is a pair of colours \(\{c_1, c_2\} \in \binom{\Delta}{2} - \{C_{1,2}\}\) such that \((S_\varphi(x) - C_{1,2}) \cup \{c_1, c_2\} \notin S_\varphi(v)\) for every \(v \in N_d\). The restriction \(\varphi|E(G - \{x_1, x_2, xx_1, xx_2\})\) has an nd-extension \(\hat{\varphi} : E(G - x_1x_2) \to C\) with \(S_{\hat{\varphi}}(x_1) = (S_\varphi(x) - C_{1,2}) \cup \{c_1, c_2\}\): if \(\{c_1, c_2\} \cap C_{1,2} = \varphi(xx_i), i \in \{1, 2\}\), we define \(\hat{\varphi}(xx_i) := \varphi(xx_i)\). Clearly, there is \(l \in \{1, 2\}\) with \(\hat{\varphi}(xx_l) \notin C_{1,2}\); thus, we have \(\varphi(xx_l) \in S_{\varphi}(x_l) - S_{\varphi}(x_{3-l})\) and \(S_{\varphi}(x_1) \neq S_{\varphi}(x_2)\) in contradiction to Claim 5.
Claim 9 If \( n_{3,3} \geq 1 \), then \( n_{4,4} = 0 \) and \( n_{3,3} = 1 \).

Proof. Suppose \( G \) has a \((d, k, k)\)-cycle \([x, x_1, x_2]\), \( k \in [3, 4] \), and a \((d, 3, 3)\)-cycle \([x, x_3, x_4]\) with \( \{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset \). Let \( \varphi : E(G - x_1x_2) \to C \) be an nd-colouring and let \( \mu := \varphi|E(G - \{x_1x_2, x_3x_4\}) \). By Claim 5 then \( S_\mu(x_1) = S_\mu(x_2) \); as \( \mu \) is proper, there is \( l \in [3, 4] \) such that \( \mu(x_l) \notin S_\mu(x_1) \). Let us show that exchanging the colour (under \( \mu \)) on \( xx_l \) with that on either \( xx_1 \) or on \( xx_2 \) leads to an nd-colouring \( \nu : E(G - \{x_1x_2, x_3x_4\}) \to C \) with \( S_\nu(x_1) \neq S_\nu(x_2) \) and \( S_\nu(x_3) \neq S_\nu(x_4) \) (in contradiction to Claim 4). First note that \( S_\nu(x_1) \neq S_\nu(x_2) \) since exactly one of those sets contains \( \mu(xx_l) \). If there is \( i \in [1, 2] \) with \( \mu(xx_i) \notin S_\mu(x_j), j = 3, 4 \), exchange \( \mu(xx_l) \) with \( \mu(xx_i) \) to obtain \( \mu(xx_l) \in S_\mu(x_i) \). If \( \mu(xx_i) = S_\mu(x_3) \cup S_\mu(x_4), i = 1, 2 \), there is \( j \in [1, 2] \) such that \( \mu(xx_j) \notin S_\mu(x_i) \). In such a case exchange \( \mu(xx_l) \) with \( \mu(xx_j) \); as a result then \( |S_\nu(x_3) \cup S_\nu(x_4)| = 3 \) (the union does not contain only \( \mu(xx_l) \) from among \( \mu(xx_i), i = 1, 2, 3, 4 \)) so that clearly \( S_\nu(x_3) \neq S_\nu(x_4) \). \( \blacksquare \)

Claim 10 If \( x_1, x_2 \in N_5 \), \( x_1 \neq x_2 \) and \( d_1 = \deg_G(x_1) \leq \deg_G(x_2) = d_2 \), then \( n_d \geq \frac{1}{2}(5 - d_2 + \Delta - d)(4 + d_2 - 2d_1 + \Delta - d) \).

Proof. Consider an nd-colouring \( \varphi : E(G - \{x_1x_2\}) \to C \) and let \( E_i \) be the set of all special edges incident with \( x_i, i = 1, 2 \). By Claim 3 we know that \( |E_i| \leq 1 \), \( i = 1, 2 \); if \( |E_i| = 1 \) for some \( i \in [1, 2] \), let \( E_i = \{x_iy_i\} \). Note that if \( d_1 = d_2 \) and \( |E_1| = |E_2| = 1 \), it may happen that \( y_1 = x_2 \) and \( y_2 = x_1 \). Let \( \mu := \varphi|E(G - \{x_1x_2\} \cup E_1 \cup E_2) \).

Suppose that \( n_d < \frac{1}{2}(5 - d_2 + \Delta - d)(4 + d_2 - 2d_1 + \Delta - d) \). We are going to find a contradiction (either to Claim 4 or to ndI(G) > |C|) by showing that \( \mu \) has an nd-extension \( \nu : E(G - (E_1 \cup E_2)) \to C \) satisfying \( S_\nu(x_i) \neq S_\nu(y_i) \) for every \( i \in [1, 2] \) with \( |E_i| = 1 \). We define the set \( C(x_{ji}) \) of candidate colours for the edge \( xx_{ji} \), \( i \in [1, 2] \), as \( C = (S_\mu(x) \cup S_\mu(y)) \) or possibly a subset of \( C - (S_\mu(x) \cup S_\mu(y)) \) that arises by deleting one colour. The deletion applies if \( |E_i| = 1 \) and \( |S_\mu(y_i) - S_\mu(x_i)| = 1 \); the deleted colour is the unique element of \( S_\mu(y_i) - S_\mu(x_i) \) so that \( C(x_{ji}) = C - (S_\mu(x) \cup S_\mu(y)) \). From the definition it follows that \( |C(x_{ji})| \geq \Delta + 2 - \lceil (d_2-2)+(d_2-2)+1 \rceil = 5-d_2+\Delta-d, i = 1, 2 \). Since \( 5-d_2+\Delta-d \leq 5-d_1+\Delta-d \), due to Lemma 3 the number of 2-element sets \( \{c_1, c_2\} \) with \( c_i \in C(x_{ji}), i = 1, 2 \), is at least \( \frac{1}{2}(5-d_2+\Delta-d)(4+d_2-2d_1+\Delta-d) > n_d \) and there are colours \( c_i \in C(x_{ji}), i = 1, 2 \), with \( c \neq c_2 \) and \( S_\mu(x) \cup \{c_1, c_2\} \neq S_\mu(v) \) for every \( v \in N_d \).

Now let \( v \) be determined by \( \nu(xx_i) := \tilde{c}_i, i = 1, 2 \). Clearly, \( \nu \) is an nd-colouring. It remains to be shown that if \( |E_i| = 1 \) for some \( i \in [1, 2] \), then \( S_\nu(x_i) \neq S_\nu(y_i) \); notice that \( |E_i| = 1 \) implies \( |S_\nu(y_i)| \geq |S_\nu(x_i)| \). If \( |S_\nu(y_i) - S_\nu(x_i)| = 0 \), then \( S_\nu(y) = S_\nu(x), y_1 = x_2, y_2 = x_1 \) and \( \nu(xx_j) \in S_\nu(y) - S_\nu(y) = S_\nu(x) - S_\nu(x_{ji}), j = 1, 2 \). If \( S_\nu(y_i) - S_\nu(x_i) = \{c\} \), then \( c \notin C(x_{ji}), c \neq \tilde{c}_i \in C(x_{ji}), S_\nu(y_i) \subseteq S_\nu(y) \) and \( S_\nu(x_i) = S_\nu(x_i) \cup \{\tilde{c}_i\} \) so that \( c \in S_\nu(y_i) - S_\nu(x_i) \). Finally, if \( |S_\nu(y_i) - S_\nu(x_i)| \geq 2 \), then \( |S_\nu(y_i) - S_\nu(x_i)| \geq |S_\nu(y_i) - S_\nu(x_i)| - 1 \geq 1 \). \( \blacksquare \)
Claim 11 1. If \( n_{3,3} = 1, n_{3-} \geq 3 \) and \( n_{4-} \geq 4 \), then \( d \geq 13 \) and \( n_{5-} \leq \frac{d-1}{3} \).

2. If either \( n_{4,4} = 1 \), \( n_{3-} \geq 2 \) and \( n_{4-} \geq 5 \) or \( n_{4,4} \geq 2 \) and \( n_{3-} + n_{4,4} \geq 3 \), then \( d \geq 14 \) and \( n_{5-} \leq \frac{d+1}{3} \).

3. If \( n_{5,5} \geq 1 \), \( n_{3-} \geq 3 \) and \( n_{4-} \geq 4 \), then \( d \geq 15 \) and \( n_{5-} \leq \frac{d+3}{3} \).

Proof. Suppose the assumptions of Claim 11.\((k-2)\) are fulfilled for some \( k \in [3,5] \).

Let \([x, x_1, x_2]\) be a \((d, k, k)\)-cycle in \( G \), let \( \varphi : E(G-x_1x_2) \rightarrow C \) be an nd-colouring and let \( c_i := \varphi(x_ix_i), i = 1, 2 \). By Claim 5 then \( S_\varphi(x_1) = S_\varphi(x_2) \supseteq \{c_1, c_2\} \).

