Another step towards proving
a conjecture by Plummer and Toft

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Abstract. A cyclic colouring of a graph \( G \) embedded in a surface is a vertex colouring of \( G \) in which any two distinct vertices sharing a face receive distinct colours. The cyclic chromatic number \( \chi_c(G) \) of \( G \) is the smallest number of colours in a cyclic colouring of \( G \). Plummer and Toft in 1987 conjectured that \( \chi_c(G) \leq \Delta^* + 2 \) for any 3-connected plane graph \( G \) with maximum face degree \( \Delta^* \). It is known that the conjecture holds true for \( \Delta^* \leq 4 \) and \( \Delta^* \geq 24 \). The validity of the conjecture is proved in the paper for \( \Delta^* \geq 18 \).

1 Introduction

Let \( G = (V, E, F) \) be a cell-embedding of a 2-connected graph in a 2-manifold. The degree \( \deg(x) \) of \( x \in V \cup F \) is the number of edges incident with \( x \). A vertex of degree \( k \) is a \( k \)-vertex, a face of degree \( k \) is a \( k \)-face. By \( V(x) \) we denote the set of all vertices incident with \( x \in E \cup F \); similarly, \( F(y) \) is the set of all faces incident with \( y \in V \cup E \). If \( e \in E \), \( F(e) = \{ f_1, f_2 \} \) and \( \deg(f_1) \leq \deg(f_2) \), the pair \( (\deg(f_1), \deg(f_2)) \) is called the type of \( e \). A \((d_1, d_2)\)-neighbour of a vertex \( x \) is a vertex \( y \) such that the edge \( xy \) is of type \((d_1, d_2)\). Paths and cycles in \( G \) will be understood as vertex sequences in which any two vertices placed on neighbouring positions are adjacent in \( G \). A cycle in \( G \) is facial if its vertex set is equal to \( V(f) \) for some \( f \in F \). Though graphs we are dealing with are nonoriented, sometimes it will be useful to equip certain edges with one of two possible orientations. A vertex \( x_1 \) is cyclically adjacent to a vertex \( x_2 \neq x_1 \) if there is a face \( f \) with \( x_1, x_2 \in V(f) \). The cyclic neighbourhood \( N_c(x) \) of a vertex \( x \) is the set of all vertices that are cyclically adjacent to \( x \) and the closed cyclic neighbourhood of \( x \) is \( \bar{N}_c(x) := N_c(x) \cup \{x\} \). (The usual neighbourhood of \( x \) is denoted by \( N(x) \).) The cyclic degree of \( x \) is

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cd(x) := |N_c(x)|. A cyclic colouring of G is a mapping \( \varphi : V \rightarrow C \) in which \( \varphi(x_1) \neq \varphi(x_2) \) whenever \( x_1 \) is cyclically adjacent to \( x_2 \) (elements of C are colours of \( \varphi \)). The cyclic chromatic number \( \chi_c(G) \) of the graph G is the minimum number of colours in a cyclic colouring of G.

The invariant \( \chi_c(G) \) was introduced by Ore and Plummer [8] for plane graphs (and in the dual form). Sanders and Zhao [10] proved that \( \chi_c(G) \leq \lfloor \frac{3}{2}\Delta^*(G) \rfloor \) for any 2-connected plane graph G, where \( \Delta^*(G) \) is the maximum face degree of G. On the other hand, there is an infinite family of 2-connected plane graphs \( G \) satisfying \( \chi_c(G) = \lfloor \frac{3}{2}\Delta^*(G) \rfloor \). It is conjectured that \( \chi_c(G) \leq \lfloor \frac{3}{2}\Delta^*(G) \rfloor \) for any 2-connected plane graph G.

However, our interest is concentrated on 3-connected plane graphs. By a classical result of Whitney [11] all plane embeddings of a 3-connected planar graph are essentially the same. This means that \( \chi_c(G_1) = \chi_c(G_2) \) if \( G_1, G_2 \) are plane embeddings of a fixed 3-connected planar graph G; thus, we can speak simply about the cyclic chromatic number of G. On the other hand, when analysing \( \chi_c(G) \) for a 3-connected planar graph G, any edge of G can be chosen to be incident or not to be incident with the unbounded face of an embedding of G in the plane. Plummer and Toft in [9] proved that \( \chi_c(G) \leq \Delta^*(G) + 9 \) and conjectured that \( \chi_c(G) \leq \Delta^*(G) + 2 \) for any 3-connected plane graph G. Let PTC(d) denote that conjecture restricted to graphs with \( \Delta^*(G) = d \). Because of Four Colour Theorem we know that for a triangulation G we have \( \chi_c(G) \leq 4 = \Delta^*(G) + 1 \). PTC(4) is known to be true due to Borodin [2]. Hornák and Jendrol' [6] proved PTC(d) for any \( d \geq 24 \). The bound was moved to 22 by Morita [7], but the proof was probably never published in an article. Enomoto et al. [4] obtained for \( \Delta^*(G) \geq 60 \) even a stronger result, namely that \( \chi_c(G) \leq \Delta^*(G) + 1 \). The example of the (graph of) \( d \)-sided prism with maximum face degree \( d \) and cyclic chromatic number \( d + 1 \) shows that the bound is best possible. The best known general result (with no restriction on \( \Delta^*(G) \)) is the inequality \( \chi_c(G) \leq \Delta^*(G) + 5 \) of Enomoto and Hornák [3].

The conjecture is still open. This means that we do not know any G with \( \chi_c(G) - \Delta^*(G) \geq 3 \). On the other hand, all G's with \( \chi_c(G) - \Delta^*(G) = 2 \) we are aware of satisfy \( \Delta^*(G) = 4 \). Therefore, the conjecture could be strengthened so that \( \chi_c(G) \leq \Delta^*(G) + 1 \) for any 3-connected plane graph G with \( \Delta^*(G) \neq 4 \).

