Oscillation Criteria for Second Order Non-Linear Differential Equation

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Abstract

The paper offers effective oscillation criteria for solutions of the second order non-linear differential equation $(r(t)g(x(t))x'(t))' + Q(t)f(x(t)) = 0$, where $r, Q : [t_0, \infty) \to (0, \infty)$, $g : \mathbb{R} \to (0, \infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions.

1 Introduction

We consider the second order non-linear differential equation

$$(r(t)g(x(t))x'(t))' + Q(t)f(x(t)) = 0, \quad t \geqslant t_0 \geqslant 0. \tag{1}$$

Throughout the paper we assume that

(i) $r(t) \in C([t_0, \infty))$, $r(t) > 0$ for $t \geqslant t_0$,

(ii) $f(x) \in C(\mathbb{R})$, $xf(x) > 0$ for $x \neq 0$,

(iii) $g(x) \in C(\mathbb{R})$, $g(x) > 0$ for $x \in \mathbb{R}$,

(iv) $Q(t) \in C([t_0, \infty))$, $Q(t) > 0$ for $t \geqslant t_0$.

By a solution of an equation of the form (1) we mean a function $x : [t_0, \infty) \to \mathbb{R}$ with the property $x(t) \in C^2([t_0, \infty))$ which satisfies equation (1) for all $t \in [t_0, \infty)$. We consider only non-trivial solutions of (1). A solution $x(t)$ of (1) is said to be oscillatory if there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ of the points of the interval $[t_0, \infty)$ such that $\lim_{n\to\infty} \tau_n = \infty$ and $x(\tau_n) = 0$, $n \in \mathbb{N}$, otherwise it is said to be non-oscillatory. An equation (1) is said to be oscillatory if all its solutions are oscillatory, otherwise it is said to be non-oscillatory.

It is known that many problems in physics, in the study of chemically reacting systems, in celestial mechanics and in other fields of science can be modeled by second order non-linear differential equations. Therefore the asymptotic and oscillatory properties of solutions of such equations have been investigated by many authors, see e.g. [1]–[8] and references cited therein.

Investigation of the differential equation (1) in this paper is motivated by the paper [3], where some of the conditions required in the theorems contain the unknown solution $x(t)$. It seems that any verification of such conditions is questionable and may be impossible. The purpose of the paper is to remove the above mentioned conditions that depend on solution and improve some results presented in [3] in this way. The relevance of the theorems in the text is illustrated by included examples.

In the last time increases the number of papers which involve oscillatory criteria based on the idea of using of the parameter functions $H(t,s)$ (see e.g. [5], [6]). These results have great teoretical value but they are less efective in aplications. On the other hand, the results which contain the requirements only on the functions ocurring in differential equation are usually better applicable. The paper contains only results of the latter kind.

2 Bounded and oscillatory solutions

We start with the following conditions for the existence of bounded solutions of equation (1).

Theorem 2.1 *Suppose $r(t)Q(t) \in C^1([t_0,\infty))$ and $(r(t)Q(t))' \geqslant 0$ for $t_0 \geqslant t$. Let*

$$\text{(H1)} \qquad \int^{\pm\infty} f(z)g(z)\,\mathrm{d}z = \infty.$$

Then every solution $x(t)$ of (1) such that $x(t_1) = 0$ for some $t_1 \in [t_0,\infty)$ is bounded.

Proof: Assume $x(t)$ is arbitrary solution of (1) such that $x(t_1) = 0$, $t_1 \geqslant t_0$. Put

$$F(\alpha) = \int_{x(t_1)}^{\alpha} f(z)g(z)\,\mathrm{d}z.$$

Multiplying both sides of (1) by $r(t)g(x(t))x'(t)$ and integrating from t_1 to t, we obtain

$$\text{(2)} \qquad \frac{1}{2}(r(t)g(x(t))x'(t))^2 + r(t)Q(t)F(x(t)) - \int_{t_1}^{t} F(x(s))(r(s)Q(s))'\,\mathrm{d}s$$
$$= \frac{1}{2}(r(t_1)g(x(t_1))x'(t_1))^2, \quad t \geqslant t_1.$$

