# LIGHT SUBGRAPHS OF GRAPHS EMBEDDED IN THE PLANE AND IN THE PROJECTIVE PLANE – A SURVEY –

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Abstract. It is well known that every planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig states that every 3-connected planar graph contains an edge whose endvertices have degree-sum at most 13. Recently, Fabrici and Jendroľ proved that every 3-connected planar graph $G$ that contains a $k$-path, a path on $k$ vertices, contains also a $k$-path $P$ such that every vertex of $P$ has degree at most $5k$. A beautiful result by Enomoto and Ota says that every 3-connected planar graph $G$ of order at least $k$ contains a connected subgraph $H$ of order $k$ such that the degree sum of vertices of $H$ in $G$ is at most $8k-1$. Motivated by these results, a concept of light graphs has been introduced. A graph $H$ is said to be *light* in a class $\mathcal{G}$ of graphs if at least one member of $\mathcal{G}$ contains a copy of $H$ and there is an integer $w(H,\mathcal{G})$ such that each member $G$ of $\mathcal{G}$ with a copy of $H$ also has a copy of $H$ with degree sum $\sum\limits_{v\in V(H)} deg_G(v) \le w(H,\mathcal{G})$.

In this paper we present a survey of results on light graphs in different families of plane and projective plane graphs and multigraphs. A similar survey dealing with the family of all graphs embedded in surfaces other than the plane and the projective plane has been prepared as well.

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## 1. INTRODUCTION

The study of the structure of plane graphs (i.e. planar graphs embedded in the plane without edge crossings) has its origin in the time of L. Euler. It is connected with the lovely result (*Euler's polyhedron formula* discovered in 1750) stating that in a convex polyhedron with $n$ vertices, $e$ edges and $f$ faces, $n - e + f = 2$. It was apparently discovered by Euler and first proved by Legendre (see, e.g. [BLW], [Mk]). The graph theory version of the formula is expressed in

**Theorem 1.1 (Euler's Polyhedron Formula).** *In a plane connected graph with $n$ vertices, $e$ edges and $f$ faces,*

$$n - e + f = 2. \tag{1.1}$$

□

Euler's formula was not systematically exploited to any extent until the late nineteenth century. Only then a renaissance of interest in metric and combinatorial properties of solids started. A renewed interest in geometry and combinatorics of convex solids culminated in the landmark book [SR] by Steinitz and Rademacher published in 1934. This book presented an extraordinary result, known as *Steinitz' theorem.* Unfortunately, the theorem was couched in such an archaic language that it was not appreciated for many years. Only the reformulation of the theorem by Grünbaum in 1963 released a torrent of results. This resulted in a cross-fertilization of geometry with both graph theory and combinatorics, with benefits to all three areas. To be able to formulate this *"Fundamental Theorem on Polyhedral Graph Theory"* in modern terminology we need two definitions. The *graph of a polyhedron $P$* is the graph consisting of vertices and edges of $P$. A graph $G$ is *polyhedral* if it is isomorphic to the graph of some convex polyhedron.

**Theorem 1.2 (Steinitz' Theorem).** *A graph is polyhedral if and only if it is planar and 3-connected.* □

This result is deeper than it might at first appear: proofs and excellent discussions may be found in ([Mk], [G1], [G3], [Zi]). What makes the theorem so remarkable is its implication that a general class of 3-dimensional structures is equivalent to a certain class of 2-dimensional ones - that is, studying convex polyhedra combinatoricaly does not require thinking of them in 3-dimensional space. It is sufficient to investigate their graphs. Hence any knowledge in 3-connected planar graphs indicates properties of convex polyhedra.

Probably the most important impulse to study the structure of plane graphs came from one of the most celebrated combinatorial problems - *The Four Colour Conjecture* (4CC), posed in 1852. This conjecture was proved in 1976 by Appel and Haken [AH] and the result is well known as

**Theorem 1.3 (The Four Colour Theorem).** *Vertices of every planar graph can be coloured with four colours in such a way that adjacent vertices are coloured with different colours.* ☐

When trying to solve 4CC, Birkhoff in 1912 reviewed several ideas due to earlier writers, and welded them into a systematic method of investigation. The line of enquiry which he suggested lead to the solution of the problem in 1976 (see Chapter 9 in [BLW]). This line is a common method for proofs of many theorems concerning properties of plane graph, see e.g. [AH], [Bo2], [Bo4], [Bo8], [Bo13], [BW2], [EHJ], [HM], [HJ2], [MS], [PT], [WL1].

If there are plane graphs which are counterexamples to a theorem, then there must be among them a graph with the smallest number of vertices; such a graph is said to be *irreducible* (or *minimal counterexample*) with respect to the theorem. The basic idea is to obtain more and more restrictive conditions which an irreducible graph must satisfy, in hope that eventually we shall have enough conditions either to construct the graph explicitly, or, alternatively, to prove that it cannot exist. These restrictive conditions are usually described in language of configurations.

A *configuration* $H$ in a plane graph $G$ is a connected portion of $G$ (i.e. a connected subgraph $H$ of $G$ together with degrees of vertices of $H$ in $G$). We call the set $\mathcal{C}$ of configurations *unavoidable* in a given family $\mathcal{G}$ of plane graphs if at least one member of $\mathcal{C}$ is present in every graph from $\mathcal{G}$. As an example we note that the set of vertices of degree less than six is in this sence unavoidable in the family of all plane graph. We also define a configuration $H$ to be *reducible* if it cannot be contained in any ireducible graph with respect to the conjecture.

Heawood in 1890 observed that if there was an irreducible plane graph with respect to the 4CC then it would belong to the set $\mathcal{A}$ of plane triangulations of minimum degree 5. Hence to prove 4CC it is sufficient to find a finite unavoidable set $\mathcal{U}$ of reducible configurations in $\mathcal{A}$. Appel and Haken [AH] were successful in finding such a set consisting of 1879 configurations (see also [HS], [WW]). In 1989 Appel and Haken [AH] announced that proofs of 4CC with only 1482, 1405, and 1256 configurations are possible. In an independent proof of 4CC Robertson, Sanders, Seymour, and Thomas [RSST] present an unavoidable set of 633 reducible configurations.

Already in 1904 Wernicke showed that every plane triangulation of minimum degree 5 contains either two adjacent vertices of degree 5 or a vertex of degree 5 adjacent to a vertex of degree 6, see [We]. In our modern language this reads as follows: "The set $\mathcal{A}$ of all plane triangulations of minimum degree 5 has an unavoidable set of two configurations, namely, an edge with both end vertices of degree 5, and an edge with one end vertex of degree 5 and the second end vertex of degree 6."

In 1922 Franklin [Fr] extended Wernicke's result proving that each plane graph from the set $\mathcal{A}$ contains a vertex of degree 5 adjacent either to two other vertices of degree 5, or to a vertex of degree 5 and a vertex of degree 6, or to two vertices of degree 6.

Attracted by 4CC and the result of Franklin, H. Lebesgue (the top personality of the *Mathematical Analysis* of the 20th century) realized that it would be very helpful to identify unavoidable sets of configurations for different families of plane graphs. In his 1940 paper [Le] he presented several such lists. His fundamental theorem (see Theorem 2.1 in Section 2 below) provides an unavoidable set of configurations for the family of all 3-connected plane graphs.

However, there is a limit to what can be achieved with Lebesgue's approach, and so stronger methods have been devised over the last three decades in order to solve various long-standing structural and colouring problems on plane graphs (for a survey of a portion of this area, see [Bo13]). As a result, many unavoidable sets of configurations for different families of plane graphs have been discovered. They are scattered in many papers. It would be an honourable achievement to collect and classify them.

We concentrate ourself on unavoidable sets $\mathcal{C}$ of graphs for given families $\mathcal{G}$ of plane graphs that have the following property: Whenever a graph $G$ from $\mathcal{G}$ has a subgraph $H$ from $\mathcal{C}$ then it also contains a copy $K$ of $H$ such that every vertex of $K$ has, in $G$, degree bounded by a constant $\varphi(\mathcal{C}, \mathcal{G})$ that depends only on the set $\mathcal{C}$ and the family $\mathcal{G}$. We call the graphs from the family $\mathcal{C}$ *light* in the family $\mathcal{G}$.

It is a well known fact easily deduced from Euler's Formula that each plane graph contains a vertex of degree at most five. Such a vertex can be interpreted as a path on one vertex. Thus such a path on one vertex is light in the class of all plane graphs. In 1955 A. Kotzig showed that each 3-connected plane graph contains an edge (e.i. a path with two vertices) of degree sum at most 13. We say that such paths with two vertices are light in the class of these graphs. The complete bipartite graphs $K_{2,s}$ for $s \geq 3$ show that the paths with two vertices are not light in the class of all plane graphs. In 1997 Fabrici and Jendrol' proved that for each $k$ the paths with $k$ vertices are light in the family of all 3-connected plane graphs and no other plane graph $H$, different from a path is light in this family of graphs.

In this paper we give a survey of results on light subgraphs in several families of graphs embedded in the plane and the projective plane. Light subgraphs in graphs embedded in surfaces other than the plane or the projective plane are considered in another survey paper [JV10]. However, in Section 9 we briefly mention a recent progress together with the most important results concerning light subgraphs in graphs edmbedded in surfaces.

## 2. NOTATION AND PRELIMINARIES

All graphs considered throughout the paper have no loops or multiple edges. Multigraphs can have multiple edges and loops. An embedding of a planar graph (planar multigraph) into the plane $\mathbb{M}$ is called a *plane graph* (a *plane multigraph*, respectively). If a planar (multi)graph $G$ is embedded in $\mathbb{M}$, then the maximal connected regions of $\mathbb{M} - G$ are called the *faces* of $G$.

The *facial walk* of a face $\alpha$ of a connected plane multigraph $G$ is the shortest closed walk traversing all edges incident with $\alpha$. The *degree (or size)* of a face $\alpha$ is the length of its facial walk. The degree of a face $\alpha$ in $G$ is denoted by $\deg_G(\alpha)$ or $\deg(\alpha)$ if $G$ is known from the context. The degree of the vertex $v$ of a connected plane multigraph $G$

is the number of incidences of edges with $v$, where loops are counted twice. Analogously, the notation $\deg_G(v)$ or $\deg(v)$ is used for the degree of a vertex $v$. Vertices and faces of degree $i$ are called $i$-vertices and $i$-faces (or $i$-gons), respectively. The numbers of $i$-vertices and $i$-faces of a connected plane multigraph $G$ are denoted $n_i(G)$ and $f_i(G)$, respectively, or $n_i$ and $f_i$, if $G$ is known. We use $\delta(G)$ to denote the minimum vertex degree of $G$.

We call an edge $h$ an *$(a, b)$-edge* if the endvertices of $h$ are an $a$-vertex and a $b$-vertex. By $e_{a,b}(G)$ or $e_{a,b}$ we denote the number of $(a, b)$-edges in a plane multigraph $G$.

An $r$-face $\alpha$, $r \geq 3$, is said to be the $(a_1, a_2, \dots, a_r)$-face if vertices $x_1, x_2, \dots, x_r$, in order incident with $\alpha$ have degrees $a_1, a_2, \dots, a_r$. An $(a_1, a_2, a_3)$-face (an $(a_1, a_2, a_3, a_4)$-face, an $(a_1, a_2, a_3, a_4, a_5)$-face) is called an $(a_1, a_2, a_3)$-triangle, (an $(a_1, a_2, a_3, a_4)$-quadrangle, or an $(a_1, a_2, a_3, a_4, a_5)$-pentagon, respectively). Now we are able to state the classical theorem of Lebesgue [Le] already mentioned in Section 1.