For \( p, q \in \mathbb{Z} \) let \( [p, q] := \{ z \in \mathbb{Z} : p \leq z \leq q \} \) and \( [p, \infty) := \{ z \in \mathbb{Z} : p \leq z \} \). The concatenation of finite sequences \( A = (a_1, \ldots, a_m) \) and \( B = (b_1, \ldots, b_n) \) is the sequence \( AB := (a_1, \ldots, a_m, b_1, \ldots, b_n) \). Because of the obvious associativity of concatenation we can use the symbol \( \prod_{i=1}^{k} A_i \) for the concatenation of \( k \in [0, \infty) \) finite sequences in the order given by the sequence \((A_1, \ldots, A_k)\). If \( A_i = A \) for all \( i \in [1, k] \), \( \prod_{i=1}^{k} A_i \) is replaced by \( A^k \), where \( A^0 = (\cdot) \) is the empty sequence.

Let \( d \in [5, \infty) \) and \( k \in [1, 5] \). A \((d, k)\)-minimal graph is a 3-connected plane graph \( G \) that satisfies (i) \( \Delta^*(G) = d \), (ii) \( \chi_c(G) > d + k \) and (iii) \( \chi_c(H) \leq d + k \) for any 3-connected planar graph \( H \) such that \( \Delta^*(H) \leq d \) and the pair \( (|V(H)|, |E(H)|) \) is lexicographically smaller then the pair \( (|V(G)|, |E(G)|) \). A configuration \( C \) is said to be \((d, k)\)- reducible if it does not appear in any \((d, k)\)-minimal graph.

Let \( G \) be an embedding of a 2-connected graph and let \( v \) be its vertex of degree \( n \). Consider a sequence \((f_1, \ldots, f_n)\) of faces incident with \( v \) in a cyclic order around \( v \)
(there are altogether $2n$ such sequences) and the sequence $D = (d_1, \ldots, d_n)$ in which $d_i = \deg(f_i)$ for $i \in [1, n]$. The sequence $D$ is called the type of the vertex $v$ provided it is the lexicographical minimum of the set of all such sequences corresponding to $v$, i.e., of the set $\bigcup_{n=1}^{n}(\{ \prod_{i=1}^{n-1}(d_{i+j}) \} \cup \{ \prod_{i=1}^{n-1}(d_{i-j}) \})$, where indices are taken modulo $n$ in the interval $[1, n]$. It is easy to see that $cd(v) = \sum_{i=1}^{n}(d_i - 2)$. The multiset $dm(v) := \{d_1, \ldots, d_n\}$ is the degree multiset of the vertex $v$. A contraction of an edge $xy \in E(G)$ consists in a continuous identification of the vertices $x$ and $y$ forming a new vertex $x \leftrightarrow y$ and the removal of the created loop together with all possibly created multiedges; if $G/xy$ is the result of such a contraction, then, clearly, $\Delta^*(G/xy) \leq \Delta^*(G)$. An edge $xy$ of a 3-connected plane graph $G$ is contractible if $G/xy$ is again 3-connected.

## 2 Auxiliary results

The lexicographical minimum of $(|V(G)|, |E(G)|)$ over 3-connected plane graphs $G$ with $\Delta^*(G) = d$ is $(d+1, 2d)$ and is attended by a plane embedding $\Pi_d$ of the graph of $d$-sided pyramid. Since $\chi(\Pi_d) = d + 1 = \Delta^*(\Pi_d) + 1$, if there is a graph violating PTC (with maximum face degree $d \in [5, 23]$), there must be a 3-connected plane graph $G$ that is $(d, 2)$-minimal. We are now going to prove that the structure of such a graph is quite restricted. For that purpose the following assertions will be useful:

**Lemma 1 (Halin [5])** Any 3-vertex of a 3-connected plane graph $G$ with $|V(G)| \geq 5$ is incident with a contractible edge.

**Lemma 2 (a consequence of results of Ando et al. [1])** If a vertex of degree at least four of a 3-connected plane graph $G$ with $|V(G)| \geq 5$ is not incident with a contractible edge, it is adjacent to three 3-vertices.

**Lemma 3** If $d \in [6, \infty)$, the following configurations are $(d, 2)$-reducible:

1. a 3-vertex $x$ with $cd(x) \leq d + 1$;
2. a vertex $x$ with $\deg(x) \geq 4$ and $cd(x) \leq d+1$ that is incident with a contractible edge;
3. a vertex $x$ with $\deg(x) \geq 4$ and $cd(x) \leq d + 1$ that is adjacent to a 3-vertex $y$ with $cd(y) \leq d + 2$;
4. a triangle $t$ incident with exactly one 3-vertex such that the face adjacent to $t$ along the edge joining vertices of degree at least four is of degree at most $d - 1$;
5. a separating 3-cycle;
6. an edge of type $(3, d_2)$ with $d_2 \in [3, 4]$;
7. the configuration $C_i$ of Fig. i, $i \in [1, 7]$, where encircled numbers represent degrees of corresponding vertices, vertices without degree specification are of an arbitrary degree and dashed lines are parts of facial cycles.

**Proof.** 1.–4. The statements have already been proved in [6] (Lemma 3.1(e), 3.3(i), 3.3(ii) and 3.4). For the rest of the proof suppose there is a $(d, 2)$-minimal graph $G$ that contains a configuration $C$ described in Lemma 3.5, 3.6 or 3.7.
5. If $C$ is a separating 3-cycle $x_1x_2x_3$, let $G_1$ and $G_2$ be components of the graph $G - \{x_1, x_2, x_3\}$. It is easy to see that the subgraph $H_i$ of $G$ induced by $V(G_i) \cup \{x_1, x_2, x_3\}$ is a 3-connected plane graph with $\Delta^*(H_i) \leq d$ and $|V(H_i)| < |V(G)|$, hence there is a cyclic colouring $\varphi_i : V(H_i) \to C$, $i = 1, 2$, where $|C| = d + 2$. Without loss of generality we may suppose that $\varphi_1(x_i) = \varphi_2(x_i)$, $i = 1, 2, 3$. Then $\psi : V(G) \to C$ determined by $\psi(x) := \varphi_i(x)$ if $x \in V(H_i)$, $i = 1, 2, 3$, is a cyclic colouring of $G$ in contradiction with $\chi_c(G) > d + 2$.