Provided that \( C_{1,2} := S_\varphi(x_1) - \{c_1, c_2\}, C^- := C - S_\varphi(x), C^+ := \{\varphi(xy) : y \in N_{5-}\} - C_{1,2}, \check{C} := \{\check{C} \subseteq C^- \cup C^+ : |\check{C}| = |C^+|, |\check{C} \cap \{c_1, c_2\}| \leq 1\} \), we have \( |C_{1,2}| = k - 3 \) and \( |C^+| \geq n_{5-} - (k - 3) \geq n_{4-} + 3 - k \geq 2 \) (the last inequality can be easily checked from the assumptions of Claim 11.\((k-2)\)). Since \( |C^-| \geq \Delta + 2 - d \geq 2 \), \( C^- \cap C^+ = \emptyset \) and \( \{c_1, c_2\} \subseteq C^- \cup C^+ \), using Claim 6 we obtain

\[
|\check{C}| = \left(\frac{|C^-| + |C^+|}{|C^+|}\right) - \left(\frac{|C^-| + |C^+|-2}{|C^+|-2}\right) \geq 2|C^+| + 1 \geq 2n_{5-} + 7 - 2k.
\]

Our task is to prove (apart from the inequality on \( d \)) that \( n_{5-} \leq \frac{d+2k-7}{3} \). So, for a proof by contradiction suppose that \( n_{5-} > \frac{d+2k-7}{3} \). Then \( |\check{C}| > \frac{d+2k-7}{3} + 7 - 2k = d - \frac{d+2k-7}{3} > d - n_{5-} \geq n_4 \); since \((S_\varphi(x) - C^+) \cap (C^- \cup C^+) = \emptyset \), there is \( \tilde{C} \in \check{C} \) satisfying \( S_\varphi(x) - C^+ \cup \tilde{C} \neq S_\varphi(v) \) for every \( v \in N_4 \). Let \( Y := \{y \in N_{5-} : \varphi(xy) \notin C_{1,2}\} \) (so that \( |Y| = |C^+| \)). Further, let \( E^+ := \{xy : y \in Y\} \), let \( E^+ \) be the subset of \( E_\mu(G) \) formed by all the edges incident with a vertex of \( Y \) and let \( \mu := \varphi|E(G - (E^+ \cup E^+))\). We shall show that \( \mu \) has an nd-extension \( \nu : E(G-E^+) \rightarrow C \) satisfying \( S_\nu(y_1) \neq S_\nu(y_2) \) for every \( y_1y_2 \in E^+ \) (in contradiction to Claim 4).

For that purpose it is sufficient to define \( \nu \) in such a way that \( S_\nu(x) = S_\nu(x) - C^+ \cup \tilde{C} \). Let us first construct a set \( \hat{C}\{(xy) \in C \) of candidate colours for an edge \( xy \in E^+ \). If there is (a unique) \( i \in [1,2] \) such that \( c_i \in \tilde{C} \), then \( \hat{C}(xx_i) := \tilde{C} \) and \( \hat{C}(xx_{3-i}) := \tilde{C} - \{c_j\} \), otherwise \( \hat{C}(xx_j) := \tilde{C}, j = 1, 2 \). If \( y_1 \in Y - \{x_1, x_2\} \) and \( xy_1 \in E^+ \), then \( \hat{C}(xy_1) \) is either \( \tilde{C} - S_\mu(y_1) \) or a subset of \( \tilde{C} - S_\mu(y_1) \) obtained by deleting exactly one colour. The deletion applies if there is \( y_1y_2 \in E^+ \) such that \( |S_\mu(y_2) - S_\mu(y_1)| = 1 \) and the deleted colour is the unique element of \( S_\mu(y_2) - S_\mu(y_1) \) so that \( \hat{C}(xy_1) = \tilde{C} - S_\mu(y_2) \). However, if \( y_2 \in Y - \{x_1, x_2\} \) and \( |S_\mu(y_2) - S_\mu(y_1)| = 1 \) (which implies \( |S_\mu(y_1) - S_\mu(y_2)| = 1 \)), the above deletion applies only for defining exactly one (arbitrarily chosen) of the sets \( \hat{C}(xy_i), i = 1, 2 \).

Suppose we are able to find a system \( \{\hat{c}(xy) : y \in Y\} \) of distinct representatives of the family \( \{\hat{C}(xy) : y \in Y\} \), i.e., \( \hat{c}(xy) \in \hat{C}(xy) \) for every \( y \in Y \). Then \( \nu \) determined by \( \nu(xy) := \hat{c}(xy) \) is a required nd-extension of \( \mu \). Indeed, \( \nu \) is proper, hence \( |Y| = |C^+| = |\tilde{C}| \) yields \( \{\nu(xy) : y \in Y\} = \tilde{C} \) and \( S_\nu(x) = S_\mu(x) - C^+ \cup \tilde{C} \). Also, if \( y_1 \in Y - \{x_1, x_2\} \) and \( y_1y_2 \in E^+ \), then
$S_\nu(y_1) \neq S_\nu(y_2)$ can be proved similarly as in the proof of Claim 10. We have $|S_\nu(y_2)| \geq |S_\nu(y_1)|$ and for $l := |S_\nu(y_2) - S_\nu(y_1)|$ we analyse three possibilities $l = 0, l = 1$ and $l \geq 2$. Moreover, if $y_2 \in Y - \{x_1, x_2\}$, $S_\mu(y_3-i) - S_\mu(y_1) = \{c\}$ and $\tilde{C}(xy_i) = \tilde{C} - S_\mu(y_3-i) = \tilde{C} - (S_\mu(y_1) \cup \{c\})$ for some $i \in [1, 2]$, then $c \in S_\nu(y_3-i) - S_\nu(y_1)$.

Now, to show the existence of a system of distinct representatives of the family $\{\tilde{C}(xy) : y \in Y\}$, let us check that the requirements of Hall’s Theorem are fulfilled. We have to prove that if $\emptyset \neq Z \subseteq Y$, then $|\bigcup_{z \in Z} \tilde{C}(xz)| \geq |Z|$. First note that from the definition of $\tilde{C}(xz)$ it follows that $|\tilde{C}(xz)| \geq |\tilde{C} - (\deg_G(z) - 1)| \geq |\tilde{C}| - 4$. Therefore, if $\tilde{z} \in Z \subseteq Y$ and $|Z| \leq |Y| - 4$, then $|\bigcup_{z \in Z} \tilde{C}(xz)| \geq |\tilde{C}(\tilde{z})| \geq |\tilde{C}| - 4 = |Y| - 4 \geq |Z|$.

As we have seen above, there exists $i \in [1, 2]$ such that $\tilde{C}(x_{3i}) = \tilde{C}$ and $|\tilde{C}(x_{3i-1})| \geq |\tilde{C} - 1$; we may suppose without loss of generality that $i = 1$ which implies $|\tilde{C}(x_3)| \geq |\tilde{C}| + 1 - j$, $j = 1, 2$. From the definition of $\tilde{C}(xz)$ we also know that if $y_1, y_2 \in Y - \{x_1, x_2\}$ and $y_1y_2 \in E'$, there is $l \in [1, 2]$, such that $|\tilde{C}(xy)| \geq |\tilde{C}| + 1 - \deg_G(y_2) - 2$. As a consequence it is easy to check that (for any $k \in [3, 5]$) there are vertices $x_3, x_4 \in Y - \{x_1, x_2\}$, $x_3 \neq x_4$, such that $|\tilde{C}(x_{3j})| \geq |\tilde{C}| + 1 - j$, $j = 3, 4$. For example, if $k = 4$ and $n_{44} \geq 2$, we can take $x_3, x_4$ so that $\deg_G(x_3) = \deg_G(x_4) = 4$ and $x_3x_4 \in E'$.

Suppose that $Z \subseteq Y$, $|Z| = |Y| - p$ for some $p \in [1, 3]$. Then $|Z \cap \bigcup_{j=1}^{p+1} \{x_j\}| = |Z| + p - 1 = |Z| \cup \bigcup_{j=1}^{p+1} \{x_j\}| \geq |Y| + 1 - |Y| = 1$ and there is $q \in [1, p+1]$ with $x_q \in Z$. Then, however, $|\bigcup_{z \in Z} \tilde{C}(xz)| \geq |\tilde{C}(x_qz)| \geq |\tilde{C}| + 1 - q \geq |\tilde{C}| + 1 - (p + 1) = |\tilde{C}| - p = |Y| - p = |Z|$.

We have also $|\bigcup_{z \in Z} \tilde{C}(xz)| \geq |\tilde{C}(x_{3i})| = |\tilde{C}| = |Y|$.

Finally, for each $k \in [3, 5]$ the assumptions of Claim 11. $(k - 2)$ yield $k + 1 \leq n_{5-} \leq \frac{d + 2k - 7}{3}$ (we have just proved the last inequality). Thus, $d \geq k + 10$.

Claim 12 Suppose that $n_{k,k} = 0$, $k = 3, 4, 5$.

1. If there is $j \in [1, 2]$ such that $n_{i,j} \geq 2$ for all $i \in [j, 4]$, then $d \geq \frac{n_{5-}^2 + 5n_{5-} + 2}{2} \geq 19$ and $n_{5-} \leq \frac{-5 + \sqrt{577}}{2}$.

2. If there is $j \in [1, 2]$ such that $n_{i,j} \geq 2$ for all $i \in [j, 2]$, $n_{3-} = 3$ and $n_4 = 0$, then $d \geq 13$.

Proof. 1. Suppose first that $n_1 = 0$ (which means that $j = 2$), $n_2 \geq 2$, $\{x_1, x_2, x_3, x_4\} \subseteq N_{5-}$, $\deg_G(x_1) = \deg_G(x_2) = 2$, $\deg_G(x_3) \leq 3$ and $\deg_G(x_4) \leq 4$. Let $E^+ := \{xy : y \in N_{5-}\}$, $E^- := \{yy' \in E_\delta(G) : y \in N_{5-}\}$, let $\varphi : E(G - (E^+ \cup E^-)) \rightarrow C$ be an nd-colouring and let $C^- := C - S_\nu(x), \tilde{C} := \tilde{C}(x)$.

Then $|C^-| \geq \Delta + 2 - (d - n_{5-})$ and $|\tilde{C}| \geq \frac{n_{5-}^2 + 2 + \Delta - d}{n_{5-}} \geq \frac{n_{5-} + 2}{n_{5-}}$.