**Theorem 2.1 (Lebesgue's Theorem).** *Every 3-connected plane graph contains at least one of the following faces*

(i) *an $(a, b, c)$-triangle for*
$a = 3$ *and* $3 \leq b \leq 6$ *and* $3 \leq c$, *or* $a = 3$ *and* $b = 7$ *and* $7 \leq c \leq 41$, *or*
$a = 3$ *and* $b = 8$ *and* $8 \leq c \leq 23$, *or* $a = 3$ *and* $b = 9$ *and* $9 \leq c \leq 17$, *or*
$a = 3$ *and* $b = 10$ *and* $10 \leq c \leq 14$, *or* $a = 3$ *and* $b = 11$ *and* $11 \leq c \leq 13$, *or*
$a = 4$ *and* $b = 4$ *and* $4 \leq c$, *or* $a = 4$ *and* $b = 5$ *and* $5 \leq c \leq 19$, *or*
$a = 4$ *and* $b = 6$ *and* $6 \leq c \leq 11$, *or* $a = 4$ *and* $b = 7$ *and* $7 \leq c \leq 9$, *or*
$a = 5$ *and* $b = 5$ *and* $5 \leq c \leq 9$, *or* $a = 5$ *and* $b = 6$ *and* $6 \leq c \leq 7$, *or*

(ii) *a $(3, b, c, d)$-quadrangle for*
$b = 3$ *and* $c = 3$ *and* $d \geq 3$, *or* $b = 3$ *and* $c = 4$ *and* $4 \leq d \leq 11$, *or*
$b = 4$ *and* $c = 3$ *and* $4 \leq d \leq 11$, *or* $b = 3$ *and* $c = 5$ *and* $5 \leq d \leq 7$, *or*
$b = 5$ *and* $c = 3$ *and* $5 \leq d \leq 7$, *or* $b = 4$ *and* $c = 4$ *and* $4 \leq d \leq 5$, *or*
$b = 4$ *and* $c = 5$ *and* $d = 4$, *or*

(iii) *a $(3, 3, 3, 3, d)$-pentagon for* $3 \leq d \leq 5$. □

Let $V(H)$ denote the set of vertices of a graph $H$. If $H$ is a subgraph of a graph $G$, then the *weight* $w_G(H)$ of $H$ in $G$ is the sum of degrees in $G$ of vertices of $H$.

$$w_G(H) = \sum_{v \in V(H)} \deg_G(v).$$

Moreover, the weight $w_G(e)$ of an $(a, b)$-edge $e$ is

$$w_G(e) = a + b$$

If $G$ is known from context, then we simply write $w(H) = w_G(H)$ and $w(e) = w_G(e)$.

A path and a cycle on $k$ distinct vertices are described as a $k$-path and a $k$-cycle, respectively. A $k$-path is denoted by $P_k$. The *length* of a path or a cycle is the number of its edges. A $k$-path $P_k$ with vertices $v_1, v_2, ..., v_k$ in order is also called an $(a_1, a_2, ..., a_k)$-path, if $\deg(v_i) = a_i$ for all $i = 1, 2, ..., k$.

Let $K_{1,3}$, be a subgraph of a graph $G$, we call it a *$(d; a, b, c)$-star* if its central vertex has degree $d$ and its three leaves have degrees $a, b$, and $c$ in $G$.

For a connected plane multigraph $G$, let $V, E$, and $F$ be the vertex set, the edge set, and the face set of $G$, respectively. Since

$$\sum_{\alpha\in F}\deg(\alpha)=\sum_{v\in V}\deg(v)=2|E|,$$

from (1.1) we can easily derive

$$\sum_{\alpha\in F}(6-\deg(\alpha))+2\sum_{v\in V}(3-\deg(v))=12 \tag{2.1}$$

$$\sum_{v\in V}(6-\deg(v))+2\sum_{\alpha\in F}(3-\deg(\alpha))=12 \tag{2.2}$$

$$\sum_{v\in V}(4-\deg(v))+\sum_{\alpha\in F}(4-\deg(\alpha))=8 \tag{2.3}$$

Let $\mathcal{P}(\delta,\rho)$ be the family of all 3-connected plane graphs (i.e. polyhedral graphs, see Theorem 1.2) with minimum vertex degree at least $\delta$ and minimum face size at least $\rho$. $\mathcal{P}(\delta,\bar{\rho})$ stands for the family of all graphs from $\mathcal{P}(\delta,\rho)$ in which every face is a $\rho$-face. Analogously the family $\mathcal{P}(\bar{\delta},\rho)$ is introduced. Note that the family $\mathcal{P}(3,\bar{3})$ is the family of all plane *triangulations* and $\mathcal{P}(3,\bar{4})$ is the family of all plane *quadrangulations*. By $\mathcal{P}(\delta,\rho;R)$ we mean a subfamily of $\mathcal{P}(\delta,\rho)$ the members of which fulfil the additional requirements $R$. It is an easy consequence of the equalities (2.1), (2.2) and (2.3) that $\mathcal{P}(\delta,\rho)$ is nonempty only when $(\delta,\rho)\in\{(3,3),(3,4),(4,3),(3,5),(5,3)\}$.

Let $\mathcal{M}(\delta,\rho)$ be the family of all connected plane multigraphs with minimum vertex degree at least $\delta$ and minimum face size at least $\rho$. A plane multigraph $G$ from the family $\mathcal{M}(3,3)$ is called to be a *normal plane map*. If a plane multigraph from $\mathcal{M}(3,3)$ has only 3-faces then it is called a plane *semitriangulation*. Note that semitriangulations can contain loops and multiple edges while triangulations do not have them.

Using (1.1) one can easily obtain

$$|E|\le 3|V|-6 \tag{2.4}$$

for each normal plane map with $|V|\ge 3$.

For two graphs $H$ and $G$ we write $G\cong H$ if the graphs $H$ and $G$ are isomorphic. For a graph $K$ we say that $G$ contains a copy of $K$ if $G$ has a subgraph $H$ such that $H\cong K$.

## 3. Light edges

The theory of light subgraphs has its origin in two beautiful theorems of Kotzig [Ko1]. They state that every 3-connected plane graph contains an edge of weight (i.e. the sum of degrees of its endvertices) at most 13 in general, and at most 11 in the absence of 3-vertices, respectively. These bounds are best possible, as can be seen from the 3-connected plane graphs obtained by placing 20 and 12 small pyramids on the faces of the icosahedron and dodecahedron, respectively, as well as for infinitely many other 3-connected plane graphs.

Kotzig's result was further developed in various directions. We shall discuss some of them in next sections. Here we only mention that in 1972 Erdös conjectured (see [G4]) that Kotzig's theorem is valid for all planar graphs with minimum vertex degree at least 3. This conjecture was proved (but never published) by Barnette (see [G4]) and independently by Borodin [Bo2]. The theorem of Kotzig was published in 1955 in Slovak. Therefore, its original proof [Ko1] is not readily accessible, but Grünbaum [G3] in 1975 gave a sketch of a proof in the English language. Other proofs can be deduced from [Bo2], [Bo3], [Bo5], [Bo10], [Je5]. Here we present a simple proof according to [Je4] that uses the Discharging method, a method used in the proof of the Four Color Theorem (see [AH], [RSST]). (Note that the idea of discharging is due to Heesch [H1]). This method is a common technique for proving results on planar graphs. We prove the theorem in a bit stronger form that implies the truth of Erdös' conjecture.

**Theorem 3.1 (Kotzig's theorem).** *Every normal plane map contains a* $(3,a)$*-edge with* $3 \le a \le 10$*, or a* $(4,b)$*-edge with* $4 \le b \le 7$*, or a* $(5,c)$*-edge with* $5 \le c \le 6$*. The bounds 10, 7 and 6 are best possible.*

**Proof.** Let $G$ be a counterexample on a set $V$ of $n$ vertices that has the maximum number of edges, say $m$, among all counterexamples on $n$ vertices. Let $f$ be the number of faces of $G$. For the purposes of the proof, edges of the desired type are called *light* edges.

By the choice of $G$, it must be a semitriangulation. Suppose $G$ has a $k$-face $\alpha$ with $k \ge 4$. Because light edges are not present in $G$ each edge has an endvertex of degree at least 6, so $\alpha$ is incident with two vertice $x$ and $y$ that are not consecutive on the boundary of $\alpha$ and both have degrees at least 6.

Inserting a diagonal $xy$ into the face $\alpha$ we obtain a graph $G^*$ having the same vertex set $V$ as $G$ but one edge more, a contradiction.

Because $G$ is a semitriangulation, (2.2) may be rewritten

$$\sum_{v\in V}(\deg(v) - 6) = -12. \tag{3.1}$$

Consider an initial charge function $\varphi : V \to \mathbb{Q}$ such that $\varphi(v) = \deg(v) - 6$ for $v \in V$. Therefore (3.1) is equivalent to

$$\sum_{v\in V}\varphi(v) = -12.$$

We use the following rule in order to transform $\varphi$ into a new charge function $\psi : V \to \mathbb{Q}$ by redistributing charges locally so that $\sum\limits_{v\in V}\varphi(v) = \sum\limits_{v\in V}\psi(v)$.

**Rule.** If $e = uv$ is an edge of $G$ with $\deg(u) \ge 7$ and $\deg(v) \le 5$. Then the vertex $u$ sends along $e$ to $v$ the charge $\frac{6-\deg(v)}{\deg(v)}$. Let $\psi(x)$ denote the resulting charge at a vertex $x$. Since charge sent to $v$ is deducted from $u$, we have

$$\sum_{x\in V}\psi(x) = -12.$$

We are going to show that $\psi$ is a nonnegative function, which will trivially be a contradiction. To this end consider several cases.

**Case 1.** Let $v$ be a $k$-vertex for $3 \le k \le 6$. As $G$ does not contain light edges, $v$ receives a charge $\frac{6-k}{k}$ from each of its neighbours. Hence $\psi(v) = \varphi(v) + k\frac{6-k}{k} = k - 6 + 6 - k = 0$.

**Case 2.** Let $u$ be a $k$-vertex for $k \ge 7$. Because $G$ is a semitriangulation and does not contain light edges, at most half of the neighbors of $u$ can have degree at most 5. Hence the vertex $u$ sends a charge to at most $\lfloor \frac{k}{2} \rfloor$ vertices, all of degrees $\le 5$.

**2.1.** $k = 7$. A transfer from $u$ is possible only to 5-vertices, therefore

$$\psi(u) \ge \varphi(u) - \frac{1}{5}\left\lfloor \frac{7}{2} \right\rfloor = 7 - 6 - \frac{3}{5} > 0$$

.

**2.2.** $k \in \{8, 9, 10\}$. The vertex $u$ sends charges only along $(k, 4)$-edges or $(k, 5)$-edges. Since charge $\frac{1}{2}$ would be sent along $(k, 4)$-edges and only charge $\frac{1}{5}$ along $(k, 5)$-edges we have

$$\psi(u) \ge \varphi(u) - \frac{1}{2}\left\lfloor \frac{k}{2} \right\rfloor = k - 6 - \frac{1}{2}\left\lfloor \frac{k}{2} \right\rfloor \ge 0.$$

**2.3.** If $k \ge 11$, then $u$ sends charge $\frac{1}{5}$ along $(k, 5)$-edges, charge $\frac{1}{2}$ along $(k, 4)$-edges and charge 1 along $(k, 3)$-edges. Since at most $\lfloor \frac{k}{2} \rfloor$ neighbours of $u$ have degree at most 5, we obtain

$$\psi(u) \ge \varphi(u) - \left\lfloor \frac{k}{2} \right\rfloor = k - 6 - \left\lfloor \frac{k}{2} \right\rfloor \ge 0.$$

The bounds 10 and 6 are best possible as can be seen from the graphs mentioned in Section 2. An example showing that also 7 is best possible can be found in [Bo4].

This finishes the proof. □

Now we turn our attention to the problem of estimating the number of light edges (i.e. the edges having weight at most 13) in families of plane graphs and multigraphs. Let $e_{i,j}(G) = e_{i,j}$ be the number of edges in a planar multigraph $G$ that join $i$-vertices with $j$-vertices.

Theorem 3.1 of Kotzig states that $\sum_{i+j\le 13} e_{i,j} > 0$ for every 3-connected planar graph. Grünbaum [G4] conjectured that for every 3-connected plane graph the following is true:

$$20e_{3,3} + 15e_{3,4} + 12e_{3,5} + 10e_{3,6} + \frac{20}{3}e_{3,7} + 5e_{3,8} + \frac{10}{3}e_{3,9} + 2e_{3,10}$$

$$+12e_{4,4} + 7e_{4,5} + 5e_{4,6} + 4e_{4,7} + \frac{8}{3}e_{4,8} + \frac{2}{3}e_{4,9}$$

$$+4e_{5,5} + 2e_{5,6} + \frac{1}{3}e_{5,7} + 12e_{6,6} \ge 120.$$

Jucovič [Ju1] proved that each plane triangulation satisfies

$$20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + \frac{20}{3}e_{3,7} + 5e_{3,8} + \frac{5}{2}e_{3,9} + 2e_{3,10}$$

$$+20e_{4,4} + 11e_{4,5} + 5e_{4,6} + 6e_{4,7} + 5e_{4,8} + 3e_{4,9}$$

$$+8e_{5,5} + 2e5,6 + 2e_{5,7} + 2e_{5,8} \geq 120.$$

Later Jucovič [Ju2] proved that this inequality holds for all 3-connected planar graphs.