6. Now let $G$ contain a triangle $xy_1y_2$ adjacent to a quadrangle $y_1y_2z_2z_1$. Without loss of generality we may suppose that neither of the two faces incident with $y_1y_2$ is unbounded. By Lemma 3.1 we have $\deg(y_i) \geq 4$, $i = 1, 2$, and consequently, by Lemma 3.4, $\deg(x) \geq 4$. If the graph $G' := G - y_1y_2$ is 3-connected, it has a cyclic colouring using at most $d + 2$ colours which is also a cyclic colouring of $G$, a contradiction. Therefore, $G'$ has to be 2-connected. Consider a cutset $\{v_1, v_2\}$ of $G'$. Clearly, $\{v_1, v_2\} \cap \{y_1, y_2\} = \emptyset$, so there is a component $C(y_i)$ of the graph $G'' := G' - \{v_1, v_2\}$ containing the vertex $y_i$, $i = 1, 2$. From 3-connectedness of $G$ it follows that any vertex of $G''$ belongs either to $C(y_1)$ or to $C(y_2)$, hence $C(y_1) \neq C(y_2)$, $x \in \{v_1, v_2\}$ and $\{v_1, v_2\} \subseteq \{x, z_1, z_2\}$ (otherwise there is a path joining $y_1$ to $y_2$ in $G''$). Thus we may suppose without loss of generality that $v_1 = x$ and $v_2 = z_j$ for some $j \in [1, 2]$. Then both $x$ and $z_j$ are incident with the unbounded face $f$ of $G$. Because of Lemma 3.5 the vertices $x$ and $z_j$ are not adjacent in $G$, otherwise $(x, y_j, z_j, x)$ would be a separating 3-cycle of $G$. Therefore, the facial cycle of the unbounded face of $G$ is of the form $(x)P_i^1(z_j)P_i^2(x)$, where both paths $P_i^1$ and $P_i^2$ are nonempty. For $i = 1, 2$ consider the cycle $C^i := (x)P_i^1(z_j, y_j, x)$, the plane subgraph $G^i$ of $G$ induced by all vertices lying in the closed disc bounded by the closed Jordan curve corresponding to $C^i$, and join vertices $x$ and $z_j$ of $G^i$ by an arc lying in the unbounded face of $G^i$. It is easy to see that we obtain a 3-connected plane graph $H^i$ with $\Delta^*(H^i) \leq d$ and $|V(H^i)| < |V(G)|$, hence there is a cyclic colouring $\varphi^i : V(H^i) \to C$; if $f^i$ is the unbounded face of $H^i$, then $V(f^1) \cup V(f^2) = V(f)$ has at most $d$ vertices, and so we may suppose without loss of generality that $\varphi^1(v) = \varphi^2(v)$ for any $v \in \{x, y_j, z_j\}$ (note that $xy_jz_j$ is a 3-face of both $H^1$ and $H^2$) and $\varphi^1(V(f^1) - \{x, z_j\}) \cap \varphi^2(V(f^2) - \{x, z_j\}) = \emptyset$. As in Lemma 3.5, the colouring $\psi : V(G) \to C$ with $\psi(x) := \varphi_i(x)$ if $x \in V(H_i)$, $i = 1, 2$, yields a contradiction.

Fig. 1: $cd(x_1)=d+2$
7. If $C = C_i$, $i \in \{1, 3, 5, 6, 7\}$, the configuration $C$ contains a 3-vertex $x_1$ incident with a contractible edge $u_i x_1$; the oriented edge $(u_i, x_1)$ is indicated by an arrow. The graph $G' := G / u_i x_1$ is a 3-connected plane graph satisfying $\Delta^*(G') \leq d$ and $|V(G')| = |V(G)| - 1$, hence there is a cyclic colouring $\varphi : V(G') \to C$. This colouring will be used to find a cyclic colouring $\psi : V(G) \to C$ to obtain a contradiction with $\chi_c(G) > d + 2$. If not stated explicitly otherwise, we put $\psi(u) := \varphi(u)$ for any $u \in V(G) - \{u_i, x_1\}$ and $\psi(u_i) := \varphi(u_i \to x_1)$ (so that we have to determine only $\psi(x_1)$).

$i = 1$: If there is a colour that appears twice on vertices of $N_c(x_1)$ (under $\varphi$), from $cd(x_1) = d + 2$ we see that at least one colour is available as $\psi(x_1)$. Henceforth suppose that $|\varphi(N_c(x_1))| = d + 2$. Put $W := \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $C_j := \varphi(V(f_j) - W)$, $j = 1, 2, 3$, then $C_2 \cap C_3 = \emptyset$. If there is $j \in \{2, 3\}$ such that $C_j - C_1 \neq \emptyset$, we take $\psi(x_j) \in C_j - C_1$ and define $\psi(x_1) := \varphi(x_1)$. To conclude this case notice that $C_2 - C_1$ and $C_3 - C_1$ cannot be both empty, since then $C_j \subseteq C_1$, $j = 2, 3$, and $\deg(f_1) = |C_1| + 4 \geq |C_2| + |C_3| + 4 = d + 1$, a contradiction.
\( i = 2 \): Since, by Lemma 3.6, \( \deg(f_j) \geq 5 \), the configuration \( C_2 \) is \((d, 2)\)-reducible by Lemma 3.2 of [6].

\( i = 3 \): As for \( i = 1 \) it is sufficient to analyse the case in which \( |\varphi(N_c(x_1))| = d + 2 \). Putting \( W := \{ x_0, x_1, x_2, y_1, y_2 \} \) and \( C_j := \varphi(V(f_j) - W) \), \( j = 0, 1, 2 \), we obtain \( C_0 \cap C_2 = \emptyset \). If \( C_2 - C_1 \neq \emptyset \), we are done by taking \( \psi(x_2) \in C_2 - C_1 \) and \( \psi(x_1) := \varphi(x_2) \). On the other hand, \( C_2 - C_1 = \emptyset \) implies \( C_1 \subseteq C_2 \), and so defining \( \psi(x_1) := \varphi(x_0) \) leaves at least one colour available for \( \psi(x_0) \).

\( i = 4 \): For the proof see Lemma 3.1(c) and 3.1(d) of [6].