Assume that $d < \frac{n_{5-}^2 + 5n_{5-} + 2}{2}$. Then $n_d \leq d - n_{5-} < \frac{n_{5-} + 2}{2} \leq |\tilde{C}|$ and there is $\tilde{C} \in \tilde{C}$ such that $S_\nu(x) \cup \tilde{C} \neq S_\nu(y)$ for every $y \in N_d$. Let $y_p \neq x$ be the neighbour of $x_i$, $i = 1, 2$. We want to show that $\varphi$ has an nd-extension $\nu : E(G - E') \rightarrow C$ such that $S_\nu(y) \neq S_\nu(y')$ whenever $yy' \in E'$ (in contradiction to Lemma 4).
The set $\bar{C}(xy) \subseteq \bar{C}$ of candidate colours for $xy, y \in N_{5-}$, is defined as distinct from $\bar{C} - S_\varphi(y)$ if and only if there is $yy' \in E'$ (so that $y' \notin N_{5-}$) with $|S_\varphi(y') - S_\varphi(y)| = 1$, in which case $\bar{C}(xy) := \bar{C} - S_\varphi(y')$. Let us show that we are able to find a system $\{\bar{c}(xy) : y \in N_{5-}\}$ of distinct representatives of the family $\{\bar{C}(xy) : y \in N_{5-}\}$; then defining $\nu(xy) := \bar{c}(xy)$, $y \in N_{5-}$, leads to $S_\nu(y) \neq S_\nu(y')$ for $yy' \in E'$. Since $|\bar{C}(xx_i)| \geq |\bar{C}| - 1$ and $|\bar{C}(xx_i)| \geq |\bar{C}| + 1 - i$, $i = 2, 3, 4$, similarly as in the proof of Claim 11 we see that the requirement $|\bigcup_{z \in Z} \bar{C}(xz)| \geq |Z|$ of Hall's Theorem is fulfilled for any nonempty proper subset $Z$ of $N_{5-}$. (If $x_1 \in Z$, then $|\bigcup_{z \in Z} \bar{C}(xz)| \geq |\bar{C}(xx_1)| \geq |\bar{C}| - 1 \geq |Z|$.)

If $\varphi(x_1y_1) \neq \varphi(x_2y_2)$, then $|\bigcup_{z \in N_{5-}} \bar{C}(xz)| \geq |\bar{C}(xx_1) \cup \bar{C}(xx_2)| \geq |(\bar{C} - \{\varphi(x_1y_1)\}) \cup (\bar{C} - \{\varphi(x_2y_2)\})| = |\bar{C}| = |N_{5-}|$. If $\varphi(x_1y_1) = \varphi(x_2y_2) \in S_\varphi(x)$, then $\varphi(x_1y_1) \notin C' \supseteq \bar{C}, C(x_i) = \bar{C} - \{\varphi(x_1y_1)\} = \bar{C}$, $i = 1, 2$, and $|\bigcup_{z \in N_{5-}} \bar{C}(xz)| \geq |\bar{C}| = |N_{5-}|$.

Assume that $\varphi(x_1y_1) = \varphi(x_2y_2) \notin S_\varphi(x)$. If there is $i \in [1, 2]$ with $x_{yi} \in E(G)$, consider the colouring $\hat{\varphi} : E(G - (E^+ \cup E')) \to C$ obtained by exchanging the colours (under $\varphi$) on the edges $x_{yi}$ and $x_{yi}$. For any $v \in V(G) - \{x, x_i\}$ then $\hat{S}_\varphi(v) = S_\varphi(v)$; the structure of $\hat{S}_\varphi(x)$ and $\hat{S}_\varphi(x_1)$ is not important, colour sets of $x$ and $x_1$ are under construction. Since $\hat{\varphi}(x_1y_1) \neq \hat{\varphi}(x_2y_2)$, we can proceed as above (with $\hat{\varphi}$ instead of $\varphi$ and with the same sets $\hat{C}(xy)$ of candidate colours, $y \in N_{5-}$) to see that $|\bigcup_{z \in N_{5-}} \bar{C}(xz)| \geq |N_{5-}|$.

If $x_{yi} \notin E(G), i = 1, 2$, consider instead of $H := G - (E^+ \cup E')$ the planar graph $H'$ with $V(H') := V(G), E(H') := E(G) - (E^+ \cup E' \cup \{x_1y_1, x_2y_2\}) \cup \{x_1, y_2\}$ and let $\varphi' : E(H') \to C$ be an nd-colouring. Further, let $\varphi'' : E(H) \to C$ be defined by $\varphi''(x_{yi}) := \varphi'(x_{yi}), i = 1, 2$, and $\varphi''(e) := \varphi'(e)$ for $e \in E(H) \cap E(H')$. For any $v \in V(G) - \{x, x_1, x_2\}$ then $S_{\varphi''}(v) = S_{\varphi}(v)$. Since $\varphi''(x_1y_1) \neq \varphi''(x_2y_2)$, we can proceed as above (with $\varphi''$ instead of $\varphi$).

If $j = 1$, the situation is easier, because then $n_1 \geq 1$ and vertices $x_1, x_2$ can be chosen so that $\deg(x_1) = 1$ and $\deg(x_2) \leq 2$. In such a case $|\bar{C}(xx_i)| \geq |\bar{C}| + 1 - i$, $i = 1, 2, 3, 4$, and we are done in the same manner as in the proof of Claim 11.

Thus we have shown that $d \geq \frac{n_{5-}^2 + 5n_{3-} - 2}{2} \geq \frac{16 + 20 + 2}{2} = 19$ and the former inequality yields $n_{5-} \leq \frac{-5 + \sqrt{52 + 17}}{2}$.

2. Proceed similarly as in the proof of Claim 12.1 with $N_{3-} = \{x_1, x_2, x_3\}$ (where $\deg_G(x_1) \leq \deg_G(x_2) \leq \deg_G(x_3) \leq 3$) instead of $N_{5-}$ to see that $d \geq \frac{n_{5-}^2 + 5n_{3-} - 2}{2} = 13$.

Claim 13 Suppose that $n_{k,k} = 0, k = 3, 4, 5, n_{2-} = 1, n_{3-} \geq 3$ and $n_{4-} \geq 4$.

Let $N_{3-} = \{x_1\}, E^+ = \{xy : y \in N_{3-}\}, E' = \{yy' \in E_3 : y \in N_{3-}\}$ and let $\varphi : E(G - (E^+ \cup E')) \to C$ be an nd-colouring.

1. If there is $x_2 \in N_3$ such that $S_\varphi(x_1) \cap S_\varphi(x_2) \neq \emptyset$, then $d \geq \frac{(n_{5-} - 1 + \Delta - d)}{n_{5-}} + n_{5-}$.

2. If there is $x_2 \in N_3$ such that $S_\varphi(x_1) \cap S_\varphi(x_2) = \emptyset$, then $n_{5-} \leq \frac{d + 1}{3}$.
Proof. 1. Evidently, in this case we have \( \deg_G(x_1) = 2 \). Consider the (unique) colour \( c \in S_\varphi(x_1) \cap S_\varphi(x_2) \). For a proof by contradiction assume that \( d < \left( \frac{n_{5-}+1+\Delta-d}{n_{5-}} \right) + n_{5-} \). With \( \hat{C} := S_\varphi(x) \cup \{c\} \) then \( |C - \hat{C}| \geq \Delta + 2 - (d - n_{5-} + 1) = n_{5-} + 1 + \Delta - d \) and \( n_d \leq d - n_{5-} < \left( \frac{n_{5-}+1+\Delta-d}{n_{5-}} \right) \), hence there is a set of colours \( \hat{C} \in \left( \frac{C - \hat{C}}{n_{5-}} \right) \) such that \( S_\varphi(x) \cup \hat{C} \neq S_\varphi(v) \) for every \( v \in \mathbb{N}_d \).

The set \( \hat{C}(xy) \subseteq \hat{C} \) of candidate colours for an edge \( xy \in E^+ \) is defined in the same way as in the proof of Claim 12. Consequently, \( |\hat{C}(xx_i)| \geq |\hat{C}| + 1 - i, \) \( i = 1, 2 \) (notice that \( c \notin \hat{C} \)), and \( |\hat{C}(xy)| \geq |\hat{C}| - (\deg_G(y) - 1) \) for every \( y \in \mathbb{N}_{5-} - \{x_1, x_2\} \). Thus, there is a 2-element set \( \{x_3, x_4\} \subseteq \mathbb{N}_{5-} - \{x_1, x_2\} \) satisfying \( |\hat{C}(xx_i)| \geq |\hat{C}| + 1 - i, i = 3, 4 \), and there exists a system \( \{\hat{c}(xy) : y \in \mathbb{N}_{5-}\} \) of distinct representatives of the family \( \{\hat{C}(xy) : y \in \mathbb{N}_{5-}\} \). The extension \( \nu : E(G - E^+) \to C \) of \( \varphi \) with \( \nu(xy) := \hat{c}(xy) \) is such an nd-colouring that \( S_\nu(y) \neq S_\nu(y') \) whenever \( yy' \in E' \) and we have obtained a contradiction to Claim 4.

2. Suppose that \( S_\varphi(x_2) = \{c_1, c_2\} \) and let \( \tilde{C} := \{C \subseteq \mathbb{N}_{5-} : |C| = n_{5-}, |C \cap \{c_1, c_2\}| \leq 1\} \). We have \( |C - S_\varphi(x)| \geq \Delta + 2 - (d - n_{5-}) \) so that \( |C - S_\varphi(x)| - n_{5-} \geq 2 \) and, by Claim 6,

\[
|\tilde{C}| = \left( \frac{|C - S_\varphi(x)|}{n_{5-}} \right) - \left( \frac{|C - S_\varphi(x)| - 2}{n_{5-} - 2} \right) \geq 2n_{5-} + 1.
\]

If \( n_{5-} > \frac{d-1}{3} \), then \( n_d \leq d - n_{5-} < 2n_{5-} + 1 \) and there is \( \tilde{C} \in \tilde{C} \) with \( S_\varphi(x) \cup \tilde{C} \neq S_\varphi(v) \) for each \( v \in \mathbb{N}_d \). If the set of candidate colours \( \tilde{C}(xy), y \in \mathbb{N}_{5-}, \) is defined as before (in the proof of Claim 12), then using the assumption \( S_\varphi(x_1) \cap S_\varphi(x_2) = \emptyset \) it is easy to see that \( |\tilde{C}(xx_i)| \geq |\tilde{C}| - 1, i = 1, 2 \) and \( \tilde{C}(xx_1) \cup \tilde{C}(xx_2) = \tilde{C} \). Moreover, there is a 2-element set \( \{x_3, x_4\} \subseteq \mathbb{N}_{5-} - \{x_1, x_2\} \) such that \( |\tilde{C}(xx_i)| \geq |\tilde{C}| + 1 - i, i = 3, 4 \). To see that the family \( \{\tilde{C}(xy) : y \in \mathbb{N}_{5-}\} \) fulfills the condition \( \bigcup_{z \in Z} \tilde{C}(xz) \geq |Z| \) for any nonempty proper subset \( Z \) of \( \mathbb{N}_{5-} \) we proceed analogously as in the proof of Claim 11 (and using the fact that \( |\tilde{C}(xx_1)| \geq |\tilde{C}| - 1 \)).

