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On Product Measures in Complete Bornological Locally Convex Spaces*

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Abstract

A construction of product measures in complete bornological locally convex topological vector spaces is given. Two theorems on the existence of the bornological product measure are proved. A Fubini-type theorem is given.

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1 Introduction

Tensor product of vector-valued measures was studied e.g. in [6], [7] and [10]. It is well known that the tensor product of two vector measures need not always exist, even in the case of measures ranged in the same Hilbert space and being the linear mapping (used in its definition) the corresponding inner product, cf. [8]. Several authors have given sufficient conditions for the existence of the tensor product measure, including the case of measures valued in locally convex spaces. In [19], a bilinear integral is defined in the context of locally convex spaces which is related to Bartle integral, cf. [1], and which allows to state the existence of the product measures valued in locally convex spaces under certain conditions. The bornological character of the bilinear integration theory in [19] shows the fitness of making a development of bilinear integration theory in the context of the complete bornological locally convex spaces. Note the paper of Ballvé and Jiménez Guerra, cf. [2], where we can find also a list of reference papers to this problem.

In this paper two theorems on the existence and the integral representation of the bornological product measures are proved, and a Fubini theorem is stated for functions valued in complete bornological locally convex topological vector spaces.

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2 Preliminaries

In this section we collect the needed definitions and results from [12], [13] and [14].

2.1 Complete bornological locally convex spaces

The description of the theory of complete bornological locally convex topological vector spaces (C. B. L. C. S., for short) may be found in [16], [17] and [18].

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be Hausdorff C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, $\mathfrak{B}_{\mathbf{Z}}$.

One of the equivalent definitions of C. B. L. C. S. is to define these spaces as the inductive limits of Banach spaces. Recall that *a Banach disk* in $\mathbf{X}$ is a set which is closed, absolutely convex and the linear span of which is a Banach space. Let us denote by $\mathcal{U}$ the set of all Banach disks in $\mathbf{X}$ such that $U \in \mathfrak{B}_{\mathbf{X}}$. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$,

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U,$$

cf. [17], where $\mathbf{X}_U$ is a linear span of $U \in \mathcal{U}$ and the family $\mathcal{U}$ is directed by inclusion and forms the basis of bornology $\mathfrak{B}_{\mathbf{X}}$ (analogously for $\mathbf{Y}$ and $\mathcal{W}$, $\mathbf{Z}$ and $\mathcal{V}$). The basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has the *vacuum vector* [1] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-, $\mathfrak{B}_{\mathbf{Z}}$ bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ with vacuum vectors $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, $V_0 \in \mathcal{V}$, $V_0 \neq \{0\}$, respectively.

We say that a sequence of elements $\mathbf{x}_n \in \mathbf{X}$, $n \in \mathbb{N}$ (the set of all natural numbers), *converges bornologically* (with respect to the bornology $\mathfrak{B}_{\mathbf{X}}$ with the basis $\mathcal{U}$) to $\mathbf{x} \in \mathbf{X}$, if there exists $U \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathbf{x}_n - \mathbf{x} \in U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$.

Example 2.1 A classical bornology consists of all sets which are bounded in the von Neumann sense, i.e. for a locally convex topological vector space X equipped with a family of seminorms Q, the set B is *bounded* (or belongs to the von Neumann bornology) if and only if for every $q \in Q$ there exists a constant C_q such that $q(x) \leq C_q$ for every $x \in B$.

2.2 Operator spaces

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$ and $\mathcal{V}$. For

[1] in literature we can find also as terms as the *ground state* or *marked element* or *mother wavelet* depending on the context

$(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, we write $(U_1, W_1, V_1) \ll (U_2, W_2, V_2)$ if and only if $U_1 \subset U_2$, $W_1 \supset W_2$, and $V_1 \supset V_2$.

We use Φ, Ψ, Γ to denote the *classes of all functions* $\mathcal{U} \to \mathcal{W}$, $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$ *with orders* $<_\Phi$, $<_\Psi$, $<_\Gamma$ defined as follows: for $\varphi_1, \varphi_1 \in \Phi$ we write $\varphi_1 <_\Phi \varphi_2$ whenever $\varphi_1(U) \subset \varphi_2(U)$ for every $U \in \mathcal{U}$ (analogously for $<_\Psi$, $<_\Gamma$ and $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$, respectively).

Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$. Analogously, $L(\mathbf{Y}, \mathbf{Z}) \subset \Psi$ and $L(\mathbf{X}, \mathbf{Z}) \subset \Gamma$. The bornologies $\mathfrak{B}_\mathbf{X}$, $\mathfrak{B}_\mathbf{Y}$, $\mathfrak{B}_\mathbf{Z}$ are supposed to be stronger than the corresponding von Neumann bornologies, i.e. the vector operations on the spaces $L(\mathbf{X}, \mathbf{Y})$, $L(\mathbf{Y}, \mathbf{Z})$, $L(\mathbf{X}, \mathbf{Z})$ are compatible with the topologies, and the bornological convergence implies the topological convergence.