\( i = 5 \): In this case \( \varphi(x_2 \leftrightarrow x_1) \) can be used as either \( \psi(x_1) \) or \( \psi(x_2) \). By Lemma 3.1 we have \( \deg(f_1) = \deg(f_2) = d \), and so we may suppose (similarly as for \( i = 1 \) or \( i = 3 \)) that \( |\varphi(N_c(x_1))| = d + 2 \) and \( |\varphi(N_c(x_2) - \{x_1\})| = d + 1 \). Since \( N_c(z) \subseteq \tilde{N}_c(y) \), this allows us to define \( \psi(x_1) := \varphi(x_2 \leftrightarrow x_1) \), \( \psi(x_2) := \varphi(y) \), \( \psi(y) := \varphi(z) \) and \( \psi(z) := \varphi(y) \).

\( i = 6, 7 \): By Lemma 3.7.1 and 3.7.3 (for \( i = 7 \)) we have \( \deg(f_1) = \deg(f_2) = \deg(f) = d \) and \( cd(v) = d + 3 \) for any \( v \in \{ x_1, x_2, z_1, z_2 \} \). If there is a colour (of \( C \)) not present in \( \varphi(\tilde{N}_c(x_2) - \{x_1\}) = \varphi(N_c(x_1)) \), we use it as \( \psi(x_1) \). Henceforth we suppose that the vertex \( x_2 \) is saturated – all colours of \( C \) appear on vertices of its closed cyclic neighbourhood; as \( x_1 \) is not coloured under \( \varphi \), on vertices of the cyclic neighbourhood of \( x_2 \) one colour appears twice and \( d \) colours appear once. If \( \varphi(z_j) \notin \varphi(V(f)) \) and \( c \in C - \varphi(N_c(z_j) - \{x_1\}) \), then we are done (i.e., we obtain a contradiction) by putting \( \varphi(z_j) := c, \psi(x_j) := \varphi(z_j) \) and \( \psi(x_{3-j}) := \varphi(x_2 \leftrightarrow x_1) \). Therefore, we assume that \( \varphi(z_j) \notin \varphi(V(f)) \) implies the vertex \( x_j \) is saturated, \( j = 1, 2 \). There is \( j \in [1, 2] \) such that the \( x_2 \)-duplicated colour, i.e., one that appears twice on vertices of \( N_c(x_2) \), is either \( \varphi(t_j) \) or \( \varphi(z_j) \). If \( \varphi(t_j) \) is \( x_2 \)-duplicated, then obviously \( \varphi(z_j) \notin \varphi(V(f)) \), so \( z_j \) is saturated, at most one of \( \varphi(t_{3-j}) \) and \( \varphi(z_{3-j}) \) is \( z_j \)-duplicated and \( \{ \varphi(t_{3-j}), \varphi(z_{3-j}) \} \neq \varphi(V(f)) \). If, say, \( \varphi(t_{3-j}) \notin \varphi(V(f)) \), then, having in mind that \( \varphi(t_{3-j}) \notin \varphi(V(f)) \), we can take \( \psi(y_j) := \varphi(t_{3-j}) \) and \( \psi(x_1) := \varphi(y_j) \). Now let \( \varphi(z_j) \) be \( x_2 \)-duplicated; as a consequence, \( z_{3-j} \) is saturated. If one of \( \varphi(t_{3-j}), \varphi(z_{3-j}) \) is out of \( \varphi(V(f_j)) \), we use it as \( \psi(y_j) \) and put \( \psi(x_1) := \varphi(y_j) \). On the other hand, provided \( \{ \varphi(t_{3-j}), \varphi(z_{3-j}) \} \subseteq \varphi(V(f_j)) \), there is a colour \( c \in C - \varphi(\tilde{N}_c(z_j) - \{x_1\}) \), which allows us to define \( \psi(z_j) := c \) together with either \( \psi(z_{3-j}) := \varphi(z_j) \) and \( \psi(x_1) := \varphi(z_{3-j}) \) (if \( \varphi(t_j) \) is \( z_{3-j} \)-duplicated) or \( \psi(y_{3-j}) := \varphi(t_j) \) and \( \psi(x_1) := \varphi(y_{3-j}) \) (otherwise).  \[ \blacksquare \]

Note that the configurations of Lemma 3, except for \( C_5 \) and \( C_7 \), are even \((5, 2)\)-reducible.

Our main theorem will be proved by Discharging Method. Namely, we shall suppose that there is a \((d, 2)\)-minimal graph \( G = (V, E, F) \) for some \( d \in [18, \infty) \). From Euler’s Theorem \( |V| - |E| + |F| = 2 \) it is easy to derive that \( \sum_{v \in V} c_0(v) = 2 \) for the mapping \( c_0 : V \to \mathbb{Q} \) (called the initial charge) with

\[
c_0(v) := 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)}.
\]

Putting \( \Sigma(c_0, W) := \sum_{v \in W} c_0(v) \) for \( W \subseteq V \) we have \( \Sigma(c_0, V) = 2 \). We are able to find consecutively in four phases charge mappings \( c_i : V \to \mathbb{Q} \), \( i = 1, 2, 3, 4 \), such
that $\Sigma(c_i, V) = 2$, which means that passing from $c_{i-1}$ to $c_i$ is simply a redistribution of charges of vertices that is governed by redistribution rules. The restriction on the structure of $G$ yielded by Lemma 3 enables us to prove that $c_d(v) \leq 0$ for any $v \in V$, which represents a contradiction with $\Sigma(c_4, V) = 2$.

If a vertex $v \in V$ is of type $(d_1, \ldots, d_n)$, then

$$c_0(v) = \gamma(d_1, \ldots, d_n) := 1 - \frac{n}{2} + \frac{1}{\sum_{i=1}^{n} d_i}.$$

Clearly, if $\pi$ is a permutation of the set $[1, n]$, then $\gamma(d_{\pi(1)}, \ldots, d_{\pi(n)}) = \gamma(d_1, \ldots, d_n)$.

Let the weight of a sequence $D = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ be defined by $\text{wt}(D) := \sum_{i=1}^{n} d_i$.

For $n \in [2, \infty)$, $q \in [0, n - 2]$, $(d_1, \ldots, d_{n-1}) \in [1, \infty)^{n-1}$ and $w \in [\sum_{i=1}^{n-1} d_i + 1, \infty)$ let $S_q(d_1, \ldots, d_{n-1}; w)$ be the set of all sequences $D = (d_1, \ldots, d_q, d_{q+1}, \ldots, d_n) \in \mathbb{Z}^n$ satisfying $d'_i \geq d_i$ for any $i \in [q+1, n-1]$ and $\text{wt}(D) \geq w$. An analogue of the following statement has been proved as Lemma 4 in [6] (with a different definition of $\gamma$).