Finally, we have also \( |\bigcup_{x \in \mathbb{N}_{5-}} \tilde{C}(xz)| \geq |\tilde{C}(xx_1) \cup \tilde{C}(xx_2)| = |\tilde{C}| = |\mathbb{N}_{5-}| \).

So, \( \nu : E(G - E^+) \to C \) with \( \nu(xy) := \tilde{c}(xy) \in \tilde{C}(xy), y \in \mathbb{N}_{5-}, \) is an nd-colouring with \( S_\nu(y) \neq S_\nu(y') \) for \( yy' \in E' \), which is in contradiction to Lemma 4.

Claim 14 If \( n_{k,k} = 0, k = 3, 4, 5, n_1 \geq 1, n_2 - 2 \geq 2 \) and \( n_4 - 3 \geq 3 \), then \( n_d \geq \left( \frac{5+\Delta-d}{3} \right) - 1 \geq 9 \).

Proof. Pick three distinct neighbours \( x_1, x_2, x_3 \) of \( x \) such that \( \deg_G(x_1) = 1 \), \( \deg_G(x_2) \leq 2 \) and \( \deg_G(x_3) \leq 4 \). Let \( E^+ := \{xx_i : i \in [1, 3]\} \), let \( E' \) be the set of all special edges incident with \( x_3 \) (so that \( |E'| \leq 1 \)) and consider an nd-colouring \( \varphi : E(G - (E^+ \cup E')) \to C \). Suppose that if \( E' \neq \emptyset \), then \( E' = \{x_3y\} \). We have
\(|C - S_\varphi(x)| \geq \Delta + 2 - (d - 3) = 5 + \Delta - d.\) If \(n_d < \left(\frac{5 + \Delta - d}{3}\right) - 1\), there is a set \(\breve{C} \in \left[\frac{C - S_\varphi(x)}{3}\right]\) such that \(C - S_\varphi(x) \cup \breve{C} \neq S_\varphi(v)\) for all \(v \in N_d\), \(\breve{C} \neq S_\varphi(x)\) and \(E' \neq \emptyset \Rightarrow \breve{C} \neq S_\varphi(y).\) (Note that if \(E' \neq \emptyset\), the condition \(\breve{C} \neq S_\varphi(x)\) is fulfilled “automatically” as a consequence of the inequality \(|S_\varphi(x)| \leq 2.\) The set \(\breve{C}(x_i)\) of candidate colours for \(xx_i\) is defined as \(\breve{C} - S_\varphi(x)\) except maybe for \(i = 3.\) Namely, if \(E' \neq \emptyset\) and \(|S_\varphi(y) - S_\varphi(x)| = 1\), then \(\breve{C}(x_3) := \breve{C} - S_\varphi(y).\)

It is easy to see that the family \(\{\breve{C}(x_i) : i \in [1, 3]\}\) has a system of distinct representatives \(\{c_i : i \in [1, 3]\}.\) (It is only important to realise that \(\breve{C}(x_3) \neq \emptyset.\)) Then the extension \(\nu : E(G - E') \to C\) with \(\nu(x_i) := c_i \in \breve{C}(x_i), i = 1, 2, 3,\) is a 4-colouring satisfying \(E' \neq \emptyset \Rightarrow S_\varphi(x_3) \neq S_\varphi(y)\) in contradiction either to the fact that \(G\) is an ndi-minimal graph (if \(E' = \emptyset\)) or to Claim 4 (if \(E' \neq \emptyset\)).

**Claim 15**

1. If \(y \in N_2,\) then \(\max(\deg_G(\text{face}(x, y)), \deg_G(\text{face}(y, x))) \geq 4.\)

2. If \(y \in N_1,\) then \(\deg_G(\text{face}(x, y)) = \deg_G(\text{face}(y, x)) \geq 2n + 3.\) If, moreover, \(n_2 \geq 1,\) then \(\deg_G(\text{face}(x, y)) \geq 2n + 4.\)

**Proof.**

1. There are no multiedges in \(G.\)

2. Let \(f\) be the unique face incident with \(y.\) Since \(\breve{C}\) is a standard embedding of \(G, f\) is incident with all vertices in \(N_1.\) Thus, \(\breve{E}(f)\) contains \(2n_1\) oriented edges incident with vertices in \(N_1\) and at least three other oriented edges. (Clearly, \(G \neq K_{1, 3}\), as ndi \((K_{1, 3}) = \Delta.\))

Provided that \(n_2 \geq 1,\) the result follows from the fact that \(l \geq 2n + 4.\) (A vertex in \(N_2\) is incident with a face of degree at least 4 in the plane graph \(\breve{G} - N_1).\)

Let \(y \in N_{k-}\) and let \(t(y, x)\) be the total amount sent from \(y\) to \(x\) by Rules 1.1, 1.2 and 1.3. Further, let

\[
\begin{align*}
u(1)(2) & := \frac{7}{24}, \quad \nu(1)(3) := \frac{1}{6}, \quad \nu(1)(4) := \frac{1}{12}, \quad \nu(1)(5) := \frac{1}{30}, \\
u(2)(3) & := \frac{5}{12}, \quad \nu(2)(4) := \frac{1}{4}, \quad \nu(2)(5) := \frac{1}{10}.
\end{align*}
\]

**Claim 16**

1. If \(y \in N_1,\) then \(t(y, x) = s(y, x) \leq \frac{2n_1 + 2}{4n_1 + 6}.\) If, moreover, \(n_2 \geq 1,\) then \(t(y, x) \leq \frac{n_2 + 3}{2n_1 + 4}.\)

2. If \(k \in [2, 5]\) and \(y \in N_k,\) then \(s(y, x) \leq \nu(1)(k).\)

**Proof.**

1. Let \(f\) be the unique face incident with \(y.\) From Claim 15.2 it follows that \(l := \deg_G(f) \geq 2n_1 + 3,\) and, if \(n_2 \geq 1,\) then \(l \geq 2n_1 + 4.\) So, \(s(y, x) = \frac{1}{5} - \frac{1}{2} + \frac{1}{27} + \frac{1}{27} = \frac{1}{2} + \frac{1}{2} + \frac{1}{27} > 0\) and \(y\) sends \(s(y, x)\) to \(x\) by Rule 1.1. Clearly, Rules 1.2 and 1.3 cannot apply to \(y,\) hence \(t(y, x) = s(y, x) \leq \frac{1}{2} + \frac{1}{2} = \frac{2n_1 + 3}{4n_1 + 6}.\)

Further, under the assumption \(n_2 \geq 1\) we have \(t(y, x) \leq \frac{1}{2} + \frac{1}{2} = \frac{n_2 + 3}{2n_1 + 4}.\)

2. If \(k = 2,\) then, by Claim 15.1, \(s(y, x) \leq \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{1}{2}.\) If \(k \in [3, 5],\) then \(s(y, x) \leq \frac{1}{k} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2} = u(1)(k).\)
Claim 17 If \( k \in [2, 5] \), \( y \in N_k \) and \( y \) does not send an amount to \( x \) by Rule 1.2, then \( t(y, x) \leq u_1(k) \).

Proof. If \( y \) sends an amount to \( x \) only by Rule 1.1, then, by Claim 16.2, \( t(y, x) = s(y, x) \leq u_1(k) \).

If \( y \) sends an amount to \( x \) by Rule 1.3, then \( y \) has a \( k \)-neighbour \( y' \), neither of faces incident with \( yy' \) is a 3-face and \( 0 < s(y, y') \leq \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{1}{8} \), hence \( k = 3 \). As \( s(y, x) \leq \frac{1}{3} - \frac{1}{2} + \frac{1}{23} + \frac{1}{24} = \frac{1}{8} \), we have \( t(y, x) = \max(0, s(y, x)) + \frac{1}{2}s(y, y') \leq \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{6} = u_1(3) \).

Claim 18 If \( k \in [3, 5] \), \( y_1, y_2 \in N_k \) and \( y_1y_2 \in E(G) \), then \( \sum_{i=1}^2 t(y_i, x) \leq u_2(k) \).

Proof. Admit first that \( xy_1y_2 \) is a face and let \( f_1, f_2, g \) be the face other than \( xy_1y_2 \) incident with \( xy_1 \), \( xy_2 \) and \( y_1y_2 \), respectively.

Suppose \( g \) is also a 3-face. If \( k = 3 \), there is \( j \in [1, 2] \) such that \( f_j \) is not a 3-face (there are no multiedges in \( G \)), hence \( s(y_j, x) \leq \frac{1}{2} - \frac{1}{2} + \frac{1}{24} = \frac{1}{8} \) (Claim 17), \( s(y_1, y_2) = \frac{1}{6} \) and, by Rule 1.2, we have \( \sum_{i=1}^2 t(y_i, x) \leq \left( \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{6} \right) + \left( \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{12} \right) = \frac{1}{2} < u_2(3) \). If \( k \in [4, 5] \), then \( \sum_{i=1}^2 t(y_i, x) \leq \frac{6-k}{6k} \left( 1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{6-k}{k} = u_2(k) \).