2.3 Set functions

Let T and S be two non-void sets. Let Δ and ∇ be two δ-rings of subsets of sets T and S, respectively. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ (resp. $\delta(\mathcal{A})$) denotes the σ-ring (resp. δ-ring) generated by the system $\mathcal{A}$. Denote by $\Sigma = \sigma(\Delta)$ and $\Xi = \sigma(\nabla)$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0, \infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$, i.e. $p_U = \inf_{\mathbf{x} \in \lambda U} |\lambda|$ (if U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = \infty$.). Similarly, p_W and p_V denotes the Minkowski functionals of the sets $W \in \mathcal{W}$ and $V \in \mathcal{V}$, respectively.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0, \infty]$ *a* (U, W)*-semi-variation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$, given as

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i \right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in \mathbf{X};\ \mathbf{x}_i \in U, i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \dots, I\}$. It is well-known that $\hat{\mathbf{m}}_{U,W}$, is a submeasure, i.e. a monotone, subadditive set function, and $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W}$ *a scalar* (U, W)*-semivariation* of $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$, defined by

$$\|\mathbf{m}\|_{U,W}(E) = \sup p_W \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{U,W}, \quad E \in \Sigma,$$

where $\|L\|_{U,W} = \sup_{\mathbf{x} \in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ \|\lambda_i\| \leq 1, i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \dots, I\}$. Note that the scalar semivariation $\|\mathbf{m}\|_{U,W}$ is also a submeasure.

Analogously, we may define a (W, V)-semivariation $\hat{\mathbf{l}}_{W,V}$ and a scalar (W, V)-semivariation $\|\mathbf{l}\|_{W,V}$ of a charge $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$.

For a more detail description of the basic $L(\mathbf{X},\mathbf{Y})$-measure set structures when both $\mathbf{X}$ and $\mathbf{Y}$ are C. B. L. C. S., cf. [12].

Definition 2.2 Let $(U,W)\in\mathcal{U}\times\mathcal{W}$. Denote by

(a) $\Delta_{U,W}$ the greatest δ-subring of Δ of subsets of finite (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ and $\Delta_{\mathcal{U},\mathcal{W}}=\{\Delta_{U,W};(U,W)\in\mathcal{U}\times\mathcal{W}\}$ the lattice with the order given with inclusions of $U\in\mathcal{U}$ and $W\in\mathcal{W}$, respectively;

(b) $\Delta^u_{U,W}$ the greatest δ-subring of Δ on which the restriction $\mathbf{m}_{U,W}:\Delta^u_{U,W}\to L(\mathbf{X}_U,\mathbf{Y}_W)$ of the measure $\mathbf{m}:\Delta\to L(\mathbf{X},\mathbf{Y})$ is uniformly countable additive, with $\mathbf{m}_{U,W}(E)=\mathbf{m}(E)$, for $E\in\Delta^u_{U,W}$ and $\Delta^u_{\mathcal{U},\mathcal{W}}=\{\Delta^u_{U,W};(U,W)\in\mathcal{U}\times\mathcal{W}\}$ the lattice with the order given with inclusions of $U\in\mathcal{U}$ and $W\in\mathcal{W}$, respectively;

(c) $\Delta^c_{U,W}$ the greatest δ-subring of Δ where $\hat{\mathbf{m}}_{U,W}$ is continuous and $\Delta^c_{\mathcal{U},\mathcal{W}}=\{\Delta^c_{U,W};(U,W)\in\mathcal{U}\times\mathcal{W}\}$ the lattice with the order given with inclusions of $U\in\mathcal{U}$ and $W\in\mathcal{W}$, respectively.

Analogously for $\nabla_{W,V}$, $\nabla^u_{W,V}$, $\nabla^c_{W,V}$, with $(W,V)\in\mathcal{W}\times\mathcal{V}$, and $\nabla_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$, $\nabla^c_{\mathcal{W},\mathcal{V}}$.

Lemma 2.3 *The lattices $\Delta^c_{\mathcal{U},\mathcal{W}}$, $\Delta^u_{\mathcal{U},\mathcal{W}}$ are sublattices of $\Delta_{\mathcal{U},\mathcal{W}}$. Analogously for $\nabla_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$ and $\nabla^c_{\mathcal{W},\mathcal{V}}$.*

Concerning the continuity on $\Delta_{U,W}$, $\nabla_{W,V}$, cf. [20]. Denote by $\Delta_{U,W}\otimes\nabla_{W,V}$ the smallest δ-ring containing all rectangles $A\times B$, $A\in\Delta_{U,W}$, $B\in\nabla_{W,V}$, where $(U,W)\in\mathcal{U}\times\mathcal{W}$, $(W,V)\in\mathcal{W}\times\mathcal{V}$.

If $\mathcal{D}_1$, $\mathcal{D}_2$ are two δ-rings of subsets of T, S, respectively, then clearly $\sigma(\mathcal{D}_1\otimes\mathcal{D}_2)=\sigma(\mathcal{D}_1)\otimes\sigma(\mathcal{D}_2)$. For every $E\in\delta(\mathcal{D}_1\otimes\mathcal{D}_2)$ there exist $A\in\mathcal{D}_1$, $B\in\mathcal{D}_2$, such that $E\subset A\times B$. For $E\subset T\times S$, $s\in S$, put

$$E^s=\{t\in T;\ (t,s)\in E\}.$$

2.4 Measure structures

The Dobrakov integral, cf. [3], is defined in Banach spaces. Since $\mathbf{X}$ and $\mathbf{Y}$ are inductive limits of Banach spaces, there is a natural question whether an integral in C. B. L. C. S. may be defined as a finite sum of Dobrakov integrals in various Banach spaces, the choice of which may depend on the function which we integrate. In [12] it is shown that such an integral may be constructed. The sense of this seemingly complicated theory is that, at the present, this is the only integration theory which completely generalizes the Dobrakov integration to a class of non-metrizable locally convex topological vector spaces. A suitable class of operator measures in C. B. L. C. S. which allow such a generalization is a class of all σ_Φ-additive measures.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$ we say that a charge $\mathbf{m}$ is of *σ-finite (U, W)-semivariation* if there exist sets $E_i \in \Delta_{U,W}$, $i \in \mathbb{N}$, such that $T = \bigcup_{i=1}^{\infty} E_i$.