For the class of normal plane maps, which includes the class of 3-connected plane graphs, Borodin [Bo5] obtained the following result. (Recall that a normal plane map is a plane multigraph in which every vertex degree and every face size is at least 3.)

**Theorem 3.2** [Bo5]. *For each normal plane map it holds that*

$$40e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + \frac{20}{3}e_{3,7} + 5e_{3,8} + \frac{5}{2}e_{3,9} + 2e_{3,10}$$

$$+\frac{50}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + \frac{5}{3}e_{4,7} + \frac{16}{3}e_{5,5} + 2e_{5,6} \geq 120;$$

*Moreover, each coefficient of this inequality is best possible.* □

The sharpness of coefficients in Theorem 2 and in those below is understood in the sence that neither of coeficients may be decreased keeping all the other $\alpha_{ij}$ constant without violating the correspondent relation.

A final answer to the above mentioned Grünbaum's conjecture gives the following result by Fabrici and Jendrol' [FJ1]:

**Theorem 3.3.** [FJ1]. *For every 3-connected plane graph there is*

$$20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + \frac{20}{3}e_{3,7} + 5e_{3,8} + \frac{5}{2}e_{3,9} + 2e_{3,10}$$

$$+\frac{50}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + \frac{5}{3}e_{4,7} + \frac{16}{3}e_{5,5} + 2e_{5,6} \geq 120;$$

*Moreover, each coefficient is best possible.* □

The two inequalities for normal maps and 3-connected planar graphs, respectively, differ only in the coefficient of $e_{3,3}$. As was later proved by Borodin, Theorem 3.3 is valid for all normal plane maps with the exception of exactly one multigraph.

In the subclass of all normal plane maps with minimum degree at least 4 we have $e_{3,j} = 0$ for $3 \leq j \leq 10$. Borodin proved that in the resulting inequality all coefficients are best possible.

**Theorem 3.4** [Bo5]. *For every normal plane map of minimum degree four, it holds that*

$$\frac{50}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + \frac{5}{3}e_{4,7} + \frac{16}{3}e_{5,5} + 2e_{5,6} \geq 120$$

*Moreover, each of these coefficients is best possible.* $\square$

Already in 1904 in his work on the Four Color Problem Wernicke [We] has established the inequality $e_{5,5} + e_{5,6} > 0$ for every graph $G \in \mathcal{P}(5,3)$. Further contributions are due to Grünbaum [G2], [G4], Fisk (see [GS]), Grünbaum and Shephard [GS], and Borodin [Bo6]. Borodin and Sanders [BS] found the best possible light edge inequality for plane graphs of minimum degree five.

**Theorem 3.5** [BS]. *For every normal plane map of minimum vertex degree at least 5 and minimum face size 3 it holds that*

$$\frac{14}{3}e_{5,5} + 2e_{5,6} \geq 120.$$

*Moreover, the coefficients* $\frac{14}{3}$ *and* 2 *are best possible.* $\square$

If in Theorem 3.5 we use $e_{4,j} = 0$ for $4 \leq j \leq 7$ then the resulting inequality differs from that of Theorem 3.4 only in the coefficient of $e_{5,5}$.

We finish this section with few very recent results.

A planar graph which can be embedded in the plane in such a way that every vertex lies on the boundary of the same region is called an *outerplanar* graph. Hackmann and Kemnitz [HK] recently proved

**Theorem 3.6** [HK]. *Every outerplanar graph with minimum degree at least 2 contains an* $(2, b)$*-edge for* $2 \leq b \leq 3$ *or a* $(2, 4, 2)$*-path.* $\square$

**Theorem 3.7** [JM]. *Every planar graph of minimum degree five contains a* $(5; a, b, c)$*-star for* $a = 5$ *and* $5 \leq b \leq 6$ *and* $5 \leq c \leq 7$*, or* $a = b = c = 6$.
*Moreover the bounds 6 and 7 are best possible.* $\square$

The next theorem is a strengthenning of Theorems 3.1 and 3.7.

**Theorem 3.8** [FHJ]. *Every planar graph* $G$ *of minimum degree at least 3 contains*

(i) *a* $(3, b)$*-edge for* $3 \leq b \leq 10$*, or*
(ii) *an* $(a, 4, b)$*-path for*
  $a = 4$ *and* $4 \leq b \leq 10$*, or*
  $a = 5$ *and* $5 \leq b \leq 9$*, or*
  $6 \leq a \leq 7$ *and* $6 \leq b \leq 8$*, or*
(iii) *a* $(5; a, b, c)$*-star for*
  $4 \leq a \leq 5$ *and* $5 \leq b \leq 6$ *and*
  $5 \leq c \leq 7$*, or* $a = b = c = 6$.

*Moreover, for every* $S \in \{(3,10)$*-edge,* $(4,4,9)$*-path,* $(5,4,8)$*-path,* $(6,4,8)$*-path,* $(7,4,7)$*-path,* $(5;5,6,7)$*-star,* $(5;6,6,6)$*-star*$\}$ *there is a 3-connected plane graph* $H$ *containing* $S$ *and no other configuration from the above list.* ☐

Theorem 3.8 has an application in a problem of colouring of vertices of the square of planar graph posed by Wegner [Wg] in 1977 (see also Jensen and Toft [JT], p.51). Recall that the square $G^2$ of a graph $G$ is a graph with the same vertex set $V(G)$. Two vertices $x$ and $y$ are adjacent in $G^2$ if and only if their distance in $G$ is at most 2.

**Corollary 3.8.1** [FHJ]. *Let* $G$ *be a planar graph of maximum degree* $\Delta \geq 11$. *Then for the chromatic number* $\chi(G^2)$ *of the square of* $G$ *there holds:*

$$\chi(G^2) \leq 2\Delta + 19.$$

*Proof.* Suppose there is a counterexample. Let $H$ be one with the minimum number of vertices. Evidently the minimum degree of $H$ is at least 3. By Theorem 3.8 the graph $H$ contains an $(a,b)$-edge $e = xy$ with $\deg_H(x) = a$, $\deg_H(y) = b$ and $\deg_{H^2}(x) \leq 2\Delta + 18$ with values $a = 3$ and $b \leq 10$, or $a = 4$ and $b \leq 7$, or $a = 5$ and $b \leq 6$, and $H^2$ being the square of $H$. If the edge $e$ is contracted into a vertex $z$ and all multiple edges are simplyfied, the resulting graph $H^*$ has the same maximum degree but it is not a counterexample. The square of $H^*$ has a colouring with $2\Delta + 19$ colours. This colourings induces a colouring with $2\Delta + 19$ colours of the graph $H$ in which all vertices except of $x$ and $y$ are coloured. Then the colour of vertex $z$ is assigned to the vertex $y$. As the vertex $x$ has, in $H^2$, at most $2\Delta + 18$ neighbours it can be coloured with one of the available $2\Delta + 19$ colours, a contradiction. ☐

Note that the bound of Corollary 3.8.1 is better than $2\Delta + 25$, the bound obtained recently by van den Heuvel and McGuinnes [HM].

The requirement on minimum degree at least 3 in the above theorems cannot be relaxed as one can see from the graphs $K_{1,r}$ and $K_{2,r}$ for $r \geq 3$.

**Theorem 3.9** [ABG]. *Every connected planar graph of order at least 2 contains either*

(i) *two vertices having degree sum* $\leq 4$, *or*
(ii) *two 3-vertices at distance two, or*
(iii) *an edge* $e$ *of weight at most 11 incident with two 3-faces, or*
(iv) *an edge* $g$ *of weight at most 9 incident with a 3-face, or*
(v) *an edge* $h$ *of weight at most 7.*

*Moreover the bounds 11, 9, and 7 are tight.* ☐

For other results in this direction, see Borodin [Bo3], [Bo5], [Bo6], [Bo13], Borodin and Sanders [BS], Fabrici, Harant, and Jendrol' [FHJ], Jucovič [Ju2], Sanders [Sa], and Zaks [Za1], [Za2]. Note that a result weaker than the theorem of Kotzig was known already to Lebesgue. In 1940 he proved that every $G \in \mathcal{P}(3,3)$ contains an edge $e$ with $w(e) \leq 14$, see [Le] and Theorem 2.1.

## 4. LIGHT SUBGRAPHS OF ORDER THREE

Trying to solve the Four Color Map Problem Franklin [Fr] in 1922 proved that every 3-connected plane graph $G$ of minimum degree at least 5, contains a 3-path $P_3$ with weight 17, the bound being best possible. Only recently, in 1993, Ando, Iwasaki, and Kaneko obtained the analogous result for all 3-connected plane graphs. Namely, they proved

**Theorem 4.1** ([AIK]). *Every $G \in \mathcal{P}(3,3)$ contains a 3-path $P_3$ with weight at most 21. Moreover the bound 21 is sharp.*

The following result has been proved by Jendrol' [Je2] which strengthens Theorem 3.1 of Kotzig [Ko1].

**Theorem 4.2** ([Je2]). *Every $G \in \mathcal{P}(3,3)$ contains an $(a,b,c)$-path where*

(i) $3 \le a \le 10$ *and* $b = 3$ *and* $3 \le c \le 10$, *or*
(ii) $4 \le a \le 7$ *and* $b = 4$ *and* $4 \le c \le 7$, *or*
(iii) $5 \le a \le 6$ *and* $b = 5$ *and* $5 \le c \le 6$, *or*
(iv) $a = 3$ *and* $3 \le b \le 4$ *and* $4 \le c \le 15$, *or*
(v) $a = 3$ *and* $5 \le b \le 6$ *and* $4 \le c \le 11$, *or*
(vi) $a = 3$ *and* $7 \le b \le 8$ *and* $4 \le c \le 5$, *or*
(vii) $a = 3$ *and* $3 \le b \le 10$ *and* $c = 3$, *or*
(viii) $a = 4$ *and* $b = 4$ *and* $4 \le c \le 11$, *or*
(ix) $a = 4$ *and* $b = 5$ *and* $5 \le c \le 7$, *or*
(x) $a = 4$ *and* $6 \le b \le 7$ *and* $4 \le c \le 5$.

*Moreover, in each of the cases (iii), (iv), (vi), and (vii) the upper bounds on parameters $a, b, c$ can be obtained simultaneously. Furthermore, there is a graph in $\mathcal{P}(3,3)$ having only $(4,7,4)$-paths and $(7,4,7)$-paths from the above list.* ☐

The requirement of 3-connectedness in the above theorems is fundamental because of the following theorem. We present a proof of this theorem with a proof technique that is typical in this area.

**Theorem 4.3** [Je6]. *For every connected plane graph $H$ of order at least 3 and every integer $m \ge 3$ there exists a 2-connected plane graph $G$ such that each copy of $H$ in $G$ contains a vertex $A$ with*

$$deg_G(A) \ge m$$

.

**Proof.** Augment $H$ to a triangulation $T$ with vertex set $V(H)$. Let $[uvw]$ be an outer 3-face of $T$. For $m \ge 2$, let $D_m$ be a plane graph obtained from the double $2m$-pyramid with poles $z_1$ and $z_2$ by deleting every second edge of the equatorial cycle $C_{2m}$. If we insert into a 3-face $[z_1xy]$ of $D_m$ the triangulation $T$ so that the vertex $u$ coincides with $r_1$, the vertices $v$ and $w$ with $x$ and $y$, respectively, we obtain the required graph $G$. If $H$ has at least three vertices then each copy of $H$ in $G$ contains at least one of the vertices $z_1$ or $z_2$, which have degree at least $2m$. ☐

The graph $G$ from the proof of Theorem 4.3 contains a 3-face incident with two 3-vertices. Borodin [Bo11] observed that if such faces are excluded the requirement of 3-connectivity can be omitted. More precisely he has proved.