Denote

$$k = \frac{1}{2}(r(t_1)g(x(t_1))x'(t_1))^2.$$

Now by (2) it follows that

$$r(t)Q(t)F(x(t)) = k - \frac{1}{2}(r(t)g(x(t))x'(t))^2 + \int_{t_1}^{t} F(x(s))(r(s)Q(s))' \,\mathrm{d}s$$
$$\leqslant k + \int_{t_1}^{t} F(x(s))(r(s)Q(s))' \,\mathrm{d}s, \quad t \geqslant t_1,$$

i.e.,

$$r(t)Q(t)F(x(t)) \leqslant k + \int_{t_1}^{t} F(x(s))r(s)Q(s)\frac{(r(s)Q(s))'}{r(s)Q(s)} \,\mathrm{d}s, \quad t \geqslant t_1.$$

Hence by Gronwall's lemma it follows that

$$r(t)Q(t)F(x(t)) \leqslant k \cdot \exp \int_{t_1}^{t} \frac{(r(s)Q(s))'}{r(s)Q(s)} \,\mathrm{d}s, \quad t \geqslant t_1. \tag{3}$$

By (3) using assumptions (i) and (iv) it is easy to verify that

$$F(x(t)) \leqslant \frac{k}{r(t_1)Q(t_1)}, \quad t \geqslant t_1,$$

so $F(x(t))$ is bounded and thus by assumption (H1) $x(t)$ is bounded.

□

Now we give two oscillation results for bounded solutions of (1).

Theorem 2.2 *Let*

(H2) $r(t) \in C^1([t_0, \infty))$ *and* $r'(t) \leqslant 0$ *for* $t \geqslant t_0$,

(H3) $g(x) \in C^1(\mathbb{R})$ *and* $xg'(x) \geqslant 0$ *for* $x \neq 0$,

and

(H4) $\displaystyle\int^{\infty} Q(s)\,\mathrm{d}s = \infty.$

Then every bounded solution of equation (1) *is oscillatory.*

Proof: Let $x(t)$ be a bounded non-oscillatory solution of (1). Then there exists $t_1 \in [t_0, \infty)$ such that $x(t)$ has a fixed sign for all $t \in [t_1, \infty)$. According to assumptions $r(t) \in C^1([t_0, \infty))$ and $g(x) \in C^1(\mathbb{R})$ we can write equation (1) in the form

$$r(t)g(x(t))x''(t) + r(t)g'(x(t))x'^2(t) + r'(t)g(x(t))x'(t) + Q(t)f(x(t)) = 0. \tag{4}$$

A. Consider first $x(t) > 0$ for all $t \in [t_1, \infty)$. Suppose $x'(t^*) = 0$ for some $t^* \in [t_1, \infty)$. By (4) using conditions (i)–(iv) we get

$$x''(t^*) = -\frac{Q(t^*)f(x(t^*))}{r(t^*)g(x(t^*))} < 0.$$

This implies that the function $x(t)$ has a local maximum at the point t^*. It is easy to see that $x'(t)$ can not have more than one zero, otherwise we should get a contradiction with the feature $x(t) \in C^2([t_0, \infty))$. Thus there exists $t_2 \in [t_1, \infty)$ such that $x'(t) \neq 0$ for all $t \in [t_2, \infty)$.

1. Suppose $x'(t) > 0$ for all $t \in [t_2, \infty)$. Then there exists $A \in \mathbb{R}$, $A > x(t_2) > 0$ such that $\lim_{t\to\infty} x(t) = A$. Thus $x(t) \in [x(t_2), A]$ for all $t \geqslant t_2$ and function $f(x)$ has a minimum $\varepsilon_1 > 0$ on an interval $[x(t_2), A]$, i.e. $f(x(t)) \geqslant \varepsilon_1$ for all $t \geqslant t_2$. Integrating (1) from t_2 to t we get

$$r(t)g(x(t))x'(t) = r(t_2)g(x(t_2))x'(t_2) - \int_{t_2}^{t} Q(s)f(x(s))\,\mathrm{d}s. \tag{5}$$

Hence

$$\int_{t_2}^{t} Q(s)f(x(s))\,\mathrm{d}s \geqslant \varepsilon_1 \int_{t_2}^{t} Q(s)\,\mathrm{d}s$$

and from (5) it follows

$$0 < r(t)g(x(t))x'(t) \leqslant r(t_2)g(x(t_2))x'(t_2) - \varepsilon_1 \int_{t_2}^{t} Q(s)\,\mathrm{d}s, \quad t \geqslant t_2. \tag{6}$$

Hence by (H4) it follows that for sufficiently large t the right-hand side of the inequality (6) is negative and we have a contradiction.