For $\varphi \in \Phi$, we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$, the charge $\mathbf{m}$ is of *σ-finite $(\mathcal{U}, \varphi(U))$-semivariation.*

We say that a charge $\mathbf{m}$ is *of σ_Φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that for every $U \in \mathcal{U}$ the charge is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive vector measure, if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 2.4 We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *(W, σ)-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W, σ)-additive vector measure.

Let $W \in \mathcal{W}$ and let $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, be a sequence of (W, σ)-additive vector measures. If for every $\varepsilon > 0$, $E \in \Sigma$, $p_W(\nu_n(E)) < \infty$ and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$ such that for every $J \geq J_0$,

$$p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$$

uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures ν_n, $n \in \mathbb{N}$, is *uniformly (W, σ)-additive on* Σ, cf. [15].

Definition 2.5 We say that the family of measures $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly $(\mathcal{W}, \sigma)$-additive on* Σ, if there exists $W \in \mathcal{W}$ such that the family of measures ν_n, $n \in \mathbb{N}$, is uniformly (W, σ)-additive on Σ.

The following definition generalizes the notion of the σ-additivity of an operator valued measure in the strong operator topology in Banach spaces, cf. [3], to C. B. L. C. S.

Definition 2.6 Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a *σ_φ-additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$ the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a *σ_Φ-additive measure* if there exists $\varphi \in \Phi$ such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows, $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ are supposed to be operator valued σ_Φ- and σ_Ψ-additive measures, respectively.

The notation Th. I.8, resp. Th. II.7, resp. Th. III.2, stands for Theorem 8 from the first, resp. Theorem 7 from the second, resp. Theorem 2 from the third part of Dobrakov sequence of papers on integration in Banach spaces, cf. [3],[4] and [5], respectively.

3 Bornological product measure

Definition 3.1 We say that a (bornological) *product measure* of a σ_Φ-additive measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and σ_Ψ-additive measure $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$ (we write $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$), if there exists one and only one σ_Γ-additive measure $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ such that

$$(\mathbf{m} \otimes \mathbf{l})(A \times B)\mathbf{x} = \mathbf{l}(B)\mathbf{m}(A)\mathbf{x}$$

for every $\mathbf{x} \in \mathbf{X}_U$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where there exists $\gamma \in \Gamma$, $\varphi \in \Phi$, $\psi \in \Psi$, such that $\gamma = \psi \circ \varphi$ and $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$.

Remark 3.2 From the Hahn-Banach theorem and the uniqueness of enlarging of the finite scalar measure from the ring to the generated σ-ring, there is implied that if

$$\mathbf{n}_1, \mathbf{n}_2 : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V),$$

are two σ_γ-additive measures ($\gamma \in \Gamma$) such that $\mathbf{n}_1(A \times B) = \mathbf{n}_2(A \times B)$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, then $\mathbf{n}_1 = \mathbf{n}_2$ on $\Delta_{U,W} \otimes \nabla_{W,V}$.

Remark 3.3 Definition 3.1 differs from that of Dobrakov [5], Definition 1, in reduction to Banach spaces. Instead of the general $\Delta \otimes \nabla$ we deal only with $\Delta_{U,W} \otimes \nabla_{W,V}$, $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$. In fact, only our case is needed for proving the Fubini theorem in [5].

Remark 3.4 Let $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Then

$$\begin{aligned} (U_1, W_1) \ll (U_2, W_2) \quad &\Rightarrow \quad \Delta_{U_2,W_2} \subset \Delta_{U_1,W_1}, \\ (W_1, V_1) \ll (W_2, V_2) \quad &\Rightarrow \quad \Delta_{W_2,V_2} \subset \Delta_{W_1,V_1}. \end{aligned}$$

In general, for a fixed $W \in \mathcal{W}$,

$$(U_1, V_1) \ll (U_2, V_2) \Rightarrow \Delta_{U_2,W} \otimes \nabla_{W,V_2} \subset \Delta_{U_1,W} \otimes \nabla_{W,V_1}$$

and we may say nothing about the uniqueness, the existence, etc. of $W \in \mathcal{W}$. However, we guarantee the uniqueness of the measure in the case if it exists.

Lemma 3.5 *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$ such that $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$. If for every $\mathbf{x} \in \mathbf{X}_U$ there exists a $\mathbf{Z}_V$-valued vector measure $\mathbf{n}_\mathbf{x}$ on $\Delta_{U,W} \otimes \nabla_{W,V}$, such that*

$$\mathbf{n}_\mathbf{x}(A \times B) = \mathbf{l}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$$

for every $A \in \Delta_{U,W}$ and $B \in \nabla_{W,V}$, then the product measure $\mathbf{m} \otimes \mathbf{l}$ exists on $\Delta \otimes \nabla$.