**Lemma 4** The maximum of $\gamma(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n)$ over all sequences $(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \in S_q(d_1, \ldots, d_{n-1}; w)$ is equal to $\gamma(d_1, \ldots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$.

**Proof.** Pick a sequence $(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \in S_q(d_1, \ldots, d_{n-1}; w)$. Decrease $d'_i$ to $d_i$ and increase $d'_i$ by $d'_i - d_i$ successively for all $i \in [q+1, n-1]$. If $a_1, a_2, a_3, a_4 \in [1, \infty)$, $a_1 + a_2 = a_3 + a_4$ and $a_1 < \min(a_3, a_4)$, then $\frac{1}{a_3} + \frac{1}{a_4} < \frac{1}{a_2} + \frac{1}{a_4}$. Moreover, with $d''_i := d''_i + \sum_{i=q+1}^{n-1} (d'_i - d_i)$ we have $\sum_{i=1}^{n-1} d_i + d''_i = \text{wt}(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \geq w$, hence $(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \in S_q(d_1, \ldots, d_{n-1}; w)$ and $\gamma(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \leq \gamma(d_1, \ldots, d_{n-1}, d''_i) \leq \gamma(d_1, \ldots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$. Here equalities apply if and only if $d'_i = d_i$ for any $i \in [q+1, n-1]$ and $d''_i = d'_i = w - \sum_{i=1}^{n-1} d_i$. $lacksquare$

## 3 Proof of Theorem

As already mentioned, for the proof by contradiction we suppose that $G = (V, E, F)$ is a $(d, 2)$-minimal graph with $\Delta^+(G) = d \in [18, \infty)$. A set $W \subseteq V$ is **positive** if $\Sigma(c_0, W) > 0$, otherwise it is nonpositive; similarly is defined a negative and a nonnegative set. If $W = \{v\}$ or $W = V(f)$, $f \in F$, we shall speak simply about a positive (nonpositive, negative, nonnegative) vertex $w$ or face $f$ respectively.

A triangle $t \in F$ is an $i$-**triangle** if the number of 3-vertices in $V(t)$ is $i$. For a vertex $v \in V$ let $N_+(v)$ denote the set of all neighbours of $v$ of degree at least four and put $n_+(v) := |N_+(v)|$. Now we are going to prove a series of claims concerning vertices of $V$ and faces of $F$ (which is implicitly assumed in those claims).

**Claim 1.** 1. If faces $f_1$ and $f_2$ are adjacent to each other, then $\deg(f_1) + \deg(f_2) \geq 8$.

2. If a vertex is of type $(d_1, d_2, d_3)$, then $d_3 \geq d + 8 - d_1 - d_2$.

3. If a vertex is positive, it is of degree 3.
4. If a vertex of type \((d_1, d_2, d_3)\) is positive, then either \(d_1 = 3\) and \(d_2 \in [5, 11]\) or \(d_1 = 4\) and \(d_2 \in [4, 5]\).

5. If a vertex of type \((3, d_2, d_3)\) is nonpositive, then \(d_2 \geq 7\).

**Proof.** 1. The inequality follows from Lemma 3.6.

For the rest of the proof consider an \(n\)-vertex \(v\) of type \((d_1, \ldots, d_n)\) and put \(d_{n+i} := d_i\) for \(i \in [1, n]\).

2. If \(\deg(v) = 3\), then \(cd(v) = d_1 + d_2 + d_3 - 6\). To obtain the desired inequality use Lemma 3.1.

3. Suppose that \(n \geq 4\). By Claim 1.1 we have \(d_i + d_{i+1} \geq 8 + \frac{1}{2}d_i + \frac{1}{2}d_{i+1} \leq \max\{\frac{1}{3} + \frac{1}{5}; \frac{1}{3} + \frac{1}{4}\} = \frac{8}{15}\) for any \(i \in [1, 2n-1]\), hence \(\sum_{i=1}^{n} \frac{1}{d_i} = \frac{1}{2} \sum_{i=1}^{n} (\frac{1}{d_{2i-1}} + \frac{1}{d_{2i}}) \leq \frac{4n}{15}\)
and \(c_0(v) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i} \leq 1 - \frac{7n}{30}\). If \(n \geq 5\), then \(c_0(v) \leq -\frac{1}{6}\). It remains to analyse the case \(n = 4\). If \(d_1 \geq 4\), then \(c_0(v) \leq -1 + 4 \cdot \frac{1}{4} = 0\). If \(d_3 \geq 4\), then \(c_0(v) \leq -1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{4} + \frac{1}{5} = -\frac{1}{60}\). Finally, suppose that \(v\) is of type \((3, d_2, 3, d_4)\).

4. If \(d_1 \geq 5\), then, by Lemma 4, \(c_0(v) \leq -\frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{d_2-3} \leq -\frac{11}{12} + \frac{1}{10} < 0\). If \(d_1 = 4\) and \(d_2 \geq 6\), then, again by Lemma 4, \(c_0(v) \leq -\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{d_2-3} \leq -\frac{1}{12} + \frac{1}{16} < 0\). If \(d_1 = 3\), then \(d_2 \geq 5\) (Claim 1.1) and with \(d_3 \geq d_2 \geq 12\) we have \(c_0(v) \leq -\frac{1}{6} + \frac{1}{12} + \frac{1}{12} = 0\).

5. If \(d_1 = 3\) and \(d_2 \leq 6\), then \(c_0(v) = -\frac{1}{6} + \frac{1}{12} + \frac{1}{d_3} \geq -\frac{1}{12} > 0\).

By Claim 1.2 and Lemma 4, provided \(v\) is a vertex of type \((d_1, d_2, d_3)\), we have 
\(c_0(v) \leq \gamma(d_1, d_2, d+8-d_1-d_2) \leq \gamma(d_1, d_2, 26-d_1-d_2) =: u(d_1, d_2)\). The positive upper bounds \(u(d_1, d_2)\) are presented in Table 1.

| \(d_1\) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 |
| \(d_2\) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 4 | 5 |
| \(u(d_1, d_2)\) | \(\frac{4}{25}\) | \(\frac{1}{17}\) | \(\frac{13}{336}\) | \(\frac{1}{40}\) | \(\frac{1}{63}\) | \(\frac{1}{195}\) | \(\frac{1}{187}\) | \(\frac{1}{18}\) | \(\frac{3}{340}\) |

Table 1

A triangle is of type \((d_1, d_2, d_3)\) if it is adjacent to three distinct faces \(f_1, f_2, f_3\) with \(\deg(f_1) = d_1 \leq \deg(f_2) = d_2 \leq \deg(f_3) = d_3\).