2. Suppose last $x'(t) < 0$ for all $t \in [t_2, \infty)$. Now using (i)–(iv), (H2) and (H3) on (4) we get

$$x''(t) = \frac{1}{r(t)g(x(t))}\Big[-Q(t)f(x(t)) - r'(t)g(x(t))x'(t) - r(t)g'(x(t))x'^2(t)\Big] < 0,$$

for all $t \geqslant t_2$. Thus $x(t)$ is decreasing and concave on $[t_2, \infty)$, i.e. $\lim_{t\to\infty} x(t) = -\infty$, this contradicts our assumption of $x(t)$ to be positive on $[t_1, \infty)$.

B. Consider now $x(t) < 0$ for all $t \in [t_1, \infty)$. Suppose $x'(t^*) = 0$ for some $t^* \in [t_1, \infty)$. Similarly as above using conditions (i)–(iv) on (4) we get

$$x''(t^*) = -\frac{Q(t^*)f(x(t^*))}{r(t^*)g(x(t^*))} > 0.$$

This implies that the function $x(t)$ has a local minimum at the point t^*. Similarly as above we can say that $x'(t)$ has at most one such point t^*. Thus there exists $t_2 \in [t_1, \infty)$ such that $x'(t) \neq 0$ for all $t \in [t_2, \infty)$.

1. Suppose that $x'(t) > 0$ for all $t \in [t_2, \infty)$. Using assumptions (i)–(iv), (H2) and (H3) on (4) we get

$$x''(t) = \frac{1}{r(t)g(x(t))}\Big[-Q(t)f(x(t)) - r'(t)g(x(t))x'(t) - r(t)g'(x(t))x'^2(t)\Big] > 0,$$

for all $t \geqslant t_2$. Thus $x(t)$ is increasing and convex on $[t_2, \infty)$, i.e. $\lim_{t\to\infty} x(t) = \infty$, this contradicts the assumption of $x(t)$ to be negative on $[t_1, \infty)$.
2. Suppose $x'(t) < 0$ for all $t \in [t_2, \infty)$. Then there is $B \in \mathbb{R}$, $B < x(t_2) < 0$ such that $\lim_{t\to\infty} x(t) = B$. Hence $x(t) \in [B, x(t_2)]$ for all $t \geqslant t_2$ and function $f(x)$ has a maximum $\varepsilon_2 < 0$ on an interval $[B, x(t_2)]$, i.e. $f(x(t)) \leqslant \varepsilon_2$ for all $t \geqslant t_2$. Integrating (1) from t_2 to t similarly as was shown above we obtain

$$0 > r(t)g(x(t))x'(t) \geqslant r(t_2)g(x(t_2))x'(t_2) - \varepsilon_2 \int_{t_2}^{t} Q(s)\,\mathrm{d}s, \quad t \geqslant t_2. \tag{7}$$

By (H4) it follows that for sufficiently large t the right-hand side of the inequality (7) is positive and it is a contradiction.
Hence every bounded solution of (1) is oscillatory.

□

The following example is illustrative:
Example 1. We consider the differential equation

$$\big((x^2(t)+1)x'(t)\big)' + \frac{x^3(t)}{3} + x(t) = 0, \quad t \geqslant 0. \tag{8}$$

This is the equation of the form (1), where $r(t) \equiv 1$, $Q(t) \equiv 1$, $g(x) = x^2 + 1$ and $f(x) = \frac{x^3}{3} + x$ and it is easy to verify that conditions of Theorem 2.2 are satisfied and so all bounded solutions of (8) are oscillatory. One of them is the function

$$x(t) = \frac{\sqrt[3]{12\cos t + 4\sqrt{4 + 9\cos^2 t}}}{2} - \frac{2}{\sqrt[3]{12\cos t + 4\sqrt{4 + 9\cos^2 t}}}.$$

■

Theorem 2.3 *Assume that* (H4) *is satisfied and*

(H3*) $\quad g(x) \in C^1(\mathbb{R})$,

(H5) $\quad r(t) \in C^1([t_0, \infty))$ *and* $r(t)$ *is bounded on* $[t_0, \infty)$.