Proof. For $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$ put

$$(\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V})(E)\mathbf{x} = \mathbf{n}_{\mathbf{x}}(E).$$

We have to prove that

(a) $\mathbf{n}_{\alpha\,\mathbf{x}_1+\beta\,\mathbf{x}_2}(E) = \alpha\,\mathbf{n}_{\mathbf{x}_1}(E) + \beta\,\mathbf{n}_{\mathbf{x}_2}(E)$, and

(b) $\lim_{\mathbf{x}\to\mathbf{0}} \mathbf{n}_{\mathbf{x}}(E) = 0$,

for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$, $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_U$ and all scalars $\alpha, \beta \in \mathbb{K}$.

Denote by $\mathcal{R}$ the ring of all finite unions of rectangulars of the form $A \times B$, where $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$. Denote by

$$\mathbf{var}_V(z'\mathbf{n}_{\mathbf{x}}, \cdot) : \Delta_{U,W} \otimes \nabla_{W,V} \to [0, \infty]$$

the variation of the real measure $z'\mathbf{n}_{\mathbf{x}} : \Delta_{U,W} \otimes \nabla_{W,V} \to [0, \infty]$, for $z' \in V^0$ where V^0 is the polar of the set $V \in \mathcal{V}$. We will use the following fact:

(c) Let $z' \in V^0$ and $E \in \Delta_{U,W} \otimes \nabla_{W,V}$. Then the inequality

$$|\langle \mathbf{n}_{\mathbf{x}}(E_1) - \mathbf{n}_{\mathbf{x}}(E_2), z' \rangle| \le \mathbf{var}_V(z'\mathbf{n}_{\mathbf{x}}, E_1 \triangle E_2),$$

for $E_1, E_2 \in \Delta_{U,W} \otimes \nabla_{W,V}$, and [11], Theorem D, § 13, imply that for every $\varepsilon > 0$ there exists a set $F \in \mathcal{R}$, such that

$$|\langle \mathbf{n}_{\mathbf{x}}(E) - \mathbf{n}_{\mathbf{x}}(F), z' \rangle| < \varepsilon.$$

Let $\alpha, \beta, \mathbf{x}_1, \mathbf{x}_2$ be given. Then (a) holds for $E \in \mathcal{R}$ since $\mathbf{n}_{\mathbf{x}}(A \times B) = \mathbf{l}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, the values $\mathbf{l}_{W,V} \otimes \mathbf{m}_{U,W}$ are linear operators and $\mathbf{n}_{\mathbf{x}}$ is an additive function. From (c) and the Hahn-Banach theorem for Banach spaces it follows that (a) holds for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$.

To show that (b) holds, let $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and consider $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, such that $E \subset A \times B$. Let $F \in \mathcal{R} \cap (A \times B)$. Without loss of generality we may suppose that

$$F = \bigcup_{i=1}^{r} (A_i \times B_i), \quad \text{where } A_i \in \Delta_{U,W},\ B_i \in \nabla_{W,V},$$

and B_i are pairwise disjoint, $i = 1, 2, \ldots, r$. But then

$$\begin{aligned}|\langle \mathbf{n}_{\mathbf{x}}(F), z' \rangle| &\le p_V(\mathbf{n}_{\mathbf{x}}(F)) = p_V\left(\sum_{i=1}^{r} \mathbf{n}_{\mathbf{x}}(A_i \times B_i)\right) = p_V\left(\sum_{i=1}^{r} \mathbf{l}(B_i)\mathbf{m}(A_i)\mathbf{x}\right)\\ &\le p_U(\mathbf{x}) \cdot \|\mathbf{m}\|_{U,W}(A) \cdot \hat{\mathbf{l}}_{W,V}(B)\end{aligned}$$

for every $z' \in V^0$. Since $B \in \nabla_{W,V}$, the uniform boundedness principle implies that

$$\|\mathbf{m}\|_{U,W}(A) = \sup_{\mathbf{x}\in U} \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(A) = \sup_{\mathbf{x}\in U} \sup_{y'\in W^0} \mathbf{var}_W(y'\mathbf{m}(\cdot)\mathbf{x}, A) < \infty.$$

Thus,

$$\lim_{\mathbf{x}\to\mathbf{0}} |\langle \mathbf{n}_{\mathbf{x}}(F), z'\rangle| = 0$$

uniformly for $F \in \mathcal{R} \cap (A \times B)$ and $z' \in V^0$, $V \in \mathcal{V}$. Using (c) we easily obtain (b) for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$. □

Lemma 3.6 *Let* $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. *Then*

(i) for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ *and every* $\mathbf{x} \in \mathbf{X}_U$ *the function* $s \mapsto \mathbf{m}(E^s)\mathbf{x}$, $s \in S$, *is bounded and* $\nabla_{W,V}$*-measurable;*

(ii) for every $E \in \Delta^u_{U,W} \otimes \nabla_{W,V}$ *the function* $s \mapsto \|\mathbf{m}(E^s)\|_{U,W}$, $s \in S$, *is bounded and* $\nabla_{W,V}$*-measurable;*

(iii) for every $E \in \Delta^c_{U,W} \otimes \nabla_{W,V}$ *the function* $s \mapsto \hat{\mathbf{m}}_{U,W}(E^s)$, $s \in S$, *is bounded and* $\nabla_{W,V}$*-measurable.*