**Theorem 4.4** [Bo11]. *Each normal plane map $G$ having no 3-faces incident with two 3-vertices has the following two properties:*

(i) *$G$ has either a 3-path $P_3$ with $w(P_3) \leq 18$ or a vertex of degree $\leq 15$ adjacent to two 3-vertices*
(ii) *$G$ has either a 3 path $P'_3$ with $w(P'_3) \leq 17$ or an edge $e$ with $w(e) \leq 7$.* □

Already in 1940 Lebesgue [Le] proved that each 3-connected plane graph $G$ of minimum degree 5 contains a 3-cycle $C_3$ bounding a 3-face $\alpha$ with $w(C_3) \leq 19$. Kotzig [Ko2] improved this result to $w(C_3) \leq 18$ and conjectured in [Ko3] that 17 is the best upper bound. Borodin confirms this in

**Theorem 4.5** [Bo1]. *Every $G \in \mathcal{P}(5,3)$ contains a 3-face $\alpha$ with $w_G(\alpha) \leq 17$. This bound is best possible.* □

This theorem has a beautiful corollary. It confirms the conjecture of Grünbaum from 1975, see [G3], that every 5-connected plane graph is cyclically 11-connected. Let us sketch a proof of this statement. First recall the a graph is *cyclically $k$-connected* for some $k \geq 1$ if there is no set $S$ of less than $k$-edges with property that $G - S$ has two components each containing a cycle. Let $G$ be a graph from $\mathcal{P}(5,3)$. Consider a face $\alpha$ of weight at most 17 of $G$. Choose $S$ to be the set of edges incident with vertices of $\alpha$ but not on the boundary of $\alpha$. Clearly $|S| \leq 11$ and the graph $G - S$ has two componts each containing a cycle.

Note that already earlier, in 1972, Plummer [Pl] proved that the cyclic connectivity of these graphs is at most 13.

Many papers have studied the structural properties of different classes of plane triangulations, see e.g. [Ko2], [Ko3], Borodin [Bo6], [Bo9], [Bo12], [BB], Jendrol' [Je5], Sanders [Sa], Borodin and Sanders [BS]. Recenty Jendrol' [Je5] proved the following theorem which includes earlier results by Lebesgue [Le], Kotzig [Ko2], [Ko3] and Borodin [Bo9].

**Theorem 4.6** [Je5]. *Each plane triangulation of order at least five contains an $(a,b,c)$-triangle, where*

(i) *$a = 3$ and $b = 4$ and $4 \leq c \leq 35$, or*
(ii) *$a = 3$ and $b = 5$ and $5 \leq c \leq 21$, or*
(iii) *$a = 3$ and $b = 6$ and $6 \leq c \leq 20$, or*
(iv) *$a = 3$ and $b = 7$ and $7 \leq c \leq 16$, or*
(v) *$a = 3$ and $b = 8$ and $8 \leq c \leq 14$, or*
(vi) *$a = 3$ and $b = 9$ and $9 \leq c \leq 14$, or*
(vii) *$a = 3$ and $b = 10$ and $10 \leq c \leq 13$, or*
(viii) *$a = 4$ and $b = 4$ and $c \geq 4$, or*
(ix) *$a = 4$ and $b = 5$ and $5 \leq c \leq 13$, or*
(x) *$a = 4$ and $b = 6$ and $6 \leq c \leq 17$, or*
(xi) *$a = 4$ and $b = 7$ and $7 \leq c \leq 8$, or*
(xii) *$a = 5$ and $b = 5$ and $5 \leq c \leq 7$, or*
(xiii) *$a = 5$ and $b = 6$ and $c = 6$.*

*Moreover, if $c(k)$ denotes the upper bound on $c$ in the above case $(k)$, then $c(\text{i}) \geq 30, c(\text{ii}) \geq 18, c(\text{iii}) = 20, c(\text{iv}) \geq 7, c(\text{v}) = 14, c(\text{vi}) \geq 11, c(\text{vii}) \geq 12, c(\text{viii}) = \infty, c(\text{ix}) \geq 10, c(\text{x}) \geq 10, c(\text{xi}) \geq 7$ and $c(\text{xii}) = 7$.* □

Note that Theorem 4.6 considers all cases since plane triangulations have no multiple edges and hence every plane triangulation with at least five vertices has no $(3,3)$-edge.

Theorem 4.6 points out that if a plane triangulation has no $(4,4)$-edge, then it contains a light triangle. Borodin [Bo12] has gone further. He proved

**Theorem 4.7** [Bo12]**.** *If in a plane triangulation $T$ there is no path consisting of $k$ vertices of degree 4 for some $k \geq 1$, then*

(i) *$T$ contains a 3-face $\alpha$ with weight*

$$w(\alpha) \leq \max\{37, 10k+17\} \text{ and}$$

(ii) *either there is a $(3,4)$-edge or a 3-face $\beta$ with weight*

$$w(\beta) \leq \max\{29, 5k+8\},$$

*where all bounds are best possible.* □

In a connection with results mentioned in this section, the following problem seems to be interesting.

**Problem 4.1.** *Find the best versions of Theorem* 4.2. *and Theorem* 4.6

In each of the cases (iii), (iv), (vi) and (vii) of Theorem 4.2 the upper bounds on parameters are tight and can be obtained independently from the others. Furthermore, there is a graph $G \in \mathcal{P}(3,3)$ having only (4,7,4)-paths and (7,4,7)-paths from the list of the Theorem.

Similarly we are interested in the best version of Theorem 4.6. As we have mentioned, in the cases (iii), (v), (viii), (xii) and (xiii) the assertions of Theorem 4.6 are best possible. For the other cases we believe the following Conjecture is true:

**Conjecture 4.1** [Je4]**.** *If $c(k)$ denotes the upper bound on $c$ in the case $(k)$ of Theorem 4.6, then $c(\text{i}) = 30, c(\text{ii}) = 18, c(\text{iv}) = 14, c(\text{vi}) = 2, c(\text{vii}) = 12, c(\text{ix}) = 10$ and $c(\text{xi}) = 7$.* □

For a plane graph $G$ from $\mathcal{P}(3,3)$, let $f_{i,j,k}$ be the number of 3-faces that are incident with an $i$-vertex, a $j$-vertex, and a $k$-vertex. As proved by Lebesgue [Le], $f_{5,5,5} + \frac{2}{3}f_{5,5,6} + \frac{3}{7}f_{5,5,7} + \frac{1}{4}f_{5,5,8} + \frac{1}{9}f_{5,5,9} + \frac{1}{3}f_{5,6,6} + \frac{2}{21}f_{5,6,7} \geq 120$ holds for every graph $G \in \mathcal{P}(5,3)$. This result of Lebesgue and Theorem 4.5 are strengthened in the next Theorem proved by Borodin [Bo7] except fact that the optimality of the coefficient 5 of $f_{5,5,7}$. The best possibility of it was proved by Borodin and Sanders in [BS].

**Theorem 4.8.** *If $G$ is a graph from $\mathcal{P}(5,3)$, then*

$$18f_{5,5,5} + 9f_{5,5,6} + 5f_{5,5,7} + 4_{5,6,6} \geq 144.$$

*Moreover, all of these coefficients are best possible.* □

## 5. SUBGRAPHS WITH RESTRICTED DEGREES

In this section we prove two basic results that are typical for this topic and have served as a starting point for a theory of light subgraphs which we shall describe in further sections. Because of Theorem 4.3 we only deal with the family of 3-connected plane graphs. The first result is due to Fabrici and Jendrol'.

**Theorem 5.1** [FJ2]. *If $G$ is a 3-connected plane graph having a $k$-path where $k$ is a positive integer, then $G$ contains a $k$-path $P_k$ such that all vertices of $P_k$ have degree at most $5k$ in $G$. Moreover, the bound $5k$ is tight.*

**Proof.** **(a)** *The upper bound.* Suppose the theorem is not true, and let $G$ be a counterexample on $n$ vertices that has the most edges among all counterexamples on $n$ vertices. Let us call a vertex *major (minor)* if its degree is $> 5k$ ($\leq 5k$, respectively). The crucial point of the proof is in the following

**Claim** $(*)$ *Every major vertex is incident only with triangular faces.*

Otherwise, there is a major vertex $u$ incident with a face $\alpha$ of size at least 4. A diagonal $uv$ can be insert into $\alpha$ that joins $u$ with a vertex $v$ of $\alpha$ not adjacent with $u$. The graph $G^*$ so obtained is again a counterexample, because the edge $AB$ cannot appear on any $k$-path consisting of minor vertices only. But $G^*$ has one edge more than $G$, which contradicts the maximality of $G$. This completes the proof of Claim $(*)$.

Let $M$ be a subgraph of $G$ induced by major vertices of $G$. Since $M$ is also a plane graph, it contains a vertex $x$ such that

$$\deg_M(x) \leq 5. \tag{5.1}$$

On the other hand, because $x$ is major, we have

$$\deg_G(x) \geq 5k+1.$$

Due to Claim $(*)$ there is a subgraph of $G$ on the vertex $x$ and its neighbours that is a wheel with $\deg_G(x)$ spokes. The ends of spokes different from $x$ form a cycle with at least $5k+1$ vertices. Among them there are at most five major vertices, by (5.1). By the pigeonhole principle, there is a path on the wheel consisting of $k$ minor vertices, a contradiction.

**(b)** *The sharpness of the bound.* Now we shall construct a 3-connected plane graph $G$ in which every $k$-path has a vertex $u$ of degree at least $5k$. The construction begins with the dodecahedron. Into each of its 5-faces we insert a new vertex $x$ and join it to all vertices incident with this face. The result is a graph all faces of which are triangles $[xyz]$ with $\deg(x) = 5$ and $\deg(y) = \deg(z) = 6$. Into every triangle of this graph we insert a subdivided 3-star consisting of a central vertex $v$ and three paths $p_x, p_y, p_z$ from $v$ to $x, y, z$, respectively, with $p_x$ of length $\lceil \frac{k+2}{2} \rceil$ and $p_y$ and $p_z$ of length $\lfloor \frac{k-2}{2} \rfloor$. Now make $x$ adjacent to all vertices of $p_x$ and $p_y$, and similarly make $y$ adjacent to all of $p_y$ and $p_z$, and $z$ adjacent to all of $p_z$ and $p_x$. Observe that in the resulting graph $G$ all the vertices $x$ have degree $5k$, the vertices $y$ and $z$ have degrees at least $6k-6$ and every other vertex has degree at most 6. It is easy to see that each $k$-path contains at least one vertex of type $x, y$ or $z$. □

A natural question arises whether every 3-connected graph $G$ having a copy of a connected graph $H$ different from a path must also contain a copy of $H$ such that its vertices have bounded degrees in $G$. The answer is surprisingly negative. Fabrici and Jendrol' proved

**Theorem 5.2** [FJ2]. *If $H$ is a connected plane graph other than a path and $m$ is an integer greater than $|V(H)|$. Then there is a 3-connected planar graph $T$ such that each copy of $H$ in $T$ has a vertex $y$ with*

$$deg_T(y) \geq m.$$

**Proof.** Augment $H$ to a triangulation $T_o$ with vertex set $V(H)$. Into each triangle $[uvw]$ of $T_o$ insert a wheel with a central vertex $z$ and with $m$ spokes $zx_i$ for $1 \leq i \leq m$. Join the vertex $u$ to $x_i$ for $1 \leq i \leq \lfloor \frac{m}{3} \rfloor$, the vertex $v$ to $x_j$ for $\lfloor \frac{m}{3} \rfloor \leq j \leq \lfloor \frac{2m}{3} \rfloor$, and the vertex $w$ to $x_1$ and to $x_t$ for $\lfloor \frac{2m}{3} \rfloor \leq t \leq m$. The result is the desired triangulation $T$. One can easily check that each vertex of $T$ that lies in $T_o$ has degree at least $m$ in $T$, as does the center of each inserted wheel. The vertices of degrees less than $k$ induce $m$-cycles in $T$. Therefore, as $H$ is not a path, each copy of $H$ in $T$ contains at least one vertex of degree $\geq m$. □

Recently Madaras improved Theorem 5.1 by showing the following result.

**Theorem 5.3** [Ma1]. *If $G$ is a 3-connected plane graph containing a vertex of degree at least $k$, where $k$ is a positive integer, then $G$ contains a $k$-path $P_k$ such that $k-1$ vertices of $P_k$ have degree at most $\frac{5}{2}k$, and the remaining vertex has degree at most $5k$ in $G$.* □

## 6. MAXIMUM DEGREE PROBLEMS

The problems mentioned in the previous sections suggest the formulation of more general problems.

**Problem 6.1.** *Let $\mathcal{H}$ be a family of graphs and let $H$ be a connected graph that is a proper subgraph of at least one member of $\mathcal{H}$. Let $\varphi(H, \mathcal{H})$ be the smallest integer $k$ with the property that every graph $G$ in $\mathcal{H}$, that contains $H$ contains a copy of $H$ whose vertices all have degree at most $k$ in $G$. Determine the value of $\varphi(H, \mathcal{H})$ for given $H$ and $\mathcal{H}$.*

If such a $\varphi(H, \mathcal{H})$ does not exist then we write $\varphi(H, \mathcal{H}) = +\infty$. If $\varphi(H, \mathcal{H}) < +\infty$, then we call the graph $H$ *light* in the family $\mathcal{H}$.