Then every bounded solution $x(t)$ *of* (1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof: Assume on the contrary that $x(t)$ is bounded non-oscillatory solution of (1) such that $\lim_{t\to\infty} x(t) = 0$ is not valid. Then there exists $t_1 \in [t_0, \infty)$ such that $x(t)$ has a fixed sign for all $t \in [t_1, \infty)$. The part of the proof is the same as the proof of Theorem 2.2 excepting the cases A.2. and B.1.
Case A.2. Suppose $x(t) > 0$ for $t \geqslant t_1$ and $x'(t) < 0$ for $t \geqslant t_2 \geqslant t_1$. So $x(t)$ is bounded, positive and decreasing function on $[t_2, \infty)$. Hence there exists $C \in \mathbb{R}$, $x(t_2) > C \geqslant 0$ such that $\lim_{t\to\infty} x(t) = C$. Thus $x(t) \in [C, x(t_2)]$ for all $t \geqslant t_2$ and function $f(x)$ has a minimum $\varepsilon_3 \geqslant 0$ on an interval $[C, x(t_2)]$ but

considering $\lim_{t\to\infty} x(t) \neq 0$ we can say that $\varepsilon_3 > 0$, i.e. $f(x(t)) \geqslant \varepsilon_3$ for all $t \geqslant t_2$. Integrating (1) from t_2 to t thence it follows

$$r(t)g(x(t))x'(t) \leqslant r(t_2)g(x(t_2))x'(t) - \varepsilon_3 \int_{t_2}^{t} Q(s)\,\mathrm{d}s, \quad t \geqslant t_2. \tag{9}$$

Since function $r(t)g(x(t))x'(t)$ is decreasing then by (H4)

$$\lim_{t\to\infty} r(t)g(x(t))x'(t) = -\infty$$

and because $r(t)$ and $g(x(t))$ are bounded on $[t_2, \infty)$ then $\lim_{t\to\infty} x'(t) = -\infty$. This yields $\lim_{t\to\infty} x(t) = -\infty$, this contradicts $x(t) > 0$ for all $t \geqslant t_1$.
Case B.1. Suppose that $x(t) < 0$ for $t \in [t_1, \infty)$ and $x'(t) > 0$ for all $t \in [t_2, \infty)$. So $x(t)$ is on $[t_2, \infty)$ bounded, negative and increasing and thus there exists $D \in \mathbb{R}$, $x(t_2) < D \leqslant 0$ such that $\lim_{t\to\infty} x(t) = D$. Therefore $x(t) \in [x(t_2), D]$ for all $t \geqslant t_2$ and function $f(x)$ has a maximum $\varepsilon_4 < 0$ (considering $\lim_{t\to\infty} x(t) \neq 0$) on an interval $[x(t_2), D]$, i.e. $f(x(t)) \leqslant \varepsilon_4$ for $t \geqslant t_2$. Similarly as in the case A.2. we get

$$r(t)g(x(t))x'(t) \geqslant r(t_2)g(x(t_2))x'(t) - \varepsilon_4 \int_{t_2}^{t} Q(s)\,\mathrm{d}s, \quad t \geqslant t_2. \tag{10}$$

Since function $r(t)g(x(t))x'(t)$ is increasing then by (H4)

$$\lim_{t\to\infty} r(t)g(x(t))x'(t) = \infty$$

and hence by (iii) and (H5) it follows $\lim_{t\to\infty} x'(t) = \infty$ and it yields $x(t) \to \infty$ as $t \to \infty$ which contradicts $x(t) < 0$ for all $t \geqslant t_1$ and the proof is finished.

□

The following example illustrates the meaning of Theorem 2.3.
Example 2. We consider the differential equation

$$\left(e^{-\sin t} e^{x(t)} x'(t)\right)' + x(t) = 0, \quad t \geqslant 0. \tag{11}$$

This is the equation of the form (1), where $r(t) = e^{-\sin t}$, $Q(t) \equiv 1$, $g(x) = e^{x}$ and $f(x) = x$. In this case conditions (H2) and (H3) are not valid and thus Theorem 2.2 is not applicable. But it is easy to see that conditions of Theorem 2.3 hold and hence we can say that each bounded solution of equation (11) is either oscillatory or tends to 0. Note that one solution of equation (11) is the function $x(t) = \sin t$.