If \( g \) is not a 3-face, then \( s(y_1, y_2) \leq \frac{1}{k} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = \frac{24-5k}{24k} \), and so Rule 1.2 can be applied only if \( k \in [3, 4] \). In such a case, by Claim 17, \( \sum_{i=1}^2 t(y_i, x) \leq 2 \cdot \left( \frac{6-k}{6k} + \frac{24-5k}{24k} \right) = \frac{48-9k}{12k} = u_2(k) \). If \( k = 5 \), then \( \sum_{i=1}^2 t(y_i, x) \leq \frac{1}{10} + \frac{1}{30} < u_2(5) \) (if \( k = 3 \)), \( \sum_{i=1}^2 t(y_i, x) \leq \frac{1}{60} < u_2(4) \) (if \( k = 4 \)) and \( \sum_{i=1}^2 t(y_i, x) = 0 < u_2(5) \) (if \( k = 5 \)).

Claim 19 1. \( c_0(x) \leq 1 - \frac{d}{6} - \frac{n_2}{24} \).

2. If \( n_1 \geq 1 \), then \( c_0(x) \leq 1 - \frac{d}{6} - \frac{2n_1^2+2n_1}{6n_1+9} \).

3. If \( n_1 = 1 \) and \( n_2 \geq 1 \), then \( c_0(x) \leq \frac{2}{3} - \frac{d}{6} \).

4. If \( n_2 \leq 1 \), then \( c_0(x) \leq 1 - \frac{d}{6} - \frac{n_2}{12} \).

Proof. 1. From Claim 15.1 it follows that if \( y \in N_2 \), then \( s(x, y) \leq \frac{1}{4} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = \frac{1}{4} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = \frac{1}{12} \). On the other hand, for any \( y \in N(x) \) we have \( s(x, y) \leq \frac{1}{2} - \frac{1}{3} + \frac{1}{6} + \frac{1}{8} = \frac{1}{12} \).

Thus, \( c_0(x) = \sum_{y \in N(x)} s(x, y) \leq n_2(\frac{1}{2} - \frac{5}{24}) + (d - n_2)(\frac{1}{3} - \frac{1}{8}) = 1 - \frac{d}{6} - \frac{n_2}{24} \).

2. Suppose that \( f \) is the face incident with all vertices in \( N_1 \). We have already seen in the proof of Claim 16.1 that \( \deg_G(f) \geq 2n_1 + 3 \). The number of vertices \( y \in N(x) \), that satisfy face\((x, y) = f \), is \( n_1 + p, p \geq 1 \) (all vertices in \( N_1 \) have the mentioned property), hence \( c_0(x) = 1 - \frac{d}{2} + \sum_{y \in N(x)} \frac{1}{\deg_G(\text{face}(x, y))} \leq 1 - \frac{d}{2} + (n_1 + p) \cdot \frac{1}{2n_1+3} + (d - n_1 - p) \cdot \frac{1}{3} = 1 - \frac{d}{6} - \frac{2n_1^2+2n_1}{6n_1+9} \leq 1 - \frac{d}{6} - \frac{2n_1^2+2n_1}{6n_1+9} \).
3. Let $N_1 = \{y_1\}$ and let $f$ be the unique face incident with $y_1$. By Claim 15.2 then $\deg_G(f) \geq 6$. Besides $y_1$ there are at least two other vertices $y \in N(x)$ with $\text{face}(x, y) = f$ or $\text{face}(y, x) = f$ and for each such $y$ we have $s(x, y) \leq \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{12} = \frac{1}{2} - \frac{1}{4}$. Moreover, $s(x, y_1) \leq \frac{1}{2} - \frac{1}{2} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} - \frac{1}{5}$, and so $c_0(x) \leq \frac{1}{5} - \frac{1}{4} + 2\left(\frac{1}{3} - \frac{1}{2}\right) + (d - 3)(\frac{1}{2} - \frac{1}{5}) = \frac{2}{3} - \frac{7}{6} - \frac{6}{5}$.

4. If $n_2 = 0$, use Claim 19.1. If $N_2 = \{y_2\}$, by Claim 15.1 there is a face $f \in F$ incident with $y_2$ such that $\deg_G(f) \geq 4$. Besides $y_2$ there is another vertex $y \in N(x)$ with $\text{face}(x, y) = f$ or $\text{face}(y, x) = f$ so that we have $s(x, z) \leq \frac{1}{2} - \frac{5}{7}$ for every $z \in \{y, y_2\}$. Thus, $c_0(x) \leq 2\left(\frac{1}{2} - \frac{5}{7}\right) + (d - 2)(\frac{1}{2} - \frac{5}{7}) = 1 - \frac{3}{5} - \frac{6}{7}$.

Consider a sequence $X = \{x_i\}_{i \in \mathbb{Z}}$ of neighbours of $x$ in a cyclic order $(x_i = x_j$ if $i \equiv j \pmod{d})$ as they are encountered when rotating around $x$ and let $f_i$ be the face incident with $x_i$, $x_{i-1}$ and $x_i$, $l \in \mathbb{Z}$.

Let $k \in [3, 5]$. A $k$-section is a maximal subsequence $(x_i, \ldots, x_j)$ of consecutive terms of $X$ with $i \in [1, d]$ and $j \in [i, d + i - 1]$ such that $\deg_G(x_l) \geq k$ for every $l \in [i, j]$, $\deg_G(f_l) = 3$ for every $l \in [i + 1, j]$ and $x_lx_{l+1}$ is not a special edge for every $l \in [i - 1, j]$.

**Claim 20** If $i \in [1, d]$, $j \in [i, d + i - 1]$, $k \in [3, 5]$ and $(x_i, \ldots, x_j)$ is a $k$-section, then $\sum_{l=i}^{j} t(x_l, x) \leq \frac{j-i+1}{2} u_1(k)$.

**Proof.** First note that if $(k \leq) \deg_G(x_i) \leq 5$, then $\deg_G(f_i) \geq 4$. Indeed, if $\deg_G(f_i) = 3$, then, since $x_{i-1}x_i$ is not a special edge, from Lemma 4.5 it follows that $\deg_G(x_{i-1}) \geq 6$; in such a case the sequence $(x_{i-1}, \ldots, x_j)$ contradicts the maximality of the sequence $(x_i, \ldots, x_j)$.

Similarly, $\deg_G(x_j) \leq 5$ implies $\deg_G(f_{j+1}) \geq 4$.

We have $u_1(k) > 0$. So, if $t(x_l, x) = 0$ for each $l \in [i, j]$, the desired inequality is trivial.

Suppose that $j = i$. If $\deg_G(x_i) \geq 6$, then $t(x_i, x) = 0$. If $\deg_G(x_i) \leq 5$, then (as we have seen above) $\deg_G(f_i) \geq 4$, $\deg_G(f_{i+1}) \geq 4$ and (since $x_i$ is not incident with a special edge) $t(x_i, x) = \max(0, s(x_i, x)) \leq \max(0, \frac{1}{2} - \frac{1}{2} + \frac{1}{8} + \frac{1}{8}) = \max(0, \frac{4-k}{4k}) \leq 1 - \frac{3}{2} \cdot \frac{5}{6k} = \frac{1}{2} u_1(k)$.

If $j = i + 1$, then $x_{i+1}x_{i+1}$ is not a special edge, and so, by Lemma 4.5, $\max(\deg_G(x_i), \deg_G(x_{i+1})) \geq 6$. Thus at least one of $t(x_i, x)$ and $t(x_{i+1}, x)$ is 0 and, if $t(x_i, x) > 0$ for $l \in [i, i + 1]$, then $t(x_i, x) \leq \frac{1}{2} - \frac{1}{2} + \frac{1}{6} + \frac{1}{8} = \frac{24-5k}{24k}$ (so that $k \in [3, 4]$). Therefore, if $\sum_{l=i}^{i+1} t(x_l, x)$ is positive, it is bounded from above by $\frac{24-5k}{24k} < \frac{6-k}{6k} = \frac{i-i-i+1}{u_1(k)}$.

Now assume that $j \geq i + 2$. From the definition of a $k$-section it follows (using Lemma 4.5) that for any $l \in [i + 1, j - 1]$ the assumption $t(x_l, x) > 0$ implies $t(x_{l-1}, x) = 0 = t(x_{l+1}, x)$; in such a case, by Claim 17, $t(x_l, x) \leq u_1(\deg_G(x_l)) \leq u_1(k)$ (recall that $\deg_G(x_l) \geq k$). Consider the index set $L = \{l \in [i + 1, j - 1] : t(x_l, x) > 0\}$.

If both $t(x_l, x)$ and $t(x_j, x)$ are positive, then $|L| \leq \left\lfloor \frac{i-i-2}{2} \right\rfloor \leq \frac{i-i-2}{2}$ and $\sum_{l=i}^{j} t(x_l, x) \leq 2 \cdot \frac{24-5k}{24k} + \frac{i-i-2}{2} u_1(k) < \frac{3}{2} \cdot \frac{5-k}{6k} + \frac{i-i-2}{2} u_1(k) = \frac{i-i+1}{2} u_1(k)$.
If exactly one of \( t(x_i, x) \) and \( t(x_j, x) \) is positive, then \( |L| \leq \left\lfloor \frac{i-j-2}{2} \right\rfloor \leq \frac{i-j-1}{2} \) and \( \sum_{i=1}^{d} t(x_i, x) \leq \frac{2k-5k}{24k} \cdot \frac{i-j-1}{2} \cdot u_1(k) \leq \frac{2k-5k}{6k} \cdot \frac{i-j-1}{2} \cdot u_1(k) = \frac{i-j+1}{2} \cdot u_1(k) \).