Here we consider the family $\mathcal{H} = \mathcal{P}(\delta, \rho)$ of all 3-connected plane graphs (i.e. the family of all polyhedral graphs [G1]) with minimum vertex degree at least $\delta$ and minimum face size at least $\rho$, where $\delta$ and $\rho$ are at least 3. In the sequel, let

$$\varphi(\delta, \rho; H) = \varphi(H, \mathcal{P}(\delta, \rho))$$

and

$$\varphi(\delta, \bar{\rho}, H) = \varphi(H, \mathcal{P}(\delta, \bar{\rho})).$$

The result for $P_1$ derived from Euler's formula can be rewritten as $\varphi(3,3;P_1) = 5$. Kotzig [Ko1] (see also Section 3) proved that each graph $G \in \mathcal{P}(3,3)$ contains an edge $e$ with $w(e) \leq 13$. This implies $\varphi(3,3;P_2) = 10$. Theorem 4.1 provides $\varphi(3,3;P_3) = 15$. Fabrici and Jendrol' (see Section 5) generalized the results for $P_1, P_2$ and $P_3$ to arbitrary $P_k$. Their Theorem 5.1 states that the path $P_k$ is light in the family $\mathcal{P}(3,3)$ for every $k$. Next theorem sumarizes results concerning the constant $\varphi(P_k, \mathcal{G})$ for several families of 3-connected plane graphs $\mathcal{G}$. (Brackets indicate the papers, where the results are proved.)

**Theorem 6.1.**

(i) $\varphi(3,3;P_k) = 5k$ *for all* $k \geq 1$. [FJ1]
(ii) $\varphi(4,3;P_k) = 5k - 7$ *for all* $k \geq 8$. [FHJW]
(iii) $5\left\lfloor \frac{k}{2} \right\rfloor \leq \varphi(3,4;P_k) \leq \frac{5}{2}k$ *for all* $k \geq 2$. [HJT]
(iv) $5k - 325 \leq \varphi(5,3;P_k) \leq 5k - 7$ *for all* $k \geq 68$. [Fa1]
(v) $\frac{5}{3}k - 80 \leq \varphi(3,5;P_k) \leq \frac{5}{3}k$ *for all* $k \geq 935$. [JO]
(vi) $2k + 2 \leq \varphi(P_k, \mathcal{F}) \leq 2k + 3$ *for all* $k \geq 2$. [HW]
*(Here* $\mathcal{F}$ *is the family of 4-connected planar graphs)* □

Note that the precise values of $\varphi(4,3;P_k)$ are known also for all $k \leq 7$, see [FHJW]. We can also show that the lower bound in (iv) cannot be smaller than $5k - 125$ if $k \geq 714$ and that the upper bound in (v) is valid for all $k \geq 2$.

Theorem 5.2 asserts that no connected planar graph distinct from a $k$-path $P_k$ for every $k$ is light in the family $\mathcal{P}(3,3)$. This motivates the following

**Problem 6.2.** *For a given infinite family* $\mathcal{G}$ *of plane graphs determine all connected planar graphs that are light in* $\mathcal{G}$. □

The next theorem provides results of this type.

**Theorem 6.2.** *If* $\mathcal{G} \in \{\mathcal{P}(3,3), \mathcal{P}(3,4), \mathcal{P}(4,3), \mathcal{F}\}$, *where* $\mathcal{F}$ *is the family of 4-connected planar graphs, then a graph* $H$ *is light in the family* $\mathcal{G}$ *if and only if* $H$ *is* $k$*-path* $P_k$ *for some* $k \geq 1$. □

The proof of Theorem 6.2 for the family $\mathcal{P}(3,4)$ is by Harant, Jendrol' and Tkáč [HJT], for $\mathcal{P}(4,3)$ by Fabrici, Hexel, Jendrol' and Walther [FHJW], and for the family $\mathcal{F}$ by Mohar [Mo1].

In the graph families $\mathcal{P}(3,3), \mathcal{P}(3,4)$, and $\mathcal{P}(4,3)$ only the paths $P_k$ are light. The situation changes significantly in the families $\mathcal{P}(5,3)$ and $\mathcal{P}(3,5)$. From Theorem 6.1 it follows that each path $P_k, k \geq 1$, is light in both families $\mathcal{P}(5,3)$ and $\mathcal{P}(3,5)$.

Next theorem excludes some families of graphs to be light

**Theorem 6.3** [Fa1], [JMST], [JO].

(i) *No plane connected graph* $H$ *with maximum degree at least 5 or with a block having at least 11 vertices is light in* $\mathcal{P}(5,\overline{3})$ *(and, hence, in* $\mathcal{P}(5,3)$*).*
(ii) *No plane connected graph* $H$ *with maximum degree at least 4 or with a block having at least 19 vertices is light in* $\mathcal{P}(3,5)$. □

From the classical result of Lebesgue [Le] (see Theorem 2.1) it follows that the 3-cycle $C_3$ is light in $\mathcal{P}(5,3)$ and the 5-cycle $C_5$ is light in $\mathcal{P}(3,5)$. Recently Jendrol' and Madaras in

[JM] proved that the star $K_{1,r}$ for $r \geq 3$ is light in $\mathcal{P}(5,3)$ if and only if $r \in \{3,4\}$. However, the problem of determining all graphs that are light in $\mathcal{P}(5,3)$ and $\mathcal{P}(3,5)$ remains open. Jendroľ et al. [JMST] proved the following theorem

**Theorem 6.4** [JMST]**.** *The r-cycle $C_r$ is light in the family $\mathcal{P}(5,\bar{3})$ of all plane triangulations with minimum degree 5 if and only if $3 \leq r \leq 10$. Moreover*

$$\begin{array}{rrr} \varphi(C_3)=7, & \varphi(C_4)=10, & \varphi(C_5)=10, \\ 10 \leq \varphi(C_6) \leq 11, & 15 \leq \varphi(C_7) \leq 17, & 15 \leq \varphi(C_8) \leq 29, \\ 19 \leq \varphi(C_9) \leq 41, & 20 \leq \varphi(C_{10}) \leq 415, & \end{array}$$

*where $\varphi(C_k) = \varphi(5,\bar{3};C_k)$.* □

In [He] there is a question for which $k \geq 11$ the cycle $C_k$ is light in the family of all 5-connected plane triangulation. The answer is rather surprising

**Theorem 6.5** [HS]**.** *The k-cycle $C_k$ is light in the family of all 5-connected plane triangulation for every $k$ at least 3.* □

Hexel and Soták conjecture that for every $k \geq 3$ the $k$-cycle $C_k$ is light in the family of all 5-connected planar graphs. More results concerning light graphs in subfamilies of plane graphs with high connectivity can be found in [Mo1] and [HW].

An analogue of Theorem 6.4 for the family $\mathcal{P}(3,\bar{5})$ is

**Theorem 6.6** [JO]**.** *The r-cycle $C_r$ is light in the family $\mathcal{P}(3,\bar{5})$ if and only if $r \in \{5,8,11,14\}$. Moreover $\varphi(3,\bar{5};C_5) = 5$, $6 \leq \varphi(3,\bar{5};C_8) \leq 7$, $10 \leq \varphi(3,\bar{5};C_{11}) \leq 11$ and $10 \leq \varphi(3,\bar{5};C_{14}) \leq 17$.* □

By Theorem 6.6 the only cycles that are candidates for being light in $\mathcal{P}(3,5)$ are $C_5, C_8, C_{11}$ and $C_{14}$. As we mentioned above, $C_5$ is light in this family. In [JO] the cycles $C_8$ and $C_{11}$ are proved to be not light there. It is an open question whether $C_{14}$ is light in $\mathcal{P}(3,5)$.

The 3-cycle is light in $\mathcal{P}(5,3)$ and not light in $\mathcal{P}(4,3)$. The question arises for which subclasses other than $\mathcal{P}(5,3)$ the 3-cycle is light? Let $\mathcal{P}(4,3,\mathcal{E}_t)$ denote the family of all graphs in $\mathcal{P}(4,3)$ having no path consisting of $t$ vertices all having degree 4. Borodin [Bo12] showed that the triangle $C_3$ is light in $\mathcal{P}(4,\bar{3},\mathcal{E}_t)$ for all $t \geq 1$. Mohar, Škrekovski, and Voss [MŠV] showed that the cycle $C_4$ is light in $\mathcal{P}(4,3,\mathcal{E}_t)$ for $t \in \{2,3,4\}$; so $C_4$ is light in $\mathcal{P}(4,3,\mathcal{E}_t)$ for $1 \leq t \leq 4$. Further, $C_4$ is not light in $\mathcal{P}(4,3,\mathcal{E}_t)$ for all $t \geq 23$. For $5 \leq t \leq 22$ this question is open. For $r \geq 5$ and $t \geq 3$ the cycle $C_r$ is not light in $\mathcal{P}(4,3,\mathcal{E}_t)$. For $t = 2$ Mohar, Škrekovski, and Voss showed

**Theorem 6.7** [MŠV]**.** *The r-cycle $C_r$ is light in the family $\mathcal{P}(4,3,\mathcal{E}_2)$ of all 3-connected plane graphs of minimum degree at least 4 and edge-weight at least 9 if and only if $r \in \{3,4,5,6\}$. Moreover,*

$$\varphi(4,3,\mathcal{E}_2;C_3) = 12,\ \varphi(4,3,\mathcal{E}_2;C_4) \leq 22,$$
$$\varphi(4,3,\mathcal{E}_2;C_5) \leq 107,\ \varphi(4,3,\mathcal{E}_2;C_6) \leq 107.$$

□

From a theorem of Borodin [Bo1] follows: $\varphi(5,3;C_3)=7$. Soták (oral communication) proved $\varphi(5,3;C_4)=11$ and $\varphi(5,3;C_5)=10$. The proof that $\varphi(5,3,C_6)\le 107$ is by Mohar et al. [MŠV]. The lightness of the 7-cycle in $\mathcal{P}(5,3)$ is proved by Madaras et al. [MaŠV]. Together with Theorem 6.7, these results imply

**Theorem 6.8.** *The r-cycle $C_r$ is light in $\mathcal{P}(5,3)$ if $r\in\{3,4,5,6,7\}$, and is not light in $\mathcal{P}(5,3)$ if $r\ge 11$. Moreover,*

$$\varphi(5,3;C_3)=7,\ \varphi(5,3;C_4)=11,\ \varphi(5,3;C_5)=10,\ \varphi(5,3;C_6)\le 107,\ \varphi(5,3;C_7)\le 359.$$ □

It is an open question whether in the class $\mathcal{P}(5,3)$ the cycles $C_8, C_9, C_{10}$ are light or not.

Jendrol' and Madaras [JM] showed for $r\ge 3$ that the star $K_{1,r}$ is light in $\mathcal{P}(5,3)$ if and only if $r\in\{3,4\}$. Mohar, Škrekovski, and Voss [MŠV] proved that this is also true in the class $\mathcal{P}(4,3,\mathcal{E}_2)$. For $t\ge 3$ the only star that is light in $\mathcal{P}(4,3,\mathcal{E}_t)$ is $K_{1,3}$, see [MŠV].

## 7. MAXIMUM DEGREE OF LIGHT FAMILIES

In section 6 we have defined a light subgraph $H$ in a family $\mathcal{H}$ of graphs. Here we introduce the concept of a light class $\mathcal{L}$ of graphs in a family $\mathcal{H}$ of graphs.

**Problem 7.1.** *Let $\mathcal{H}$ be a family of graphs, and let $\mathcal{L}$ be a finite family of connected graphs having the property that every member of $\mathcal{L}$ is a proper subgraph of at least one member of $\mathcal{H}$. Let $\varphi(\mathcal{L},\mathcal{H})$ be the smallest integer $t$ with the property that every graph $G$ in $\mathcal{H}$ that has a subgraph belonging to $\mathcal{L}$, has such a subgraph whose vertices all have degree at most $t$ in $G$. Determine the value $\varphi(\mathcal{L},\mathcal{H})$ for a given pair of families $\mathcal{L}$ and $\mathcal{H}$.*

If such a $\varphi(\mathcal{L},\mathcal{H})$ does not exist, then we write $\varphi(\mathcal{L},\mathcal{H})=+\infty$. If $\varphi(\mathcal{L},\mathcal{H})<+\infty$, then we call the family $\mathcal{L}$ *light* in the family $\mathcal{H}$. When $\mathcal{L}$ is the family $\mathcal{T}_k$ of all trees on $k$ vertices, and $\mathcal{H}=\mathcal{P}(\delta,\rho)$, we write $\tau(k,\delta,\rho)$ for $\varphi(\mathcal{T}_k,\mathcal{P}(\delta,\rho))$.