**Claim 2.** If a 3-triangle \(t\) of type \((d_1, d_2, d_3)\) is positive, then \(d_1 \in [6, 7]\), \(d_2 \geq d + 6 - d_1\) and \(\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{d + 6 - d_1} =: \beta(d_1, d)\).

**Proof.** From Claim 1.1 and \(C_1\) it follows that \(d_1 \geq 6\). Put \(d_4 := d_1\). If \(d_1 \geq 12\), then \(\Sigma(c_0, V(t)) = \sum_{i=1}^{3} \gamma(3, d_i, d_{i+1}) = -\frac{1}{2} + 2 \sum_{i=1}^{3} \frac{1}{d_i} \leq -\frac{1}{2} + 2 \cdot \frac{1}{12} = 0\). Let \(x \in V(t)\) be a vertex of type \((3, d_1, d_2)\). From \(C_1\) we obtain \(d + 3 \leq cd(x) = d_1 + d_2 - 3\), \(d_3 \geq d_2 \geq d + 6 - d_1\), and so \(\Sigma(c_0, V(t)) \leq -\frac{1}{2} + 2(\frac{1}{d_1} + \frac{2}{d + 6 - d_1}) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{2d_1 - 3}\).

With \(d_1 \in [8, 11]\) we have \(\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{8} + \frac{4}{10} = 0\), hence \(d_1 \in [6, 7]\).
Let us define absorbing vertices as follows: Any vertex of degree at least four is absorbing. A 3-vertex is absorbing if it is either of type $(5, d_2, d_3)$ with $d_2 \geq 11$ and $d_3 \geq d - 1$ or of type $(7, d_2, d_3)$ with $d_2 \geq 10$.

**Claim 3.** If a 5-face $f$ is incident with a vertex of type $(4, 5, d_3)$, then $f$ is incident with an absorbing vertex.

**Proof.** Let $C = (x_1, x_2, x_3, x_4, x_5, x_1)$ be a facial cycle of $f$ and let $f_i$ be the face adjacent to $f$ along the edge $x_i x_{i+1}$ (with indices taken modulo 5). If $\deg(x_i) \geq 4$ for some $i \in [1, 5]$, then $x_i$ is absorbing. If $\deg(x_i) = 3$ for any $i \in [1, 5]$, we may suppose without loss of generality that $\deg(f_3) = 4$. By Claim 1.2 then $\deg(f_i) \geq d - 1$ for $i = 2, 4$. By the same Claim we have $\max\{\deg(f_1), \deg(f_3)\} \geq 11$, and so at least one of the vertices $x_2, x_5$ is absorbing.

**Claim 4.** If a 7-face $f$ is adjacent to a 3-triangle, then $f$ is incident with an absorbing vertex.

**Proof.** Let $C = (x_1, x_2, \ldots, x_7, x_1)$ be a facial cycle of $f$ and let $f_i$ be the face adjacent to $f$ along the edge $x_i x_{i+1}$ (with indices taken modulo 7). If $\deg(x_i) \geq 4$ for some $i \in [1, 7]$, then $x_i$ is absorbing. Henceforth assume that $\deg(x_i) = 3$ for any $i \in [1, 7]$. Since 3-triangles adjacent to $f$ cover an even number of vertices of $f$, there is a subpath $P$ of $C$ of an odd order $k \in \{1, 3, 5\}$, without loss of generality $P = \prod_{i=1}^{k} (x_i)$, such that none of $x_i$ with $i \in [1, k]$ is incident with a 3-triangle, but $x_k$ is incident with a 3-triangle for any $i \in \{k + 1\} \cup \{7\}$. By Claim 1.2 then $\min\{\deg(f_i), \deg(f_j)\} \geq d - 1$. If $k = 1$, then the vertex $x_1$ is absorbing. If $k \in \{3, 5\}$ and $\max\{\deg(f_1), \deg(f_k)\} \geq 10$, at least one of the vertices $x_1, x_k$ is absorbing; note that, by Claim 1.2, the inequality is certainly true if $k = 3$. Finally, if $k = 5$ and $\max\{\deg(f_1), \deg(f_3)\} \leq 9$, then, again by Claim 1.2, $\min\{\deg(f_2), \deg(f_3)\} \geq 10$, and hence the vertex $x_3$ is absorbing.

A transition edge of a vertex $x$ of type $(4, 5, d_3)$ is an oriented edge $(v, w)$ whose endvertex is an absorbing vertex of the 5-face $f$ incident with $x$ that is closest to $x$ in one of two possible orientations of the cycle bounding $f$. Similarly, a transition edge of a 3-triangle $t$ adjacent to a 7-face $f$ is an oriented edge $(v, w)$ whose endvertex is an absorbing vertex of $f$ that is closest to (a vertex of) $t$ in one of two possible orientations of the cycle bounding $f$. Finally, a transition edge of a 3-triangle $t$ adjacent to a 6-face $f$ is an oriented edge $(v, w)$ with $v \in V(t)$ and $w \in V(f) - V(t)$. From Claims 1.1, 2, 3 and 4 it follows that any vertex of type $(4, 5, d_3)$ and any positive 3-triangle has exactly two transition edges. Moreover, the initial vertex of any transition edge is a 3-vertex.

Let us now present redistribution rules leading from $c_0$ to $c_4$. The first “coordinate” $i$ of a rule RR $i.j$ means that RR $i.j$ is used when passing from $c_{i-1}$ to $c_i$.

**RR 1.1** If $(v, w)$ is a transition edge of a vertex $x$ of type $(4, 5, d_3)$, then $x$ sends to $w$ the amount $\frac{1}{2}c_0(x)$ through $(v, w)$.