If \( t(x_i, x) = 0 = t(x_j, x) \), then \( |L| \leq \left\lfloor \frac{i-j}{2} \right\rfloor \leq \frac{i-j}{2} \) and \( \sum_{i=1}^{d} t(x_i, x) \leq \frac{i-j}{2} \cdot u_1(k) < \frac{i-j+1}{2} \cdot u_1(k) \).

**Claim 21** If \( k \in [3, 5] \) and there exists a \( k \)-section (in the neighbourhood of \( x \)), then \( \sum_{i=1}^{d} t(x_i, x) \leq n_1 \cdot \frac{7}{10} + n_2 \cdot \frac{7}{24} + \sum_{i=3}^{5} (n_i - 2n_{i,1})u_1(l) + \sum_{i=3}^{5} n_iu_2(l) + \frac{1}{2}(d - \sum_{i=1}^{k-1} n_i - 2 \sum_{i=k}^{5} n_{i,1})u_1(k) \).

**Proof.** Without loss of generality we may suppose that if \((x_1, \ldots, x_j)\) is the \( k \)-section with minimum \( i \), then \( i = 1 \). Create from the sequence \( X_1 := (x_1, \ldots, x_d) \) a sequence of subsequences of (consecutive terms of \( X_1 \)) by deleting all \( x_j \)'s such that either \( \deg_G(x_j) \leq k - 1 \) or one of \( x_{i-1}x_i, x_ix_{i+1} \) is a special edge. What arises is a sequence of \( k \)-sections. Let \( I \) be the index set of all \( i \in [1, d] \) such that \( x_i \) is in one of those \( k \)-sections. Then \( |I| = d - \sum_{i=1}^{k-1} n_i - 2 \sum_{i=k}^{5} n_{i,1} \) and, since any two distinct \( k \)-sections have no term in common, from Claim 20 it follows that \( \sum_{i \in I} t(x_i, x) \leq \frac{1}{2}|I|u_1(k) \).

By Claims 16.1 and 17 (with help of Lemma 4.2) neighbours of \( x \) of degree at most two send to \( x \) altogether an amount bounded from above by \( n_1 \cdot \frac{7}{10} + n_2 \cdot \frac{7}{24} \).

If \( l \in [3, k - 1] \), then, by Claim 7, there are \( n_l - 2n_{l,1} \) vertices in the neighbourhood of \( x \) that are of degree \( l \), but do not belong to a special edge. By Claim 17, each such vertex sends to \( x \) at most \( u_1(l) \).

If \( l \in [3, 5] \), then, again by Claim 7, the total amount sent to \( x \) from all its neighbours, that are incident with a special edge, is at most \( n_{l,1}u_2(l) \).

The desired inequality follows by summing all the above mentioned upper bounds. \( \blacksquare \)

**5 The final analysis**

Now we are ready to show that \( c_2(x) \leq 0 \) (recall that \( x \) is a high vertex of degree \( d \)). Note that if \( d \neq 8 \), then \( c_2(x) = c_1(x) \). Further, if \( d = 8 \) and \( c_1(x) \leq 0 \), then, by Rule 2.1, \( c_1(x) \leq c_2(x) \leq 0 \). Therefore, we will be done by proving that either \( c_1(x) \leq 0 \) or \( d = 8, c_1(x) > 0 \) and \( c_2(x) \leq 0 \).

**1** If there is \( k \in [3, 5] \) such that \( n_{k,k} \geq 1 \), then \( n_1 = 0 \) (Claim 8).

**11** If \( n_{3,3} \geq 1 \), then \( n_{3,3} = 1 \) and \( n_{4,4} = 0 \) (Claim 9), \( n_2 = 0 \) (Claim 8), \( d \geq 11 \) (Lemma 4.8, \( h(3) = 10 \)) and \( n_d \geq 5 \) (Claim 8).

**111** If \( n_{3,3} \geq 3 \) and \( n_{4,4} \geq 4 \), then \( d \geq 13 \) and \( n_{5,5} \leq \frac{d-1}{3} \) (Claim 11.1). By Claims 17 and 18, an average amount sent to \( x \) from a low vertex not incident with a special \((3,3)\)-edge, is at most \( \max(u_1(3), u_1(4), u_1(5), \frac{1}{2}u_2(5)) = \frac{1}{6} \). Therefore, by Claims 19 and 18, \( c_1(x) \leq 1 - \frac{d}{6} + \frac{7}{12} + \frac{1}{2}u_2(5) = \frac{43 - 4d}{36} < 0 \).

**112** If \( n_3 = 2 \), then \( x \) has at most \( d - 7 \) neighbours of degree 4 or 5 and each of them sends to \( x \) in average at most \( \max(u_1(4), u_1(5), \frac{1}{2}u_2(5)) = \frac{1}{12} \).
(1121) If \( d = 11 \), then \( n_d = 9 \) (Claim 8) and \( c_1(x) \leq 1 - \frac{11}{6} + \frac{7}{12} < 0 \).

(1122) If \( d \geq 12 \), then \( c_1(x) \leq 1 - \frac{d}{6} + \frac{7}{12} + (d - 7) \cdot \frac{1}{12} = \frac{12 - d}{12} \leq 0 \).

(113) If \( n_3 = 3 \) and \( n_4 = 0 \), then Claim 8 yields \( d \geq 12 \) (\( d = 11 \) would mean \( n_{11} \geq 9 \), a contradiction), each of (at most \( d - 8 \)) 5-neighbours of \( x \) sends to \( x \) in average at most \( \max(u_1(5), \frac{1}{2} u_2(5)) = \frac{1}{2} \), and so \( c_1(x) \leq 1 - \frac{d}{6} + \frac{7}{12} + \frac{1}{6} + (d - 8) \cdot \frac{1}{20} = \frac{81 - 7d}{60} < 0 \).

(12) If \( n_{4,4} \geq 1 \), then \( n_{3,3} = 0 \) (Claim 9), \( n_2 \leq 1 \) (Claim 8) and \( d \geq 11 \) (Lemma 4.8, \( h(4) = 10 \)).

(121) If either \( n_{4,4} = 1 \), \( n_{3,3} \geq 2 \) and \( n_4 \geq 5 \) or \( n_{4,4} \geq 2 \) and \( n_3 \geq 3 \), then \( d \geq 14 \) and \( n_5 \leq \frac{d + 1}{3} \) (Claim 11.2). A neighbour of \( x \) of degree 3, 4 or 5 sends to \( x \) in average at most \( \max(u_1(3), u_1(4), u_1(5), \frac{1}{2} u_2(4), \frac{1}{2} u_2(5)) = \frac{1}{2} \). Thus, by Claims 19.4 and 17, \( c_1(x) \leq 1 - \frac{d}{6} + \frac{7}{12} + n_2 \cdot \frac{7}{24} + (d - n_3 - n_5 - 2) \cdot \frac{1}{6} = \frac{76 + 3n_2 - 8d}{72} < 0 \).

(122) Suppose that the assumption of (121) is not true.

(1221) If \( n_{4,4} = 1 \), then \( n_{3,3} \leq 2 \).

(12211) If \( d = 11 \) or \( n_2 = 1 \), then, by Claim 8, \( n_d \geq 5 \). A 5-neighbour of \( x \) sends to \( x \) in average at most \( \max(u_1(5), \frac{1}{2} u_2(5)) = \frac{1}{2} \), and so \( c_1(x) < 1 - \frac{d}{6} - \frac{n_2}{12} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + (n_4 - 2) \cdot \frac{1}{12} + \frac{1}{4} + (d - n_3 - n_4 - 5) \cdot \frac{1}{20} = \frac{1}{120}(100 + 19n_2 + 14n_3 + 4n_4 - 14d) \leq \frac{1}{120}(141 - 14d) < 0 \).

(12212) If \( d \geq 12 \) and \( n_2 = 0 \), then \( n_d \geq 4 \). Similarly as above, \( c_1(x) < 1 - \frac{d}{6} + n_3 \cdot \frac{1}{6} + (n_4 - 2) \cdot \frac{1}{12} + \frac{1}{4} + (d - n_3 - n_4 - 2) \cdot \frac{1}{20} = \frac{1}{60}(59 + 7n_3 + 2n_4 - 7d) < \frac{1}{60}(77 - 7d) < 0 \).

(1222) If \( n_{4,4} = 2 \), then \( n_{3,3} = 0 \).

(12221) If \( d = 11 \), then \( n_{11} \geq 5 \). A 4-neighbour of \( x \) not incident with \( x \) sends to \( x \) in average at most \( \max(u_1(4), u_1(5), \frac{1}{2} u_2(5)) = \frac{1}{2} \). Therefore, \( c_1(x) \leq 1 - \frac{11}{6} + 2 \cdot \frac{1}{4} + \frac{7}{12} = 1 - \frac{d}{6} \).

(13) If \( n_{3,5} \geq 1 \) and \( n_{3,3} = n_{4,4} = 0 \), then \( n_2 \leq 2 \) (Claim 8) and \( d \geq 10 \) (Lemma 4.8, \( h(5) = 9 \)). We have also \( n_2 \leq d - 10 \), for otherwise Claim 8 yields \( n_d \geq \left( \frac{7^5 + 2d - 9 + 12 - d}{2} \right) - 1 = 9 \) and \( d \geq n_2 + n_5 + n_d \geq d - 9 + 2 + 9 = d + 2 \), a contradiction.

(131) If \( n_{3,3} \geq 3 \) and \( n_{4,4} \geq 4 \), then \( d \geq 15 \) and \( n_{5,5} \leq \frac{d + 3}{3} \) (Claim 11.3). In such a case, similarly as in (121) (using Claim 19.1), we have \( c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{24} + n_2 \cdot \frac{7}{24} + \frac{12 + 3n_2 - 4d}{36} \leq \frac{48 - 4d}{36} < 0 \).

(132) If \( n_{3,3} \leq 2 \), then \( c_1(x) \leq 1 - \frac{d}{6} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + (d - n_2 - n_3) \cdot \frac{1}{24} = \frac{24 + 6n_2 + 2n_3 - 3d}{24} \leq \frac{30 + 3n_2 - 3d}{24} \leq \frac{30 + 3(d - 10) - 3d}{24} = 0 \).