Obviously, $\mathcal{T}_1=\{P_1\},\mathcal{T}_2=\{P_2\},\mathcal{T}_3=\{P_3\}$, and $\{P_k\}\subset\mathcal{T}_k$ for all $k\ge 4$. Hence $\tau(k,\delta,\rho)=\varphi(\delta,\rho;P_k)$ for $1\le k\le 3$, and $\tau(k,\delta,\rho)\le\varphi(\delta,\rho;P_k)$. For $\mathcal{P}(3,3)$ Fabrici and Jendrol' proved:

**Theorem 7.1** [FJ3].

(i) $\tau(1,3,3)=5$,
(ii) $\tau(2,3,3)=10$,
(iii) $\tau(k,3,3)=4k+3$ *for any* $k\ge 3$.

The theorem can be reformulated, as follows:

**Theorem 7.1*.** *Every 3-connected planar graph $G$ of order at least $k\ge 3$ has a connected subgraph $K$ of order $k$ such that the degree in $G$ of every vertex of $K$ is at most $4k+3$. The bound $4k+3$ is best possible.* □

If the minimum degree of graphs of $\mathcal{H}$ is increased to 4 then a slightly smaller bound is obtained by Fabrici.

**Theorem 7.2** [Fa1]. *$\tau(k, 4, 3) = 4k - 1$ for any $k \geq 4$.* □

In the paper [HW] of Hexel and Walther and that of Hexel [He] the reader can find several bounds on $\tau(k, \delta, \rho)$.

Here we provide a result that gives a necessary condition for a fixed family $\mathcal{L}$ of connected plane graphs to be light in $\mathcal{P}(3, 3)$.

**Theorem 7.3.** *If $\mathcal{L}$ is a finite family of connected plane graphs $H$ such that $\triangle(H) \geq 3$ or $\delta(H) \geq 2$, then $\varphi(\mathcal{L}, \mathcal{P}(3, 3)) = +\infty$. That means that $\mathcal{L}$ is not light in $\mathcal{P}(3, 3)$.*

**Proof.** Let $K$ be the disjoint union of all graphs from the family $\mathcal{L}$. Let $T_o$ be a plane triangulation of $K$, that is the graph obtained from $K$ by inserting necessary edges into $K$ to obtain a plane triangulation. The rest is the same as in the proof of Theorem 5.2, using $K$ as $H$ and $m$ arbitrarily large. □

**Corollary 7.3.1.** *If $\mathcal{L}$ is a light family in $\mathcal{P}(3, 3)$ then $\mathcal{L}$ contains a $k$-path for some $k$.* □

Theorem 7.1 leads to the following problem:

**Problem 7.2.** *Find an optimal set $\mathcal{S}_k$ of trees on $k$ vertices such that $\varphi(\mathcal{S}_k, \mathcal{P}(3, 3)) = 4k + 3$.*

Applying Theorems 5.1 and 7.1 we can easily get the following.

**Theorem 7.4.** *If $\mathcal{L} = \{P_k, K_{1,3}\}$, then $\varphi(\mathcal{L}, \mathcal{P}(3, 3)) = 4k + 3$.* □

The next theorem generalizes this result. For $i \geq 0$, let $S_i$ denote a generalized 3-star with a central vertex of degree 3, where the three paths with common endpoint have $i + 1$ vertices. Obviously, $S_0 = K_1$ and $S_1 = K_{1,3}$.

**Theorem 7.5** [JV11]. *Let $k$ and $i$ be integers with $k \geq 3$ and $1 \leq i \leq \frac{k}{2}$. If $\mathcal{L}_i = \{P_k, S_i\}$, then $\varphi(\mathcal{L}_i, P(3, 3)) = \min\{5k, 4(k + i) - 1\}$.* □

## 8. WEIGHT PROBLEMS

For a graph $H$ contained in a graph in a family $\mathcal{H}$, one can also consider the smallest integer $w(H, \mathcal{H})$ having the property that every graph $G \in \mathcal{H}$ that contains a copy of $H$ whose vertices have degrees in $G$ that sum to at most $w(H, \mathcal{H})$. For a subgraph $K$ of $G$, let $w_G(K) = \sum\limits_{v \in V(K)} \deg_G(v)$; this is called the *weight* of $K$ in $G$, (see e.g. [G3]).

Similarly $w(H, \mathcal{H})$ can be called the *weight of $H$ in $\mathcal{H}$*, and we say that the graph $H$ is light in $\mathcal{H}$ if $w(H, \mathcal{H})$ is finite. Note that $w_G(K)$ and $\varphi_G(K)$ refer to the sum and the maximum over the same finite set. Thus $w(H, \mathcal{H})$ is finite if and only if $\varphi(H, \mathcal{H})$ is finite and the two definitions of a light graph are equivalent. Now we formulate

**Problem 8.1.** *Find the precise value of $w(H, \mathcal{H})$ for a given graph $H$ and a family of graphs $\mathcal{H}$.*

The precise value $w(H, \mathcal{H})$ is known only for a few light graphs (and families). We start our survey with a beautiful recent result of Mohar [Mo1].

**Theorem 8.1** [Mo1]. *Let $\mathcal{P}_{\text{ham}}$ be the family of all planar hamiltonian graphs. Then*

$$w(P_k, \mathcal{P}_{\text{ham}}) = 6k - 1, k \geq 1.$$

**Proof.** For a planar hamiltonian graph $H$ on $n$ vertices, let $C_n$ be a hamiltonian cycle through vertices $v_1, v_2, ..., v_n$ in order. Let $R_i$ be the part of $C_n$ on vertices $v_i, v_{i+1}, ...,$ $v_{i+k-1}$ (indices modulo $n$). Let $w(R_i) = \sum\limits_{j=i}^{i+k-1} \deg_H(v_j)$ denote the weight of $R_i$ in $H$ (that is the sum of degrees in $H$ of vertices of $R_i$). Then

$$\sum_{i=1}^{n} w(R_i) = k \sum_{v \in V(H)} \deg(v) = 2k|E(H)| \leq 2k(3n-6).$$

The last inequality is the well known corollary (2.4) of Euler's formula. Hence, one of the paths, say $R_j$, has weight at most $2k(3n-6)/n$. Since this is less than $6k$, we obtain $w(P_k, \mathcal{P}_{\text{ham}}) \leq 6k-1$. On the other hand, there are 5-connected plane triangulations which contain precisely 12 vertices of degree 5, and all other vertices are 6-vertices.Moreover, the 5-vertices are as far away from each other as we like. This shows that $w(P_k, \mathcal{P}_{\text{ham}}) \geq 6k-1$ and completes the proof. □

Clearly 2-connected outerplanar graphs are hamiltonian. When using the idea of Mohar one can prove that every 2-connected outerplanar has a $k$-path of the weight at most $4k-1$. The next theorem is by Fabrici [Fa2].

**Theorem 8.2** [Fa2]. *Let $\mathcal{O}(2)$ be the family of all 2-connected outerplanar graphs. Then for every $k \geq 3$*

$$\varphi(P_k; \mathcal{O}(2)) = k + 3 \text{ and } w(P_k; \mathcal{O}(2)) = 4k - 2.$$

□

Fabrici, Harant, and Jendrol' [FHJ] further developed the ideas of Mohar's proof to show that an upper bound that is linear in $k$ also holds for a wider family of plane graphs.

**Theorem 8.3** [FHJ]. *For $G \in \mathcal{P}(\delta, \rho)$ let $c(G)$ be the length of a longest cycle of $G$. Let $k$ be an integer $3 \leq k \leq c(G)$. If $c(G) \geq \sigma|V(G)|$ for some positive number $\sigma$, then $G$ contains a $k$-path $P_k$ such that*

$$w_G(P_k) \leq \left( \left( \frac{2\rho}{\rho - 2} - \delta \right) \frac{1}{\sigma} + \delta \right) k - 1.$$

□

For the families $\mathcal{P}(\delta, \rho)$, only trivial bounds on $w(\delta, \rho, H)$ are known except in a few cases. Trivially,

$$w(\delta, \rho; H) \leq |V(H)| \varphi(\delta, \rho; H)$$

where $w(\delta, \rho; H) = w(H; \mathcal{P}(\delta, \rho))$. This yields e.g. $w(5, 3; P_k) \leq w(4, 3; P_k) \leq 5k^2 - 7k$, for $k \geq 8$, see [FHJW]. Also $w(3, 4; P_k) \leq \frac{5}{2}k^2$, see [HJT] and $w(3, 5; P_k) \leq \frac{5}{3}k^2$, see [JO].

[Je3] Jendrol' formulated the following problem

**Problem 8.2.** *Determine the precise value of* $w(3,3;P_k)$.

The precise values are known up to $k = 3$ : $w(3,3;P_1) = 5$, $w(3,3;P_2) = 13$ (Kotzig [Ko1]), and $w(3,3;P_3) = 21$ (Ando et al. [AIK]). The bounds on $w(3,\rho;P_k)$ for $3 \leq \rho \leq 5$ in the next theorem are due to Fabrici, Harant and Jendroľ [FHJ]. The lower bound in case (i) is by Fabrici and Jendroľ [FJ3].

**Theorem 8.4** [FHJ]. *Let* $k$ *be an integer,* $k \geq 4$. *Then*

(i) $k \log_2 k \leq w(3,3;P_k) \leq k^2 + 13k$.
(ii) $\frac{1}{2} k \log_2 k + \mathcal{O}(k) \leq w(3,4;P_k) \leq \frac{1}{2}(k^2 + 13k)$.
(iii) $\frac{1}{8} k \log_2 k + \mathcal{O}(k) \leq w(3,5;P_k) \leq \frac{1}{3}(k^2 + 13k)$. □

Let us notice that there is a contrast between hamiltonian and nonhamiltonian planar graphs. Mohar's Theorem 8.1 provides a sharp linear upper bound while constructions by Fabrici and Jendroľ [FJ3] show that the exact value of $w(3,3;P_k)$ is not linear. We believe that $w(\delta,\rho;P_k) = \mathcal{O}(k \log_2 k)$. Mohar [Mo] has constructed 3-connected planar graphs proving that $w(4,3;P_k) \geq \frac{9}{16} k \log_2 k$ and $w(5,3;P_k) \geq \frac{3}{10} k \log_2 k$.

Due to a result of Tutte [Tu] that every 4-connected planar graph is hamiltonian Theorem 8.1 has the following corollary

**Theorem 8.5** [Mo1]. *Every* $n$*-vertex 4-connected planar graph contains a* $k$*-path* $P_k$ *of weight* $\leq 6k - 1$ *for every* $1 \leq k \leq n$. *Moreover, this bound is tight.* □

Problem 8.2 can be formulated more generally

**Problem 8.4.** *Determine the precise value of* $w(\delta,\rho;H)$ *for all light graphs* $H$ *in the family* $\mathcal{P}(\delta,\rho)$ *for all possible pairs* $(\delta,\rho)$.

The known precise results concerning this problem not mentioned in Section 4 are listed here: Theorem 2.1 of Lebesgue yield: $w(3,5;P_4) = 12$, and $w(3,5;P_5) = 17$. Jendroľ and Madaras [JM] proved $w(5,3;P_4) = w(5,3;K_{1,3}) = 23$, and recently Borodin and Woodall [BW1] showed that $w(5,\bar{3};C_4) = 25$, and $w(5,\bar{3};C_5) = 30$, and $w(5,3;K_{1,4}) = 30$. Madaras [Ma2] proved that $w(4,\bar{3};P_4) \leq 31$ and $w(5,\bar{3};P_5) = 29$.

Analogously we can define the value $w(\mathcal{L},\mathcal{H})$ for a finite family of connected graphs $\mathcal{L}$ and a family $\mathcal{H}$ having the properties mentioned in Section 7. Namely, every member of $\mathcal{L}$ is a proper subgraph of at least one member of $\mathcal{H}$. Let $w(\mathcal{L},\mathcal{H})$ be the smallest integer $t$ with the property that every graph $G$ in $\mathcal{H}$ that has a subgraph belonging to $\mathcal{L}$ has such a subgraph $H$ whose vertices have degrees in $G$ such that sum to at most $w(\mathcal{L},\mathcal{H})$.