**RR 1.2** If $(v, w)$ is a transition edge of a positive 3-triangle $t$, then $t$ sends to $w$ the amount $\frac{1}{2}\Sigma c_0(V(t))$ through $(v, w)$ and $c_1(x) := 0$ for any $x \in V(t)$.

**RR 1.3** If $(v, w)$ is a transition edge involved in RR 1.1 or RR 1.2 and $c_0(v) < 0$, then $v$ sends to $w$ the amount $c_0(v)$ through $(v, w)$. 
RR 3.1 A vertex $v$ of type $(3, d_2, d_3)$ with $c_2(v) > 0$, that is incident with a 1-triangle, sends to its $(3, d_3)$-neighbour $w$ the amount $c_2(v)$ through. (The rule is correct, since $c_2(v) > 0$ implies $c_0(v) > 0$, and so, by Claims 1.2 and 1.4, $d_3 > d_2$.)

RR 3.2 If $t$ is a 2-triangle with $V(t) = \{v_1, v_2, w\}$, where $v_1, v_2$ are 3-vertices, then $v_i$ sends to $w$ the amount $c_2(v_i)$ through $(v_i, w)$, $i = 1, 2$.

RR 3.3 If $v$ is a vertex of type $(4, 4, d)$ satisfying $c_2(v) > 0$ and $n_{4+}(v) = 0$ and $n_{4+}(w) \geq 1$ for the $(4, 4)$-neighbour $w$ of $v$, then $v$ sends to $w$ the amount $c_2(v)$.

RR 4.1 If $v$ is a 3-vertex with $c_3(v) > 0$ and $N_{4+}(v) = \emptyset$, if $v$ sends to $w$ the amount $c_2(v)$ through $(v, w)$. Recall that our aim is to show that $c_{4}(w) \leq 0$ for any $w \in V$. The case $\deg(w) = 3$ will be treated separately at the end of our analysis. If $\deg(w) \geq 4$ and $v \in N(w)$, let $a(v, w)$ be the total amount received by $w$ through the oriented edge $(v, w)$ (according to one of RR 1.1, 1.2, 1.3, 3.1, 3.2 and 4.1). If $\deg(v) \geq 4$, then $a(v, w) = 0$. If $\deg(v) = 3$, then $a(v, w)$ depends among other things on the type of the edge $vw$. Let $\tilde{a}(d'_1, d'_2)$ be a nonnegative upper bound for $a(v, w)$ provided $vw$ is of type $(d'_1, d'_2)$. If $\tilde{a}(d'_1, d'_2)$ is not mentioned at all, it is considered to be 0. We shall assume that $\dm(v) = \{d'_1, d'_2, d'_3\}$.

First suppose that $d'_1 = 3$. If $d'_2 = 5$, then $v$ is of type $(3, 5, d)$ (Claim 1.2), and so, because of RR 1.1 and RR 3.2, we have $a(v, w) \leq \gamma(3, 5, d) + \gamma(4, 5, d - 1) = -\frac{1}{24} + \frac{1}{d + 1} + \frac{3}{24} \leq \frac{11}{24}$. Let $d'_2 = 6$. If $c_2(v) \neq c_0(v)$, it is because of RR 1.2; in such a case, by Claim 1, $d'_3 = d$, and so, by Claim 2, $a(v, w) = c_2(v) \leq \gamma(3, 6, d) + \frac{1}{2}\gamma(6, d) = \frac{3}{3} - \frac{1}{17} \leq \frac{1}{17}$. If $c_2(v) = c_0(v)$, Claim 1.2 yields $d'_3 \geq d - 1$ and $a(v, w) = c_0(v) = \frac{1}{d + 1} \leq \frac{1}{17}$. Thus, we can take $\tilde{a}(3, 6) = \frac{1}{17}$. Similarly, we can define $\tilde{a}(3, 7) := \gamma(3, 7, 17) + \beta(7, 18)$. If $d'_2 \in [8, d]$, then $c_2(v) = c_0(v)$, $\cd(v) = d'_2 + d'_3 - \frac{3}{2} \geq d + 2$ and $d'_3 \geq d + 5 - d'_2$. Therefore, because of RR 3.1 or RR 3.2, $a(v, w) \leq \gamma(4, d', 23 - d'_3)$. Moreover, $\gamma(3, 3, 23 - d'_3) \leq \gamma(3, 3, 15) =: \tilde{a}(3, d'_3)$ for any $d'_3 \in [12, d - 3]$; for $d'_3 \in [8, 11] \cup [d - 2, d]$ we put $\tilde{a}(3, d'_3) := \gamma(3, d', 23 - d'_3)$.

Now consider the case $d'_1 = 4$. If $d'_2 = 4, 4.1$ yields $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) =: \tilde{u}(4, 4)$. If $d'_2 = 5$, then, by RR 1.1, $a(v, w) \leq 2\gamma(4, 5, 17) =: \tilde{u}(4, 5)$. If $d'_2 = 6$ and $\deg(v) = 3$, then, by RR 1.2 and Claim 2, $a(v, w) \leq \gamma(4, 6, d) + \frac{1}{2}\beta(6, d) = \frac{1}{2} + \frac{1}{2} \leq 0$ and we can take $\tilde{a}(4, 6) := 0$. If $d'_2 = 7$ and $\deg(v) = 3$, then, by RR 1.2 with Claim 2 and by RR 1.3 with Claim 1.2, $a(v, w) \leq \beta(7, 18) + \gamma(4, 7, 17) < 0$; therefore, we take again $\tilde{u}(4, 7) := 0$. If $(d'_1, d'_2) = (4, d)$, then, using $\mathcal{C}_4, \mathcal{C}_5$, RR 2.1 and RR 3.3 we can obtain $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) =: \tilde{u}(4, d)$.