Proof. Let us prove the item (i). Suppose that $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$. Take $A \in \Delta_{U,W}$ and $B \in \nabla_{W,V}$ such that $E \subset A \times B$. Denote by $\mathcal{M}$ the class of all sets $N \in \Delta_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$ for which (i) holds. Then clearly $\mathcal{M}$ contains the ring $\mathcal{R} \cap (A \times B)$, where $\mathcal{R}$ is the ring of all finite unions pairwise disjoint rectangulars $A_1 \times B_1$, for $A_1 \in \Delta_{U,W}, B_1 \in \nabla_{W,V}$. Since

$$\sup_{s\in S} p_W(\mathbf{m}(N^s)\mathbf{x}) \leq \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(A) < \infty,$$

for every $N \in \mathcal{M}$ and since each $\nabla_{W,V}$-measurable function belongs to the closure of the pointwise limits in the topology of $\mathbf{X}_U$, $U \in \mathcal{U}$, the σ-additivity of the measure $\mathbf{m}(\cdot)\mathbf{x}$ on $\Delta_{U,W}$ implies that $\mathcal{M}$ is a monotone class of sets. By [11], Theorem B, § 6, we have that

$$\mathcal{M} = \Delta_{U,W} \otimes \nabla_{W,V} \cap (A \times B),$$

and, therefore, $E \in \mathcal{M}$.

The assertions (ii) and (iii) may be proved analogously using the continuity and finiteness of semivariations $\|\mathbf{m}\|_{U,W}$ on $\Delta^u_{U,W}$ and $\hat{\mathbf{m}}_{U,W}$ on $\Delta^c_{U,W}$, respectively. □

4 Existence theorems

Theorem 4.1 *The product measure* $\mathbf{m}\otimes\mathbf{l} : \Delta\otimes\nabla \to L(\mathbf{X},\mathbf{Z})$ *exists on* $\Delta\otimes\nabla$ *if there exists* $W \in \mathcal{W}$ *such that for every* $(U,V) \in \mathcal{U}\times\mathcal{V}$*, every* $E \in \Delta_{U,W}\otimes\nabla_{W,V}$ *and every* $\mathbf{x} \in \mathbf{X}_U$*, the function* $s \mapsto \mathbf{m}(E^s)\mathbf{x}$*,* $s \in S$*, is* $\nabla_{W,V}$*-integrable. In this case*

$$(\mathbf{m}_{U,W}\otimes\mathbf{l}_{W,V})(E)\mathbf{x} = \int_S \mathbf{m}(E^s)\mathbf{x}\,\mathrm{d}\mathbf{l} \tag{1}$$

for every $E \in \Delta_{U,W}\otimes\nabla_{W,V}$ *and every* $\mathbf{x} \in \mathbf{X}_U$*.*

Proof. Suppose that the product measure $\mathbf{m}\otimes\mathbf{l} : \Delta\otimes\nabla \to L(\mathbf{X},\mathbf{Z})$ exists on $\Delta\otimes\nabla$. Let it hold for the set $W \in \mathcal{W}$ and let $\mathbf{x} \in \mathbf{X}_U$, $(U,V) \in \mathcal{U}\times\mathcal{V}$. Denote by $\mathcal{D}$ the class of all sets $G \in \Delta_{U,W}\otimes\nabla_{W,V}$ for which the function $s \mapsto \mathbf{m}(G^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable and for which the assertion (1) holds. Then clearly $\mathcal{D}$ is a subring of $\Delta_{U,W}\otimes\nabla_{W,V}$ which consists of all rectangulars $A\times B$, where $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$. Show that $\mathcal{D}$ is a δ-ring, cf. [11], Theorem E, § 33.

Let $G_n \in \mathcal{D}$, $n \in \mathbb{N}$ such that $G_n \searrow G$ and let $F \in \sigma(\Delta_{U,W}\otimes\nabla_{W,V})$. Then from the σ-additivity of the vector measure $\mathbf{m}(\cdot)\mathbf{x} : \Delta_{U,W} \to \mathbf{Y}_W$ we have that $\mathbf{m}(G_n^s)\mathbf{x} \to \mathbf{m}(G^s)\mathbf{x}$ for every $s \in S$. So, the function $s \mapsto \mathbf{m}(G^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable. Further, (1) and the σ-additivity of the vector measure

$$(\mathbf{m}_{U,W}\otimes\mathbf{l}_{W,V})(\cdot)\mathbf{x} : \Delta_{U,W}\otimes\nabla_{W,V} \to \mathbf{Z}_V$$

imply that

$$\int_F \mathbf{m}(G_n^s)\mathbf{x}\,\mathrm{d}\mathbf{l} \to (\mathbf{m}\otimes\mathbf{l})(\mathrm{F}\cap\mathrm{G})\mathbf{x},$$

where $F\cap G \in \Delta_{U,W}\otimes\nabla_{W,V}$ for every $F \in \sigma(\Delta_{U,W}\otimes\nabla_{W,V})$. Then the function $s \mapsto \mathbf{m}(G^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable and (1) holds for G. Thus, $G \in \mathcal{D}$ and, therefore, $\mathcal{D}$ is a δ-ring. Since $\mathbf{x} \in \mathbf{X}_U$ is an arbitrary vector, the first and the second assertion of the theorem is proved.