A nice result recently published by Enomoto and Ota can be read as follows: "Every 3-connected planar graph $G$ of order at least $k$ contains a connected subgraph $H$ of order $k$ such that the degree sum of vertices of $H$ in $G$ is at most $8k - 1$." More precisely, using our notation, they proved

**Theorem 8.6** [EO]. *Let* $k$ *be an integer, and* $\mathcal{T}_k$ *be the family of all trees on* $k$ *vertices. If* $k \geq 4$, *then*

$$8k - 5 \leq w(\mathcal{T}_k, \mathcal{P}(3,3)) \leq 8k - 1$$

$$8k-5 \le w(\mathcal{T}_k,\mathcal{P}(4,3)) \le 8k-3$$
$$7k-4 \le w(\mathcal{T}_k,\mathcal{P}(5,3)) \le 7k-2. \qquad \square$$

They expect the following to be true

**Conjecture 8.5** [EO]. *If $k$ is an integer and $k \ge 4$, then*

$$w(\mathcal{T}_k,\mathcal{P}(3,3)) = 8k-5. \qquad \square$$

Madaras and Škrekovski [MŠ] investigated the conditions related to weight of a fixed subgraph of plane graphs that can enforce the existence of light graphs in families of plane graphs. For the families of plane graphs and triangulations whose edges are of weight at least $w$ they study the necessary and sufficient conditions for lightness of certain graphs according to values of $w$. We like some of their results:

**Theorem 8.7** [MŠ]. *Let $\mathcal{R}(w)$ be the family of all planar graphs of minimum degree at least 3 whose edges are of weight at least $w$.*

(i) *The 4-path $P_4$ is light in $\mathcal{R}(w)$ if and only if* $8 \le w \le 13$.
(ii) *The $k$-cycle $C_k, k \in \{3,4\}$, is light in $\mathcal{R}(w)$ if and only if* $10 \le w \le 13$.
(iii) *The star $K_{1,4}$ is light in $\mathcal{R}(w)$ if and only if* $9 \le w \le 13$. $\square$

## 9. LIGHT SUBGRAPHS OF GRAPHS EMBEDDED ON SURFACES

In this section we discuss light subgraphs of connected graphs embedded into surfaces other than the plane. First we recall necessary terms.

Throughout this section we use terminology of [MT]. However, we recall some definitions. An *orientable surface* $\mathbb{S}_g$ of genus $g$ is obtained from the sphere by adding $g$ handles. Correspondingly, a *nonorientable surface* $\mathbb{N}_q$ of genus $q$ is obtained from the sphere by adding $q$ crosscaps. The Euler characteristic is defined by

$$\chi(\mathbb{S}_g) = 2-2g \text{ and } \chi(\mathbb{N}_q) = 2-q.$$

By a *surface* $\mathbb{M}$ we mean either an orientable surface $\mathbb{S}_g$ or a nonorientable surface $\mathbb{N}_q$. By the *genus* $g$ (the *nonorientable genus* $q$) of a graph $G$ we mean the smallest integer $g$ ($q$) such that $G$ has an embedding into $\mathbb{S}_g$ ($\mathbb{N}_q$, respectively).

If a graph $G$ is embedded in a surface $\mathbb{M}$ then the connected regions of $\mathbb{M}-G$ are called the *faces* of $G$. If each face is an open disc then the embedding is called a 2-*cell embedding*. If each vertex has degree at least 3 and each vertex of degree $h$ is incident with $h$ different faces then $G$ is called a *map* in $\mathbb{M}$. If, in addition, $G$ is 3-connected and the embedding has "representativity" at least three, then $G$ is called a *polyhedral map* in $\mathbb{M}$, see e.g. Robertson and Vitray [RV]. Let us recall that the *representativity* (or face width) of a (2-cell) embedded graph $G$ into a surface $\mathbb{M}$ is equal to the smallest number $k$ such that $\mathbb{M}$ contains a noncontractible closed curve that intersects the graph $G$ in $k$ points. We say that $H$ is a *subgraph* of a polyhedral map $G$ if $H$ is a subgraph of the underlying graph of the map $G$.

For a map $G$ let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set $G$, respectively. For a map $G$ on a surface $\mathbb{M}$ Euler's famous formula states

$$|V(G)|-|E(G)|+|F(G)| = \chi(\mathbb{M}).$$

In 1990, Ivančo [Iv] generalized the Theorem 3.1 of Kotzig in the following way:

**Theorem 9.1** [Iv]. *If G is a connected graph of orientable genus g and minimum degree at least 3, then G contains an edge e of weight*

$$w(e) \leq \begin{cases} 2g+13 & \text{for } 0 \leq g \leq 3 \\ 4g+7 & \text{for } g \geq 3. \end{cases}$$

*Furthermore, if G does not contain 3-cycles, then*

$$w(e) \leq \begin{cases} 8 & \text{for } g = 0 \\ 4g+5 & \text{for } g \geq 1. \end{cases}$$

*Moreover, all bounds are best possible.* ☐

An analogous result for graphs on non-orientable surfaces proved by Jendroľ and Tuhársky [JTu] is as follows

**Theorem 9.2** [JTu]. *If G is a connected graph of minimum degree at least 3 on a non-orientable genus $q \geq 1$, then G contains an edge e of weight*

$$w(e) \leq \begin{cases} 2q+11 & \text{for } 1 \leq q \leq 2, \\ 2q+9 & \text{for } 3 \leq q \leq 5, \\ 2q+7 & \text{for } q \geq 6. \end{cases}$$

*Furthermore, if G does not contain 3-cycles, then*

$$w(e) \leq 4q+5 \quad \text{for} \quad q \geq 1$$

*Moreover, all bounds are best possible.* ☐

For the projective plane, the nonorientable surface of the smallest genus, using the same ideas as in the proof of Theorem 3.1, one can easily prove

**Theorem 9.3.** *Every connected projective planar graph of minimum degree at least 3 contains a $(3, a)$-edge with $3 \leq a \leq 10$, or a $(4, b)$-edge with $4 \leq b \leq 7$ or a $(5, c)$-edge with $5 \leq c \leq 6$. The bounds 10, 7, and 6 are best possible.* ☐

The bounds in Theorem 9.1 and 9.2 can be esentially improved if embedded graphs have a "large" number of vertices. Namely, the following holds:

**Theorem 9.4** [JTV]. *Let G be a normal map on surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ and let n be the number of vertices of G. If*

(a) $\sum(\deg_G(x) - 6) > 48|\chi(\mathbb{M})|$, *or*
(b) $n > 26|\chi(\mathbb{M})|$,

*then G contains an $(a, b)$-edge such that*

(i) $a = 3$ *and* $3 \leq b \leq 12$, *or*
(ii) $a = 4$ *and* $4 \leq b \leq 8$, *or*
(iii) $5 \leq a \leq 6$ *and* $5 \leq b \leq 6$.

*The bounds 12, 8, and 6 are best possible.* ☐

Nothing seems to be done in the following

**Problem 9.1.** *Find an analogue of Theorem* 3.3 *for polyhedral maps on manifolds* $\mathbb{M}$ *of Euler characteristics* $\chi(\mathbb{M}) \leq 0$.

For the projective plane, Sanders established sharp inequalities.

**Theorem 9.5** [Sa]. *Every normal projective planar map satisfies the following inequality:*

$$\begin{aligned}40e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + \frac{20}{3}e_{3,7} + 5e_{3,8} + \frac{5}{2}e_{3,9} + 2e_{3,10} \\ + \frac{50}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + \frac{5}{3}e_{4,7} \\ + \frac{16}{3}e_{5,5} + 2e_{5,6} \geq 60\end{aligned}$$

□

*and each of these coefficients is best possible.*

In Theorem 9.3 we have the same coefficients as in the planar case (Theorem 3.2), but on the right side the value 120 has been weakened to 60.

In the subclass of all normal projective planar graphs with minimum degree at least 4 we have $e_{3,j} = 0$ for $3 \leq j \leq 10$. Sanders proved that in the resulting inequality all coefficients are best possible.

**Theorem 9.6** [Sa]. *Every normal projective plane map of minimum degree four satisfies the inequality*

$$50e_{4,4} + 33e_{4,5} + 15e_{4,6} + 5e_{4,7} + 16e_{5,5} + 6e_{5,6} \geq 180,$$

*and each of these coefficients is best possible.* □

**Theorem 9.7** [Sa]. *Every projective planar graph of minimum degree five satisfies the inequality*

$$16e_{5,5} + 7e_{5,6} \geq 210,$$

*and each of these coefficients is best possible.* □

If in Theorem 9.5 we use $e_{4,j} = 0$ for $4 \leq j \leq 7$ then an inequality is obtained which differs in the coefficient of $e_{5,5}$.

**Theorem 9.8** [Sa]. *Every projective plane graph of minimum degree five satisfies the inequality*

$$18f_{5,5,5} + 9f_{5,5,6} + 5f_{5,5,7} + 4f_{5,6,6} \geq 72,$$

*and each of these coefficients is best possible.* □

Euler's formula implies (with some terms left out) $3v_3 + 2v_4 + v_5 \geq 12$ for the plane and $3v_3 + 2v_4 + v_5 \geq 6$ for the projective plane. Most of the above inequalities differ only on the right side, where 12 appears for the normal plane graphs and 6 for the normal projective planar graphs. This is completely true if the minimum degrees are 3 or 4, respectively. The only inequality that does not follow precisely these lines is the light edge inequality for graphs of minimum degree five. For the plane, the coefficient of $e_{5,5}$ went from 8/15 to 7/15. For the projective plane, it is lowered from 8/15 to 16/35. Each other coefficient of the inequalities in the projective planar case is equal to the corresponding coefficient in the plane case.

Using the same arguments as for the planar case it is possible to prove the following analogues of Theorem 3.1 and Theorem 7.1

**Theorem 9.9** [FJ3]. *Every 3-connected projective planar graph $G$ that contains a $k$-path contains such a path whose vertices all have degree at most $5k$ in $G$. The bound $5k$ is best possible.* □

**Theorem 9.10** [JV5]. *For all $k \geq 3$ every 3-connected projective planar graph $G$ of order at least $k$ contains a connected subgraph $H$ of order $k$ whose vertices all have degree at most $4k+3$ in $G$.* □

We generalized these and other results on light subgraphs to surfaces $\mathbb{M}$ with nonpositive Euler characteristic. For details on these results, see [JV10]. In the next subsections we give a brief survey only. We mention that all other theorems of Section 7 are also true for 3-connected projective planar graphs.

**Theorem 9.11** [JV1]. *Each polyhedral map $G$ on $\mathbb{M}$ that contains a $k$-path contains such a $k$-path whose vertices all have degree at most $k\lfloor(5+\sqrt{49-24\chi(\mathbb{M})})/2\rfloor$ in $G$. Equality holds for even $k$.* □

Let $K_n$ and $K_n^-$ denote the complete graph on $n$ vertices and the graph obtained from it by deleting one edge, respectively. For odd $k$ we can show:

**Theorem 9.12** [JV4]. *For each odd $k$ greater than $\frac{4}{3}\left\lfloor\frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2}\right\rfloor+1$:*

(i) *the upper bound in Theorem 9.11 is attained at infinite many orientable surfaces and at infinite nonorientable surfaces, where these surfaces are characterized by the fact that each of these surfaces has a triangular embedding of a $K_n^-$;*
(ii) *the upper bound in Theorem 9.11 is not attained at infinite many orientable surfaces and at infinite many nonorientable surfaces, where these surfaces are characterized by the fact that each of these surfaces has a triangular embedding of a $K_n^-$ (in this case an upper bound is*

$$\left\lfloor\left(k-\frac{1}{3}\right)\frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2}\right\rfloor.$$

□

A polyhedral map $G$ is called *large* if it has a large number of vertices or large positive $k$-charge, where the *positive charge* of $G$ is $\sum\limits_{\deg_G(u)>6k}(\deg_G(u)-6k)$.

**Theorem 9.13** [JV2], [JV3]. *Every large polyhedral maps on a surface $\mathbb{M}$ of nonpositive Euler characteristic that contains a $k$-path contains such a $k$-path whose vertices all have* *degree, in $G$, at most $6k$ for $k = 1$ or even $k \geq 2$ and at most $6k - 2$ for odd $k \geq 3$. Moreover, these bounds are tight.* $\square$

The upper bound on maximum degree of vertices of light paths polyhedral maps on a surface $\mathbb{M}$ depends on $\sqrt{|\chi(\mathbb{M})|}$. In arbitrary embedding of 3-connected graphs (multigraphs) in $\mathbb{M}$ this degree bound is a linear function of $|\chi(\mathbb{M})|$.