(133) If \( n_{3,3} = n_{4,4} = 3 \), then, as above, \( c_1(x) \leq \frac{33 + 3n_2 - 3d}{24} \).

(1331) If \( n_2 \leq d - 11 \), then \( c_1(x) \leq \frac{33 + 3(d - 11) - 3d}{24} = 0 \).

(1332) If \( (2 \geq n_2 = d - 10) \), then \( d \leq 12 \) and \( n_d \geq 5 \) (Claim 8). A 5-neighbour sends to \( x \) in average at most \( \max(u_1(5), \frac{1}{2} u_2(5)) = \frac{1}{2} \), hence \( n_5 \leq d - (n_2 + n_3 + n_4) \leq d - 8 \) and \( c_1(x) \leq 1 - \frac{d}{6} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + (d - 8) \cdot \frac{1}{20} = \frac{72 + 35n_2 + 20n_3 - 14d}{120} = \frac{132 + 15n_2 - 14d}{120} = \frac{d - 18}{120} < 0 \).
(2) \(n_{k,k} = 0, \ k = 3, 4, 5\)

(21) If \(n_i \geq i\) for all \(i \in [1, 4]\), Claim 12.1 yields \(d \geq 19\) and \(n_{5,-} \leq \frac{-5+\sqrt{5d+17}}{2}\), and so it is easy to see that \(n_{5,-} \leq \frac{4d}{19}\). Therefore, by Claims 19.2 and 16.1, \(c_1(x) \leq 1 - \frac{d}{6} - \frac{2n_1+2n_2}{66} + n_1 \cdot \frac{2n_1+5}{6} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{12} + n_4 \cdot \frac{1}{12} + n_5 \cdot \frac{1}{30} = 1 + \frac{4n_1}{60n_1+9} + \frac{20n_1+15n_1+10n_1+6n_1+4n_1+20d}{120} < \frac{5}{3} + \frac{35n_{5,-} - 20d}{120} \leq \frac{5}{3} + \frac{7}{24} \cdot \frac{4d}{19} - \frac{d}{6} = \frac{5}{3} - \frac{2d}{19} < 0.\)

(22) Suppose the assumption of (21) is not fulfilled.

(221) \(n_1 = 0\)

(2211) If \(n_i \geq 1\) for all \(i \in [2, 4]\), then \(d \geq 19\) and \(n_{5,-} \leq \frac{-5+\sqrt{5d+17}}{2}\) (Claim 12.1). Thus \(c_1(x) \leq 1 - \frac{d}{6} - \frac{2n_1+2n_1}{24} + \sum_{k=2}^5 k u_1(k) = 1 + \frac{1}{60}(15n_2 + 10n_3 + 5n_4 + 2n_5 - 10d) \leq 1 + \frac{1}{12}(3n_{5,-} - 2d) \leq 1 + \frac{1}{12}(3 \cdot \frac{4d}{19} - 2d) = 1 - \frac{13d}{144} < 0.\)

(2212) Suppose the assumption of (2211) is not fulfilled.

(22121) \(n_2 = 0\)

(221211) If \(n_{5,-} \leq 1\), then \(c_1(x) \leq 1 - \frac{d}{6} + n_3 \cdot \frac{1}{6} + n_4 \cdot \frac{1}{12} + n_5 \cdot \frac{1}{30} = \frac{60-10d+10n_1+5n_1+2n_1}{60} \leq \frac{60-10d+10n_{5,-}}{60} \leq \frac{70-10d}{60} \leq 0.\)

(221212) If \(n_{5,-} \geq 2\), then \(d \geq 8\) (Lemma 4.9, \(\min(g(i,j) : i,j \in [3,5]) = 7\)).

(2212121) If \(d = 8\), then \(n_d = 6\) and \(n_3 = 0 = n_4\) (Claim 10), because \(n_d \geq 1\) implies \(n_d \geq 10\), a contradiction. Then \(c_1(x) \leq 1 - \frac{8}{6} + 2 \cdot \frac{1}{12} < 0.\)

(2212122) If \(d = 9\), then \(n_3 = 0\), since otherwise, by Claim 10, \(n_d \geq 9\), which is impossible.

(22121221) If \(n_4 \geq 1\), then \(n_d \geq 6\) (Claim 10), hence \(c_1(x) \leq 1 - \frac{9}{6} + 3 \cdot \frac{1}{12} < 0.\)

(22121222) If \(n_4 = 0\), then \(n_d \geq 3\) and \(c_1(x) \leq 1 - \frac{9}{6} + 6 \cdot \frac{1}{30} < 0.\)

(22121223) If \(d = 10\), then \(c_0(x) \leq 1 - \frac{10}{6} = -\frac{2}{3}.\)

(22121231) If \(n_3 \geq 1\) and \(n_{4,-} \geq 2\), then \(n_d \geq 6\) (Claim 10) and \(c_1(x) \leq -\frac{2}{3} + 4 \cdot \frac{1}{6} = 0.\)

(22121232) If \(n_3 = 1\) and \(n_4 = 0\), then \(n_d \geq 5\) and \(c_1(x) \leq -\frac{2}{3} + \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{30} < 0.\)

(22121233) If \(n_3 = 0\) and \(n_4 \geq 1\), then \(n_d \geq 3\) and \(c_1(x) \leq -\frac{2}{3} + 7 \cdot \frac{1}{12} < 0.\)

(22121234) If \(n_3 = n_4 = 0\), then \(c_1(x) \leq -\frac{2}{3} + 10 \cdot \frac{1}{30} < 0.\)

(22121224) \(d = 11\)

(22121241) If \(x\) is incident with at least one face of degree at least 4, then, by Claim 21 (with \(k = 3\)), \(c_1(x) \leq 1 - \frac{11}{2} + \frac{10}{3} + \frac{1}{4} + \frac{11}{2} \cdot \frac{1}{6} = 0.\)

(22121242) If \(x\) is incident with 3-faces only, then \(n_{5,-} \leq 5\) (Claim 4.5) and \(c_1(x) \leq 1 - \frac{11}{6} + 5 \cdot \frac{1}{6} = 0.\)

(2212125) If \(d \geq 12\), then \(c_1(x) \leq 1 - \frac{d}{6} + \frac{d}{2} \cdot \frac{1}{6} = \frac{12-d}{12} \leq 0.\)

(22122) If \(n_2 = 1\), then \(d \geq 8\) (Lemma 4.5).

(221221) If \(n_{5,-} \geq 2\), then \(d \geq 10\), because \(\min(g(2,i) : i \in [3,5]) = 9.\)

(2212211) If \(n_3 \geq 1\), then \(n_d \geq 3\) (Claim 10) and \(g(2,3) = 10\) implies \(d \geq 11.\)

(22122111) If \(n_3 \geq 3\) and \(n_{4,-} \geq 4\), then \(d \geq 12\), since \(d = 11\) implies either \(d \geq (n_{5,-} \cdot 1 + \Delta + \Delta) + n_{5,-} \geq (n_{5,-}^2 + 2) + n_{5,-} \geq 19\) (Claim 13.1) or \(d \geq 3n_{5,-} + 1 \geq 13\) (Claim 13.2), in both cases a contradiction (note that the assumptions either of Claim 13.1 or that of Claim 13.2 are satisfied). Moreover, we have either \(d \geq 2n_{5,-} + 1\) (Claim 13.1) or \(d \geq 3n_{5,-} + 1\) (Claim 13.2) and, consequently,
\[ n_{5-} \leq d - 7, \text{ because } n_{5-} \geq d - 6 \text{ would lead to } d \geq 2(d - 6) + 1 \text{ and } d \leq 11, \text{ a contradiction. Therefore, by Claim 19.4, } c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{12} + \frac{7}{24} + (d - 8) \cdot \frac{1}{6} = -\frac{1}{24}. \]

\[
(22122112) \text{ If } n_{4-} \leq 4, \text{ then, by Claim 21 (with } k = 5), c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{12} + \frac{7}{24} + n_3 \cdot \frac{1}{6} + n_4 \cdot \frac{1}{12} + \frac{1}{2}(d - 1 - n_3 - n_4) \cdot \frac{1}{30} = \frac{1}{120}(143 + 18n_3 + 8n_4 - 18d) \leq \frac{1}{120}(197 - 18d) < 0.
\]

\[
(2212212) \text{ If } n_3 = 0, \text{ then, again by Claim 20, } c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{12} + \frac{7}{24} + \frac{d - 1}{2} \cdot \frac{1}{12} = \frac{28 - 3d}{24} < 0.
\]

\[
(221222) \text{ If } n_{5-} = 1, \text{ then } c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{12} + \frac{7}{24} = \frac{29 - 4d}{24} < 0.
\]

\[
(221223) \text{ If } n_2 = 2 \text{ and } n_3 = 0, \text{ then } d \geq 11 \text{ (}g(2, 2) = 10). \]

\[
(2212231) \text{ If } d = 11, \text{ then } n_{11} \geq 6 \text{ (Claim 10), } n_{5-} \leq 5 \text{ and } c_1(x) \leq 1 - \frac{d}{6} - \frac{2}{24} + 2 \cdot \frac{7}{24} + n_4 \cdot \frac{1}{12} + n_4 \cdot \frac{1}{6} = \frac{1}{60}(5n_4 + 2n_5 - 20) < 0.
\]

\[
(221232) \text{ If } d \geq 12, \text{ then, by Claim 21 (with } k = 4), c_1(x) \leq 1 - \frac{d}{6} - \frac{2}{24} + 2 \cdot \frac{7}{24} + \frac{d - 2}{2} \cdot \frac{1}{12} = \frac{3d - 3d}{24} < 0.
\]

\[
(22124) \text{ If } n_2 \geq 2, \text{ } n_{5-} = 3 \text{ and } n_4 = 0, \text{ then } d \geq 11.
\]