Suppose now that there exists $W \in \mathcal{W}$ such that for the given set $E \in \Delta_{U,W}\otimes\nabla_{W,V}$, every $(U,V) \in \mathcal{U}\times\mathcal{V}$ and $\mathbf{x} \in \mathbf{X}_U$, the function $s \mapsto \mathbf{m}(E^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable. For $\mathbf{x} \in \mathbf{X}_U$ and $E \in \Delta_{U,W}\otimes\nabla_{W,V}$, put $\mathbf{n}_\mathbf{x}(E) = \int_S \mathbf{m}(E^s)\mathbf{x}\,\mathrm{d}\mathbf{l}$. Since $\mathbf{n}_\mathbf{x}(A\times B) = \mathbf{l}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, clearly $\mathbf{n}_\mathbf{x} : \Delta_{U,W}\otimes\nabla_{W,V} \to \mathbf{Z}_V$ is a σ-additive measure. Let $\mathbf{x} \in \mathbf{X}_U$ and suppose that $E_n \in \Delta_{U,W}\otimes\nabla_{W,V}$, $n \in \mathbb{N}$, are pairwise disjoint sets with the union $E = \bigcup_{n=1}^\infty E_n \in \Delta_{U,W}\otimes\nabla_{W,V}$. We have to show that $\mathbf{n}_\mathbf{x}(E) = \bigcup_{n=1}^\infty \mathbf{n}_\mathbf{x}(E_n)$, where the series unconditional V-bornological converges. Take $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$ such that $E \subset A\times B$ and consider the σ-ring $\Delta_{U,W}\otimes\nabla_{W,V}\cap(A\times B)$.

Since the measure $\mathbf{n}_\mathbf{x} : \Delta_{U,W}\otimes\nabla_{W,V}\cap(A\times B) \to \mathbf{Z}_V$ is additive by the Orlicz-Pettis theorem, see [9], IV.10.1, it is sufficient to prove that

$$\langle \mathbf{n}_\mathbf{x}(E), z'\rangle = \sum_{n=1}^\infty \langle \mathbf{n}_\mathbf{x}(E_n), z'\rangle$$

for each $z' \in V^0$, where the series unconditional V-bornological converges.

Let E_n^*, $n \in \mathbb{N}$ be some permutation of the series of the sequence E_n, $n \in \mathbb{N}$ and let $z' \in V^0$. Then for every $n \in \mathbb{N}$ and $U \in \mathcal{U}$, $W \in \mathcal{W}$, we have

$$\begin{aligned}
\left|\left\langle \mathbf{n_x}(E) - \sum_{n=1}^{\infty} \mathbf{n_x}(E_n^*), z' \right\rangle\right| &= \left|\left\langle \mathbf{n_x}\left(\bigcup_{i=n+1}^{\infty} E_i^*\right), z' \right\rangle\right| \\
&= \left|\left\langle \int_S \mathbf{m}\left(\left(\sum_{i=n+1}^{\infty} E_i^*\right)^s\right) \mathbf{x}\,\mathrm{d}\mathbf{l}, \mathrm{z}' \right\rangle\right| \\
&= \left| \int_B \mathbf{m}\left(\left(\sum_{i=n+1}^{\infty} E_i^*\right)^s\right) \mathbf{x}\,\mathrm{d}(\mathrm{z}'\mathbf{l}) \right| \\
&\le \int_S \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}\left(\left(\bigcup_{i=n+1}^{\infty} E_i^*\right)^s\right) \mathrm{d}\mathbf{var}_{\mathrm{W}}(\mathrm{z}'\mathbf{l}, \cdot).
\end{aligned}$$

Since

$$\|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}\left(\left(\bigcup_{i=n+1}^{\infty} E_i^*\right)^s\right) \searrow \emptyset,$$

where $n \to \infty$ for every $s \in S$, from the σ-additivity of the vector measure $\mathbf{m}_{U,W}(\cdot)\mathbf{x} : \Delta_{U,W} \to \mathbf{Y}_W$, we have

$$\|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}\left(\left(\bigcup_{i=n+1}^{\infty} E_i^*\right)^s\right) \le \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(B) < \infty$$

for every $s \in S$, $n \in \mathbb{N}$, and since

$$\mathbf{var}_W(z'\mathbf{l}, B) \le p_{V^0}(z') \cdot \hat{\mathbf{l}}_{W,V}(B) < \infty,$$

by the Lebesgue dominated convergence theorem we get

$$\int_S \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}\left(\left(\bigcup_{i=n+1}^{\infty} E_i^*\right)^s\right) \mathrm{d}\mathbf{var}_{\mathrm{W}}(\mathrm{z}'\mathbf{l}, \cdot) \to 0 \quad \text{as} \quad \mathrm{n} \to \infty.$$

Thus,

$$\sum_{n=1}^{\infty} \langle \mathbf{n_x}(E_i^*), z' \rangle \to \langle \mathbf{n_x}(E), z' \rangle.$$

The theorem is proved. □

Remark 4.2 For Fréchet spaces Theorem 4.1 holds also in the inverse direction, i.e. it gives the necessary and sufficient condition of the existence of the bornological product measure $\mathbf{m} \otimes \mathbf{l}$.