**Theorem 9.14** [JV6]**.** *Each 3-connected multigraph $G$ on $\mathbb{M}$ without loops and 2-faces that has a $k$-path contains such a $k$-path whose vertices have degree at most $(6k - 2\varepsilon)(1 + |\chi(\mathbb{M})|/3)$ in $G$, where $\varepsilon = 0$ if $k \geq 2$ is even, and $\varepsilon = 1$ if $k \geq 3$ is odd. The bounds are best possible.* $\square$

**Theorem 9.15** [JV7]**.** *For 3-connected graphs on $\mathbb{M}$ the precise degree bound is*

$$2 + \lfloor (6k - 6 - 2\varepsilon)(1 + |\chi(\mathbb{M})|/3) \rfloor \ for \ k \geq 4.$$

$\square$

Fabrici, Hexel, Jendrol', and Walther [FHJW] proved that each 3-connected plane graph of minimum degree at least 4 that has $k$-path contains such a $k$-path whose vertices all have degree at most $5k - 7$ in $G$. This bound is sharp for $k \geq 8$. For surfaces other than the plane, we have

**Theorem 9.16** [JV8]**.** *For $k \geq 8$, each large polyhedral map $G$ on $\mathbb{M}$ of minimum degree at least 4 that contains a $k$-path contains such a $k$-path whose vertices have degree at most $6k - 12$ in $G$. This bound is sharp for even $k$, and it must be at least $6k - 14$ for odd $k$.* $\square$

In 3-connected plane graphs of minimum degree at least 5 only the bound $5k-7$ [FHJW] is known. For large polyhedral maps on 2-manifolds $\mathbb{M}$, the degree bound is not a linear function on $k$.

**Theorem 9.17** [JV8]**.** *Let $k$ be an integer at least 66. Each large polyhedral map $G$ on $\mathbb{M}$ of minimum degree at least 5 that contains a $k$-path contains such a $k$-path whose all vertices have degree at most $6k - \log_2 k + 2$. Moreover, the exact bound is at least $6k - 72\log_2 k - 132$.* $\square$

In families of polyhedral maps of Theorem 9.10, 9.12 and 9.15 and in embeddings of 3-connected multigraphs (Theorem 9.13), and in embeddings of 3-connected graphs (Theorem 9.14) only $k$-paths are light for every $k$.

In the families of large polyhedral maps of minimum degree at least 5 on surfaces of nonpositive genus one can prove the existence of other light graphs. So the cycle $C_3$ is light there (see [JV12]), all proper spanning subgraphs $H$ of the complete graph $K_4$ are light there, while $K_4$ itself is not light (see [JV13]). The 5-cycle $C_5$ and the 5-cycle with one or two diagonals are light in this class as well (see [JV14]). For other results see [JV9].

Fabrici and Jendrol' [FJ3] proved that each 3-connected plane graph $G$ of order at least $k$ contains a connected subgraph of order $k$ whose vertices all have degree at most $4k + 3$. We have proved that this also holds for the projective plane. For polyhedral maps on $\mathbb{M}$, the degree bound for connected subgraphs of order $k$ again depends on $\sqrt{|\chi(\mathbb{M})|}$. (This result is not presented here). For polyhedral maps with many vertices we proved

**Theorem 9.18** [JV5]. *For $k \geq 2$, each polyhedral map on $\mathbb{M}$ having at least $(8k^2 + 6k - 6)|\chi(\mathbb{M})| + 1$ vertices contains a connected subgraph of order $k$ whose vertices all have degree at most $4k + 4$. This bound is best possible.* ☐

For polyhedral maps the bound depends on $\sqrt{|\chi(\mathbb{M})|}$. In arbitrary embeddings of 3-connected graphs (multigraphs) in $\mathbb{M}$ the bound is a linear function of $|\chi(\mathbb{M})|$.

**Theorem 9.19** [JV6]. *For $k \geq 2$ each 3-connected multigraph $G$ on $\mathbb{M}$ that has no loops or 2-faces and has order at least $k$ contains a connected subgraph of order $k$ whose vertices all have degree at most $\lfloor (4k+4)(1+\frac{|\chi(\mathbb{M})|}{3}) \rfloor$ in $G$. This bound is sharp.* ☐

**Theorem 9.20** [JV7]. *For 3-connected graphs on $\mathbb{M}$, the precise degree bound is*

$$2 + \left\lfloor \left(4k - 2\right) \left(1 + \frac{|\chi(\mathbb{M})|}{3}\right) \right\rfloor \quad \text{for } k \geq 5.$$

☐

**Theorem 9.21** [JV6]. *For both large 3-connected multigraphs on $\mathbb{M}$ without loops and 2-faces and for large 3-connected graphs on $\mathbb{M}$ having at least $k \geq 2$ vertices the precise degree bound is $4k + 4$.* ☐

We finish this section with an analogue of Theorem 8.6 recently proved by Kawarabayashi et al. [KNO].

**Theorem 9.22** [KNO]. *For any non-spherical surface $\mathbb{M}$, any positive integer $t$ there exist positive integers $r(\mathbb{M})$ and $n_0 = n_0(\mathbb{M}, t)$ such that if $G$ is a $n$ vertex, $n \geq n_0$, 3-connected graph embedded into $\mathbb{M}$ with representativity $r(\mathbb{M})$, then $G$ has a connected subgraph $H$ of $t$ vertices such that*

$$w_G(H) \leq \sum_{v \in V(H)} \deg_G(v) \leq 8t - 1.$$

## 10. RELATED TOPICS

The concept of the weight of an edge, of a face, of a path, or of a cycle as presented in this survey has served as a starting point for research in several other directions. We briefly mention some of them.

**1.** The idea of light edges was used by P. Erdös who formulated in 1990 at the conference in Prachatice (Czechoslovakia) the following max-min problem (see [IJ]): For a graph $G = (V, E)$ its edge weight $w(G)$ is defined as $\min\{w(e) | e \in E\}$. Let $\mathcal{G}(n, m)$ denote the family of all graphs with $n = |V|$ vertices and $m = |E|$ edges. Determine the value

$$W(n, m) = \max\{w(G) | G \in \mathcal{G}(n, m)\}.$$

Ivančo and Jendrol' [IJ] have proved some partial results. Recently Jendrol' and Schiermeyer [JS] have found a complete solution to Erdös's question and characterized all graphs on $n$ vertices and $m$ edges attaining this minimum weight.

A graph $G$ from $\mathcal{G}(n, m)$ having no isolated vertices is *degree-constrained* if $a = \frac{2m}{n} < 2\delta$, where $a$ is the average degree of $G$ and $\delta = \delta(G)$ is minimum degree of $G$. Bose, Smid, and

Wood [BSW] proved that every degree-constrained graph has an edge $uv$ with both $\deg(u)$ and $\deg(v)$ at most $\lfloor d \rfloor$ where $d = \frac{a\delta}{2\delta - a}$. Moreover they investigate matchings consisting of light edges in degree-constrained graphs.

**2.** The idea of a light path has been considered also in the family of all $K_{1,r}$-free graphs, $r \geq 3$, that are graphs without $K_{1,r}$ as an induced subgraph. Harant et al. [HJRR] proved among others the following rather surprising result.

**Theorem 10.1** [HJRR]. *If $G$ is a $K_{1,r}$-free graph on $n$ vertices, where $r \geq 3$, then each induced path of length at least $2r - 1$ and each induced cycle of length at least $2r$ in $G$ has the weight at most $(2r-2)(n-\alpha_0)$, where $\alpha_o$ is the independence number of $G$. Moreover, this bound is tight.* □

**3.** For a polyhedral map $G$ the $p$-vector is defined in [G1] to be a sequence $p = (f_i | i \geq 3)$ where $f_i$ denotes the number of $i$-gonal faces of $G$. Rosenfeld [Ro] started to investigate the problem of characterization of $p$-vectors of 3-connected non-regular plane graphs (i.e. non-regular polyhedral graphs) whose edges have the same, constant, weight. Jendrol' and Jucovič [JJ] made first steps in dealing with this problem for polyhedral maps on orientable surfaces. There is a lot of open problems in this topic, see e.g. [Je1], [JJ].

**4.** Similarly, Jucovič [IT] suggested studying polyhedral maps with constant weight of faces. By the weight of a face $\alpha$ we mean the sum of degrees of vertices incident with $\alpha$. All Platonic solids and all duals to Archimedean solids have constant face weights. Ivančo and Trenkler [IT], and Horňák and Ivančo [HI] determined the number of nonisomorphic 3-connected plane graphs having prescribed face weight $w$. There are infinitely many such graphs if and only if $16 \leq w \leq 21$ or $w = 23$ or $w = 25$. There is exactly one such graph if $w = 9$ or $w = 11$, for $w = 14$ there are four, and for $w = 15$ there are ten such graphs. For other values of $w$ at least 12 there are exactly two such graphs. For $w = 10$ no such graph exists. Nothing is known about polyhedral maps with constant face weight on surfaces other than the plane.

**5.** The idea of the weight of an edge $e = uv$ being the degree sum of the endvertices $u$ and $v$ motivated Jendrol' and Ryjáček [JR] to introduce the concept of *tolerance* of the edge $e$. The *tolerance* $\tau(e)$ of the edge $e = uv$ is defined to be the absolute value of the difference of degrees of the vertices $u$ and $v$,

$$\tau(e) = |\deg(u) - \deg(v)|.$$

Necessary and sufficient conditions for the existence of connected planar and 3-connected planar graphs with constant edge tolerance appear in [JR]. Several constructions of graphs with constant edge tolerance for general graphs appear in Acharya and Vartak [AV]. An open problem is to find necessary and sufficient conditions for graphs with constant edge tolerance embedded into surfaces different from the sphere.

**6.** Motivated by "light" results, Mohar [Mo2] considered such problems for infinite planar graphs. He uses the discharging method to prove some new results in this direction. The general outline of the method is presented in [Mo2]. Many applications are given there, including results on light subgraphs and the following: Planar graphs with only finitely many vertices of degree at most 5 and with subexponential growth contain arbitrarily large finite submaps of the tessellation of the plane or of some tessellation of the cylinder by equilateral triangles.

**7.** We feel a need to write down few remarks concerning the Lebesgue theorem (Theorem 2.1). It was published in 1940 but it remained unnoticed until 1967 when Ore's book [Or] appeared. Ore was aware of the importance of the theorem and therefore he included it into his book together with a complete proof (using Lebesgue's theory of the Euler's contributions) and with corrolaries. But only after its application in an Ore and Plummer's [OP] problem on cyclic coloring of plane graphs by Plummer and Toft [PT] the theorem started to attract people. (Note that the *cyclic colouring* is a colouring of vertices of plane graphs in such a way that the vertices incident with the same face receive different colours.) To prove the conjecture of Plummer and Toft [PT], see also [JT], that every 3-connected plane graph $G$ has a cyclic colouring of its vertices with $\Delta^* + 2$ colours, there were attempts to improve some terms in the theorem. Here $\Delta^*$ means the size of the largest face in $G$.

Recall that the unavoidable set by Lebesgue's theorem (see Theorem 2.1) consists of six infinite series of faces, namely series of $(3, b, c)$-triangles for $3 \leq b \leq 6$ and $c \geq 3$, $(4, 4, c)$-triangles for $c \geq 4$, and $(3, 3, 3, d)$-quadrangles for $d \geq 3$, and 126 individual faces. Horňák and Jendroľ [HJ1] reduced two infinite series of $(3, b, c)$-triangles for $5 \leq b \leq 6$ and $c \geq 5$ to finite ones. Namely they proved the existence of an unavoidable set of configurations consisting of four infinite series and 160 individual configurations.
Note that, in general, none of infinite series of $(3, b, c)$-triangles for $3 \leq b \leq 4, c \geq 3$, $(4, 4, c)$-triangles for $c \geq 4$, and $(3, 3, 3, d)$-quadrangles for $k \geq 3$ can be omitted. In [HJ2] Horňák and Jendroľ replaced the serie of $(4, 4, c)$-triangles with few individual terms together with a configuration consisting of a chain of three quadrangles.

On the other side Borodin [Bo14] succesfully reduced individual terms of the theorem to 95 by letting all six infinite series $(3, b, c)$-triangles for $3 \leq b \leq 6$, $(4, 4, c)$-triangles, and $(3, 3, 3, c)$-quadrangles, all with $c \geq 3$. In his paper [Bo14] Borodin posed the following problem: Find the best possible version(s) of Lebesgue's theorem.

For other discussions concerning the Lebesgue theorems readers are recommended to [Bo14], [BoL], and [HJ1]. (For a present situation concerning the conjecture of Plummer and Toft, see e.g. [EHJ].)

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