■

3 Other oscillation criteria

Now we present some sufficient conditions for (1) to be oscillatory. For this we introduce the following notation

$$R(t) = \int_{t_0}^{t} \frac{1}{r(s)}\, \mathrm{d}s,\ t \geqslant t_0. \tag{12}$$

Theorem 3.1 *Let*

(H7) $f(x) \in C^1(\mathbb{R})$

and let there exist some positive constant c, such that

(H8) $\dfrac{f'(x)}{g(x)} \geqslant c$, *for* $x \neq 0$.

Furthermore let for some integer $n \geqslant 3$

(H9) $\displaystyle \limsup_{t\to\infty} \frac{1}{R^{n-1}(t)} \int_{t_0}^{t} [R(t) - R(s)]^{n-1} Q(s)\, \mathrm{d}s = \infty.$

Then equation (1) *is oscillatory.*

Proof: Assume that $x(t)$ is a non-oscillatory solution of (1). Then there exists $T \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant T$. Define

$$W(t) = \frac{r(t)g(x(t))x'(t)}{f(x(t))},\ t \geqslant T. \tag{13}$$

Thus by (1) it follows that

$$W'(t) + \frac{W^2(t) f'(x(t))}{r(t)g(x(t))} + Q(t) = 0,\ t \geqslant T. \tag{14}$$

Hence for all $s \in [T, t]$ it can be writen as

$$\int_T^t [R(t) - R(s)]^{n-1} Q(s)\, \mathrm{d}s = -\int_T^t [R(t) - R(s)]^{n-1} W'(s)\, \mathrm{d}s$$
$$- \int_T^t [R(t) - R(s)]^{n-1} \frac{W^2(s) f'(x(s))}{r(s)g(x(s))}\, \mathrm{d}s.$$

Thus by (H8) it follows that for $t \geqslant T$

$$\int_T^t [R(t) - R(s)]^{n-1} Q(s)\, \mathrm{d}s \leqslant -\int_T^t [R(t) - R(s)]^{n-1} W'(s)\, \mathrm{d}s$$
$$-c \int_T^t [R(t) - R(s)]^{n-1} \frac{W^2(s)}{r(s)}\, \mathrm{d}s,$$

i.e.,

$$\int_T^t [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s \leqslant [R(t)-R(T)]^{n-1}W(T) \tag{15}$$
$$-(n-1)\int_T^t [R(t)-R(s)]^{n-2}\frac{W(s)}{r(s)}\,\mathrm{d}s - c\int_T^t [R(t)-R(s)]^{n-1}\frac{W^2(s)}{r(s)}\,\mathrm{d}s.$$

Now from (15) we get for $t \geqslant T$

$$\int_T^t [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s \leqslant [R(t)-R(T)]^{n-1}W(T)$$
$$+\frac{(n-1)^2}{4c}\int_T^t [R(t)-R(s)]^{n-3}\frac{1}{r(s)}\,\mathrm{d}s$$
$$-\int_T^t \frac{[R(t)-R(s)]^{n-3}}{r(s)}\left\{\sqrt{c}[R(t)-R(s)]W(s)+\frac{n-1}{2\sqrt{c}}\right\}^2\mathrm{d}s,$$

then we have

$$\frac{1}{R^{n-1}(t)}\int_T^t [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s \leqslant \left[1-\frac{R(T)}{R(t)}\right]^{n-1}W(T) \tag{16}$$
$$+\frac{(n-1)^2}{4c(n-2)R(t)}\left[1-\frac{R(T)}{R(t)}\right]^{n-2},\ t \geqslant T.$$

It is clear that for $t \in [t_0, T]$ it holds

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^T [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s \leqslant \frac{1}{R^{n-1}(t)}\int_{t_0}^T R^{n-1}(t)Q(s)\,\mathrm{d}s \tag{17}$$
$$= \int_{t_0}^T Q(s)\,\mathrm{d}s.$$

Combining (16) and (17) it follows that

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^t [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s \leqslant \left[1-\frac{R(T)}{R(t)}\right]^{n-1}W(T)$$
$$+\frac{(n-1)^2}{4c(n-2)R(t)}\left[1-\frac{R(T)}{R(t)}\right]^{n-2}+\int_{t_0}^T Q(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

It is clear that $\lim_{t\to\infty}\frac{1}{R(t)} = L \in [0,\infty)$. Thus

$$\limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^t [R(t)-R(s)]^{n-1}Q(s)\,\mathrm{d}s$$
$$\leqslant (1-LR(T))^{n-1}W(T)+\frac{L(n-1)^2}{4c(n-2)}(1-LR(T))^{n-2}+\int_{t_0}^T Q(s)\,\mathrm{d}s < \infty.$$