Let $\mathbf{g} : S \to \mathbf{Y}_W$ be a $\nabla_{W,V}$-measurable function and define the submeasure $\mathbf{l}_{W,V}(\mathbf{g}, B)$ for $B \in \sigma(\nabla_{W,V})$ by the equality

$$\begin{aligned}&\mathbf{l}_{W,V}(\mathbf{g}, B)\\ &= \sup\left\{p_V\left(\int_B \mathbf{h}\,\mathrm{d}\mathbf{l}\right); \mathbf{h} \in \sigma(\nabla_{W,V}, \mathbf{Y}_W), s \in S : p_W(\mathbf{h}(s)) \le p_W(\mathbf{g}(s))\right\}.\end{aligned}$$

Let us denote by $L^1_{W,V}(\mathbf{l})$ the space of all integrable functions with the bounded and continuous seminorm $\mathbf{l}_{W,V}(\cdot, B)$.

Let us recall Th. II.1, II.2, II.3, II.5, II.6, and moreover, when dealing with $\nabla_{W,V}$-measurable functions in paper [4], then also Th. II.16 and II.17. These facts we will use freely.

From Theorem 4.1 and definitions we easily obtain the following theorem.

Theorem 4.3 *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V} : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V)$ exists. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and let $\mathbf{f} : T \otimes S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function. Then*

$$\|\mathbf{m} \otimes \mathbf{l}\|_{U,V}(E) \le \hat{\mathbf{l}}_{W,V}(\|\mathbf{m}\|_{U,W}(E^s), S)$$

and

$$(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}(\mathbf{f}, E) \le \hat{\mathbf{l}}_{W,V}(\hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s), E^s), S).$$

In the special case of $E = A \times B$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, we have

$$\|\mathbf{m} \otimes \mathbf{l}\|_{U,V}(A \times B) \le \|\mathbf{m}\|_{U,W}(A) \cdot \hat{\mathbf{l}}_{W,V}(B) < \infty$$

and

$$(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}(A \times B) \le \hat{\mathbf{m}}_{U,W}(A) \cdot \hat{\mathbf{l}}_{W,V}(B).$$

Thus $(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}$ is a finite set function on $\Delta_{U,W} \otimes \nabla_{W,V}$.

Theorem 4.4 *Let $U \in \mathcal{U}$, $W \in \mathcal{W}$ and $V \in \mathcal{V}$. Then*

(i) the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ exists on $\Delta_{U,W} \otimes \nabla^c_{W,V}$;

(ii) $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ is a σ-additive measure in the u-(U,V)-operator bornology on $\Delta^u_{U,W} \otimes \nabla^c_{W,V}$;

(iii) the semivariation $(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}$ is continuous on $\Delta^c_{U,W} \otimes \nabla^c_{W,V}$.

Proof. (i) Let $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$. Lemma 3.6(i) implies that the function $s \mapsto \mathbf{m}(E^s)\mathbf{x}$, $s \in S$, is bounded and $\nabla^c_{W,V}$-measurable. Since

$$\{s \in S;\, \mathbf{m}(E^s)\mathbf{x} \ne 0\} \in \nabla^c_{W,V}$$

and the semivariation $\hat{\mathbf{l}}_{W,V}$ is continuous on $\nabla^c_{W,V}$, the function $s \mapsto \mathbf{m}_{U,W}(E^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable. Since $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$ are arbitrary, by Theorem 4.1 the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ exists on $\Delta_{U,W} \otimes \nabla_{W,V}$.

(ii) It is easy to see that the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ is u-(U,V)-σ-additive on $\Delta^u_{U,W} \otimes \nabla^c_{W,V}$ if and only if $E_n \in \Delta^u_{U,W} \otimes \nabla^c_{W,V}$, $n \in \mathbb{N}$, and $E_n \searrow \emptyset$ implies that $\|\mathbf{m} \otimes \mathbf{l}\|_{U,V}(E_n) \searrow 0$.

Let $E_n \in \Delta^u_{U,W} \otimes \nabla^c_{W,V}$, $n \in \mathbb{N}$ and $E_n \searrow \emptyset$. By Lemma 3.6(ii) the functions $s \mapsto \|\mathbf{m}\|_{U,W}(E^s_n)$, $s \in S$, $n \in \mathbb{N}$, are W-bounded and $\nabla^c_{W,V}$-integrable. Since

$$\{s \in S;\ \|\mathbf{m}\|_{U,W}(E^s_1) \neq 0\} \in \nabla^c_{W,V},$$

they all belong to the class $L^1_{W,V}(\mathbf{l})$.

Since $\mathbf{m}_{U,W}$ is a u-(U,V)-countable additive on $\Delta^u_{U,W}$ and since $E^s_n \in \Delta^u_{U,W}$ for every $s \in S$ and $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \|\mathbf{m}\|_{U,W}(E^s_n) = 0$$

for every $s \in S$. Then by Th. II.17 and Theorem 4.3 we get

$$\|\mathbf{m} \otimes \mathbf{l}\|_{W,V}(E_n) \leq \hat{\mathbf{l}}_{W,V}(\|\mathbf{m}\|_{U,W}(E^s_n), S) \searrow 0.$$

The assertion (iii) may be proved analogously to the second one. □

5 A Fubini-type theorem

Let $W \in \mathcal{W}$ and $(U,V) \in \mathcal{U} \times \mathcal{V}$. Denote by $\widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ the closure of the set $\sigma(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ of all $\Delta_{U,W} \otimes \nabla_{W,V}$-simple integrable functions on $T \times S$ with values in $\mathbf{X}$ in the supremum norm p_U in the Banach space of all U-bounded functions on $T \times S$. For elements from $\widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ the following Fubini-type theorem holds.