\[
(221241) \text{ If } d = 11, \text{ then } n_{11} \geq 6, n_5 \leq 2 \text{ and } c_1(x) \leq 1 - \frac{11}{6} - \frac{n_2}{24} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{50} \leq \frac{1}{60}(15n_2 + 10n_3 - 46) \leq -\frac{1}{60}.
\]

\[
(221242) \text{ If } d \geq 12, \text{ then, by Claim 21 (with } k = 5), c_1(x) \leq 1 - \frac{d}{6} - \frac{n_2}{24} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + \frac{d - 3}{2} \cdot \frac{1}{30} \leq \frac{1}{60}(57 + 15n_2 + 10n_3 - 9d) \leq -\frac{1}{10}.
\]

\[
(222) \text{ If } n_1 = 1 \text{ and } n_2 = 0, \text{ then } d \geq 8 \text{ (Lemma 4.1) and } c_0(x) \leq 1 - \frac{d}{6} - \frac{4}{15} = \frac{11}{15} - \frac{4}{d}.
\]

\[
(2221) \text{ If } n_3 \geq 1, \text{ then } d \geq 11 \text{ (}g(1, 3) = 10). \]

\[
(22211) \text{ If } n_{5-} \geq 3, \text{ then } d \geq 12, \text{ since otherwise (}d = 11\text{) Claim 10 yields } n_d \geq 9, \text{ a contradiction.}
\]

\[
(22111) \text{ If } n_{3-} \geq 3 \text{ and } n_{4-} \geq 4, \text{ then } n_{5-} \leq \frac{d - 1}{3} \text{ (from } n_{2-} = n_1 \text{ it follows that the assumptions of Claim 13.2 are fulfilled) and, by Claim 16.1,}
\]

\[
c_1(x) \leq \frac{11}{15} - \frac{d}{6} + \frac{7}{10} + \frac{d - 3}{2} \cdot \frac{1}{2} \cdot \frac{1}{30} = \frac{109 - 10d}{180} < 0.
\]

\[
(221212) \text{ If } n_{3-} = 3 \text{ and } n_4 = 0, \text{ then } c_1(x) \leq \frac{11}{15} - \frac{d}{6} + \frac{7}{10} + 2 \cdot \frac{1}{6} + (d - 8) \cdot \frac{1}{30} = \frac{45 - 4d}{30} < 0.
\]

\[
(221213) n_{3-} = 2
\]

\[
(2212131) \text{ If } d = 12, \text{ let } f \text{ be the face incident with the (unique) 1-neighbour } x_1 \text{ of } x \text{ and let } l := \deg_G(f) \text{ (so that } l \geq 5). \]

\[
(2211311) \text{ If } l \geq 6, \text{ then } t(x_1, x) \leq \frac{1}{2} - \frac{d}{12} + \frac{7}{30} + \frac{1}{2} = \frac{1}{2} + \frac{7}{3} \leq \frac{2}{3} \text{ and, by Claim 21 (with } k = 4), c_1(x) \leq \frac{11}{15} - \frac{12}{10} + \frac{2}{3} + \frac{1}{6} + \frac{12 - 2}{2} \cdot \frac{1}{12} = -\frac{1}{60}.
\]

\[
(2211312) \text{ If } l = 5, \text{ then, since } G \text{ is a standard embedding of } G, \text{ all faces incident with } x \text{ besides } f \text{ are of degree 3. Therefore, by Lemma 4.5, } n_{5-} \leq 6 \text{ and}
\]

\[
c_1(x) \leq \frac{11}{15} - \frac{12}{6} + \frac{2}{3} + \frac{1}{6} + 4 \cdot \frac{1}{12} = -\frac{1}{15}.
\]

\[
(221132) \text{ If } d \geq 13, \text{ then } c_1(x) \leq \frac{11}{15} - \frac{d}{6} + \frac{7}{10} + \frac{1}{6} + \frac{d - 2}{2} \cdot \frac{1}{12} = \frac{182 - 15d}{120} \leq -\frac{13}{120}.
\]

\[
(2222) \text{ If } n_3 = 0 \text{ and } n_4 \geq 1, \text{ then } d \geq 11.
\]

\[
(22221) \text{ If } d = 11, \text{ then } n_d \geq 7 \text{ (Claim 10) and } c_1(x) \leq \frac{11}{15} - \frac{11}{6} + \frac{7}{10} + 3 \cdot \frac{1}{12} = -\frac{3}{20}.
\]

\[
(22222) \text{ If } d \geq 12, \text{ then } c_1(x) \leq \frac{11}{15} - \frac{d}{6} + \frac{7}{10} + \frac{d - 1}{2} \cdot \frac{1}{12} = \frac{167 - 15d}{120} < 0.
\]
\[(2223)\] If \(n_3 = n_4 = 0\) and \(n_5 \geq 1\), then \(d \geq 10\) and \(c_1(x) \leq \frac{11}{10} - \frac{d}{6} + \frac{7}{10} + \frac{d-1}{6} \cdot \frac{1}{10} = \frac{85-9d}{60} < 0.\)

\[(2224)\] \(n_{5-} = 1\)

\[(22241)\] If \(d = 8\), then \(n_8 = 7\) (Lemma 2.2) and \(c_1(x) \leq \frac{11}{10} - \frac{8}{6} + \frac{7}{10} = \frac{1}{10}.\)

\[(222411)\] If \(c_1(x) \leq 0\), there is nothing to prove.

\[(222412)\] If \(c_1(x) > 0\), then, by Rule 2.1, each vertex \(y \in N_8\) with \(c_1(y) < 0\) sends a (negative) amount to \(x\). Consider an arbitrary vertex \(z \in N_8\). By Lemma 4.13 we know that \(N_1(z) = 0\). If \(n_{5-}(z) \geq 2\), then, because of Claim 10, we have necessarily \(n_{5-}(z) = n_5(z) = 2\), \(n_4(z) = 6\) and \(c_1(z) \leq 1 - \frac{8}{6} + 2 \cdot \frac{1}{10} = -\frac{4}{15}\). (Note that because of Lemma 4.8 the two 5-vertices of \(N_5(z)\) are not joined by a special edge.) If \(n_{5-}(z) = 1\), then, by Claim 19.4, \(c_1(z) \leq 1 - \frac{8}{6} - \frac{n_4(z)}{12} + n_2(z) \cdot \frac{7}{24} + n_3(z) \cdot \frac{1}{6} + n_4(z) \cdot \frac{1}{12} + n_5(z) \cdot \frac{1}{30} = \frac{-40+25n_2(z)+20n_3(z)+10n_4(z)+4n_5(z)}{120} \leq -\frac{1}{8}\).

If \(n_{5-}(z) = 0\), then \(c_1(z) \leq -\frac{1}{3}\). Thus, the vertex \(z\) sends to \(x\) the amount bounded from above by \((\frac{-4}{15} \cdot \frac{1}{6} - \frac{8}{6} \cdot \frac{1}{7} - \frac{1}{3} \cdot \frac{1}{5}) = -\frac{75}{180}\) and we have \(c_2(x) \leq \frac{1}{10} + 7 \cdot (\frac{1}{5}) < 0.\)

\[(22242)\] If \(d \geq 9\), then \(c_1(x) \leq \frac{11}{15} - \frac{d}{6} + \frac{7}{10} = \frac{43-5d}{30} < 0.\)

\[(223)\] If \(n_1 \geq 1\), \(n_2 \geq 2\) and \(n_3 = 0\), then \(n_d \geq 6\) and \(d \geq 12\).

\[(2231)\] If \(n_1 = n_2 = 1\), then, by Claim 19.3, \(c_1(x) \leq \frac{2}{3} - \frac{d}{6} + \frac{7}{10} + \frac{7}{24} + \frac{1}{12} - \frac{8}{12} \cdot \frac{1}{12} = \frac{119-10d}{120} \leq -\frac{1}{120}.\)

\[(2232)\] \(n_1 = 2\)

\[(22321)\] If \(n_4 = 0\), then, by Claims 19 and 16, \(c_1(x) \leq 1 - \frac{d}{6} - \frac{4}{7} + 2 \cdot \frac{9}{14} + (d - 8) \cdot \frac{1}{30} = \frac{152-14d}{105} \leq -\frac{16}{105}\).

\[(22322)\] If \(n_4 \geq 1\), then \(n_d \geq 9\) (Claim 14) and \(c_1(x) \leq 1 - \frac{d}{6} - \frac{4}{7} + 2 \cdot \frac{9}{14} + (d - 11) \cdot \frac{1}{12} = \frac{67-7d}{84} < 0.\)

\[(224)\] If \(n_1 \geq 1\), \(n_2 \geq 2\), \(n_3 = 3\) and \(n_4 = 0\), then \(n_d \geq 9\) (Claim 14) and \(d \geq 12\).

\[(2241)\] If \(n_1 = 1\), then, by Claim 19.3, \(c_1(x) \leq \frac{2}{3} - \frac{d}{6} + \frac{7}{10} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + (d - 12) \cdot \frac{1}{30} = \frac{116+35n_2+20n_3+16d}{120} \leq \frac{186-16d}{120} < 0.\)

\[(2242)\] If \(n_1 = 2\), then \(c_1(x) \leq 1 - \frac{d}{6} - \frac{4}{7} + 2 \cdot \frac{9}{14} + n_2 \cdot \frac{7}{24} + n_3 \cdot \frac{1}{6} + (d - 12) \cdot \frac{1}{30} = \frac{1104+245n_2+140n_3-112d}{840} \leq \frac{1314-112d}{840} \leq -\frac{25}{58} \cdot \frac{1}{30} = \frac{139-12d}{90} < 0.\)

\[(2243)\] If \(n_1 = 3\), then \(c_1(x) \leq 1 - \frac{d}{6} - \frac{8}{9} + 3 \cdot \frac{11}{18} + (d - 12) \cdot \frac{1}{30} = \frac{139-12d}{90} < 0.\)

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