It is contradiction with condiction (H9). The proof of the theorem is complete. $\square$

Theorem 3.2 *Let* (H7) *be satisfied and let there exist positive constant* c *such that* (H8) *is fullfiled. Suppose, furthermore, that*

$$\text{(H10)}\qquad \limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n Q(s) - \frac{n^2}{4c}(t-s)^{n-2} r(s)\right] \mathrm{d}s = \infty,$$

for some integer $n \geqslant 2$. *Then equation* (1) *is oscillatory.*

Proof: Let $x(t)$ be a non-oscillatory solution of (1). Then there exists $T \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant T$. Taking $W(t)$ as it is defined in (13) we obtain (14). Hence by (H8) for $t \geqslant s \geqslant T$ we can write

$$\int_T^t (t-s)^n Q(s)\,\mathrm{d}s \leqslant -\int_T^t (t-s)^n W'(s)\,\mathrm{d}s - c\int_T^t (t-s)^n \frac{W^2(s)}{r(s)}\,\mathrm{d}s,$$

i.e.,

$$\begin{aligned}
&\int_T^t (t-s)^n Q(s)\,\mathrm{d}s \\
&\qquad \leqslant (t-T)^n W(T) - n\int_T^t (t-s)^{n-1} W(s)\,\mathrm{d}s - c\int_T^t (t-s)^n \frac{W^2(s)}{r(s)}\,\mathrm{d}s \\
&\qquad = (t-T)^n W(T) - \int_T^t \left[\sqrt{\frac{c(t-s)^n}{r(s)}}W(s) + \frac{n}{2}\sqrt{\frac{r(s)(t-s)^{n-2}}{c}}\right]^2 \mathrm{d}s \\
&\qquad + \frac{n^2}{4c}\int_T^t r(s)(t-s)^{n-2}\,\mathrm{d}s.
\end{aligned}$$

Hence it follows

$$\text{(18)}\qquad \begin{aligned}\int_T^t \left[(t-s)^n Q(s) - \frac{n^2 r(s)}{4c}(t-s)^{n-2}\right]\mathrm{d}s &\leqslant (t-T)^n W(T) \\ &\leqslant (t-t_0)^n W(T), \quad t \geqslant T.\end{aligned}$$

Using the inequality (18) we get for $t \geqslant t_0$

$$\begin{aligned}
&\int_{t_0}^t \left[(t-s)^n Q(s) - \frac{n^2 r(s)}{4c}(t-s)^{n-2}\right]\mathrm{d}s \\
&= \int_{t_0}^T \left[(t-s)^n Q(s) - \frac{n^2 r(s)}{4c}(t-s)^{n-2}\right]\mathrm{d}s \\
&+ \int_T^t \left[(t-s)^n Q(s) - \frac{n^2 r(s)}{4c}(t-s)^{n-2}\right]\mathrm{d}s \\
&\leqslant (t-t_0)^n \int_{t_0}^T Q(s)\,\mathrm{d}s + (t-t_0)^n W(T).
\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n Q(s)-\frac{n^2 r(s)}{4c}(t-s)^{n-2}\right]\mathrm{d}s\leqslant\int_{t_0}^{T}Q(s)\,\mathrm{d}s+W(T)<\infty,$$

and it is contradiction with (H10).

□

For illustration we consider the following example.
Example 3. Consider the differential equation

$$\left(\frac{1}{x^2(t)+1}x'(t)\right)'+\operatorname{arctg}x(t)=0,\quad t\geqslant 0, \tag{19}$$

which satisfies the conditions of Theorem 3.2, since we have

$$\frac{f'(x)}{g(x)}=\frac{\frac{1}{x^2+1}}{\frac{1}{x^2+1}}=1 \text{ for } x\neq 0,$$

and for any integer $n\geqslant 2$ we have

$$\lim_{t\to\infty}\frac{1}{t^n}\int_0^t\left[(t-s)^n-\frac{n^2}{4}(t-s)^{n-2}\right]\mathrm{d}s=\lim_{t\to\infty}\left(\frac{t}{(n+1)}-\frac{n^2}{4(n-1)t}\right)=\infty.$$

It thus follows that every solution of (19) oscillates. One such solution is the function $x(t)=\operatorname{tg}\cos t$.

■

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