Theorem 5.1 *Let $U \in \mathcal{U}$, $W \in \mathcal{W}$ and $V \in \mathcal{V}$. Let the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ exist on $\Delta_{U,W} \otimes \nabla_{W,V}$. Let $\mathbf{f} \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ and let $F \in \Delta_{U,W} \otimes \nabla_{W,V}$ (if $\hat{\mathbf{m}}_{U,W}(T) \cdot \hat{\mathbf{l}}_{W,V}(S) < \infty$, then let $F \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$). Then*

(a) $\mathbf{f}\chi_F$ is a $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable function;

(b) for every $s \in S$ the function $\mathbf{f}(\cdot,s)\chi_F(\cdot,s)$ is $\Delta_{U,W}$-integrable;

(c) for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $s \mapsto \int_{E^s} \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}$, $s \in S$, is $\nabla_{W,V}$-integrable and

$$\int_{E^s} \mathbf{f}\chi_F \,\mathrm{d}(\mathbf{m} \otimes \mathbf{l}) = \int_S \int_{E^s} \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}$$

holds for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

Proof. Let $\mathbf{f}_n \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$, $n \in \mathbb{N}$, be a sequence of functions such that

$$\|\mathbf{f}_n - \mathbf{f}\|_{T\times S,U} \to 0.$$

Take $A_0 \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, such that $F \subset A_0 \times B_0$ (if $\hat{\mathbf{m}}_{U,W}(T) \cdot \hat{\mathbf{l}}_{W,V}(S) < \infty$, take $A_0 \in \sigma(\Delta_{U,W})$, $B \in \sigma(\nabla_{W,V})$). Then $\mathbf{f}_n(t,s) \to \mathbf{f}(t,s)$ for every $(t,s) \in T\times S$. If $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$, then $\mathbf{f}_n\chi_E \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V})$ for every $n \in \mathbb{N}$.

(a) From the definition of the semivariation $(\widehat{\mathbf{m}\otimes\mathbf{l}})_{U,V}$ and Theorem 4.3 we have

$$\begin{aligned}
&p_V\left(\int_E \mathbf{f}_n\chi_F \,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) - \int_E \mathbf{f}_k\chi_F \,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})\right)\\
&= p_V\left(\int_{B\cap F} (\mathbf{f}_n - \mathbf{f}_k)\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})\right)\\
&\le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \cdot (\widehat{\mathbf{m}\otimes\mathbf{l}})_{U,V}(F)\\
&\le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \cdot \hat{\mathbf{m}}_{U,W}(A_0)\cdot \hat{\mathbf{l}}_{W,V}(B_0)
\end{aligned}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and every $n,k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0)\cdot \hat{\mathbf{l}}_{W,V}(B_0) < \infty$, we obtain that $\mathbf{f}\chi_F$ is a $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable function and

$$\int_E \mathbf{f}_n\chi_F \,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) \to \int_E \mathbf{f}\chi_F \,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

(b) Let $s \in S$. Then

$$\begin{aligned}
&p_V\left(\int_A \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m} - \int_A \mathbf{f}_k(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\right)\\
&\le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \cdot \hat{\mathbf{m}}_{U,W}(A_0)
\end{aligned}$$

for every $A \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $n,k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0) < \infty$, then by Th. I.7 the function $\mathbf{f}(\cdot,s)\chi_F(\cdot,s)$ is $\Delta_{U,W}$-integrable and we have

$$\int_A \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m} \to \int_A \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}$$

for every $A \in \sigma(\Delta_{U,W})$. In particular,

$$\int_{E^s} \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m} \to \int_{E^s} \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

(c) Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Then using Th.I.14., we get

$$\begin{aligned}
&p_V\left(\int_B \int_{E^s} \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l} - \int_B \int_{E^s} \mathbf{f}_k(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}\right)\\
&\leq \sup_{x\in B_0} p_W\left(\int_{E^s} (\mathbf{f}_n(\cdot,s) - \mathbf{f}_k(\cdot,s))\,\mathrm{d}\mathbf{m}\right)\cdot \hat{\mathbf{l}}_{W,V}(B_0) \qquad (2)\\
&\leq \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U}\cdot \hat{\mathbf{m}}_{U,W}(A_0)\cdot \hat{\mathbf{l}}_{W,V}(B_0)
\end{aligned}$$

for every $B_0 \in \sigma(\nabla_{W,V})$ and $n,k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0)\cdot \hat{\mathbf{l}}_{W,V}(B_0) < \infty$, the relations (1) and (2) imply according to Th. I.16 ($\|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \to 0$ whenever $n,k \in \mathbb{N}$) that the function $s \mapsto \int_{E^s} \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}$, $s \in S$, is $\nabla_{W,V}$-integrable and, therefore,

$$\int_S \int_{E^s} \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l} \to \int_S \int_{E^s} \mathbf{f}(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}.$$

It is enough to note that by Theorem 4.1 there holds

$$\int_E \mathbf{f}_n\chi_F\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) = \int_S \int_{E^s} \mathbf{f}_n(\cdot,s)\chi_F(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $n \in \mathbb{N}$. The proof is complete. □

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