With $d'_1 \in [5, 7]$ the following bounds are easily derived: $\tilde{u}(5, d'_2) := 2\gamma(4, 5, 17)$ for $d'_2 \in [d - 1, d]$, $\tilde{u}(6, d'_2) := \frac{1}{2}\beta(6, 18)$, $\tilde{u}(7, d - 2) := \beta(7, 18)$, and $\tilde{u}(7, d'_2) := \frac{1}{2}\beta(7, 18)$ for $d'_2 \in [d - 1, d]$. The (positive) upper bounds $\tilde{a}(d'_1, d'_2)$ are summarised in Table 2; for our analysis it is helpful to have them ordered in a decreasing sequence $(101, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$. Finally, for $d'_1 > d''_2$ we put
\[ \bar{u}(d'_1, d'_2) := \bar{u}(d'_2, d'_1). \]

\[
\begin{array}{c|ccccccccc}
& d'_1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
& d'_2 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \in [12, d-3] \\
\hline
\bar{u}(d'_1, d'_2) & 41 & 3/8 & 1/12 & 20 & 1/357 & 1/40 & 1/65 & 2 & 1332 & 1/40 & 13/336 & 1/17 & 4/43 \\
\end{array}
\]

\[ \begin{array}{c|ccccccc}
& d'_1 & 4 & 4 & 5 & 6 & 7 & 7 \\
\hline
& d'_2 & 4 & 5 & d & d-1, d & d-2 & d-1, d \\
\hline
\bar{u}(d'_1, d'_2) & 1/18 & 3/170 & 1/18 & 3/170 & 1/37 & 5 & 23/78 & 15 & 476 \\
\end{array} \]

Table 2

Now consider an \( n \)-vertex \( w \) of type \( D = (d_1, \ldots, d_n) \) and let \((v_1, \ldots, v_n)\) be a sequence of neighbours of \( w \) in a cyclic order around \( w \) such that the edge \( v_i w \) is incident with faces \( f_i \) of degree \( d_i \) and \( f_{i+1} \) of degree \( d_{i+1} \) (if \( i \in [n+1, \infty) \)), the index \( i \) in \( v_i \), \( f_i \) or \( d_i \) is taken modulo \( n \) so as to belong to \([1, n] \). Then \( c_0(w) = 1 - \frac{n}{2} + \sum_{i=1}^{n-1} \frac{1}{d_i} = \sum_{i=1}^{n} p_i^0(w) \), where \( p_i^0(w) := \frac{1}{n} - \frac{1}{2} + \frac{1}{2d_i} + \frac{1}{2d_{i+1}} \) is the \( i \)th partial charge of the vertex \( w \) (corresponding to the edge \( v_i w \)). If \( n \geq 4 \), we have \( c_4(w) = c_0(w) + \sum_{i=1}^{n} a(v_i, w) = \sum_{i=1}^{n} (p_i^0(w) + a(v_i, w)) \leq \sum_{i=1}^{n} (p_i^0(w) + \bar{u}(d_i, d_{i+1})). \)

To bound \( p_i^0(w) \) we use the following inequality yielded by Claim 1.1: \( \frac{1}{2d_i} + \frac{1}{2d_{i+1}} \leq \max\{\frac{1}{6}, \frac{1}{10}, \frac{1}{8} + \frac{1}{2}\} = \frac{4}{15} \) for any \( i \in [1, n] \). By \( f_k := \{i \in [1, n] : d_i = k\} \) we denote the frequency of \( k \) in \( D \); we put \( f_{k+} := \sum_{i=k}^{d} f_i \).

(1) If \( n \geq 8 \), using Table 2 we see that \( p_i^0(w) + \bar{u}(d_i, d_{i+1}) \leq \frac{1}{8} - \frac{1}{2} + \frac{4}{40} + \frac{41}{408} < 0 \) for any \( i \in [1, n] \), and so \( c_4(w) < 0 \).

(2) \( n \in [5, 7] \)

(21) If \( \text{cd}(w) \leq d + 1 \), then, by Claim 1.1, \( d_i \leq d - 5 \) for any \( i \in [1, n] \). Further, by Lemma 3.3, \( \text{deg}(v_i) = 3 \) implies \( \text{cd}(v_i) \geq d + 3 \), and so from \( d_i + d_{i+1} = 8 \) it follows that \( a(v_i, w) = 0 \) and \( \frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \leq \frac{1}{6} + \frac{1}{10} = \frac{4}{15} \). Using Table 2 it is easy to check that \( d_i + d_{i+1} \geq 9 \) yields \( \frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \leq \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3} \), moreover, if \( \{d_i, d_{i+1}\} \neq \{3, 6\} \), then \( \frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \leq \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{5}{18} \).

(211) If \( n \in [6, 7] \), then \( p_i^0(w) + a(v_i, w) \leq \frac{1}{n} - \frac{1}{2} + \max\{\frac{4}{15}, \frac{1}{3}\} \leq 0 \) for any \( i \in [1, n] \) and \( c_4(w) \leq 0 \).

(212) If \( n = 5 \), then, since \( \frac{1}{5} = \frac{1}{2} + \max\{\frac{4}{15}, \frac{5}{18}\} < 0 \), \( p_i^0(w) + a(v_i, w) \) can be positive only if \( \{d_i, d_{i+1}\} = \{3, 6\} \). Let \( k := \{|i \in [1, 5] : \{d_i, d_{i+1}\} = \{3, 6\}\} \).

(2121) If \( k = 0 \), then \( c_4(w) < 0 \) as a sum of five negative summands.

(2122) \( k \geq 1 \), then, by Claim 1.1, \( f_3 \in [1, 2] \). If \( \text{deg}(v_i) = 3 \), \( v_i w \) is of type \( (3, 6) \) and \( v_i \) is not involved in \( RR \ 1.2 \), then \( a(v_i, w) \leq \gamma(3, 6, d) \leq \frac{4}{15} \); notice that the number of \( i \)'s such that \( \text{deg}(v_i) = 3 \), \( v_i w \) is of type \( (3, 6) \) and \( v_i \) is involved in \( RR \ 1.2 \) is at most \( f_6 \).

(21221) If \( f_3 = 1 \), then, by Claim 1.1 and Table 2, \( c_0(w) + \sum_{i=1}^{5} a(v_i, w) \leq (-\frac{8}{3} + \frac{1}{5} + \frac{1}{6} + 2 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{13}{78}) < 0 \).

(21222) If \( f_3 = 2 \), then, by Claim 1.1, \( f_4 = 0 \). In such a case \( a(v_i, w) = 0 \) for (the unique) \( i \in [1, 5] \) satisfying \( \min\{d_i, d_{i+1}\} \geq 5 \).
(212221) If \( k \geq 4 \), then \( w \) is of type \((3, 6, 3, 6, 6)\) and \( c_0(w) + \sum_{i=1}^{5} a(v_i, w) \leq -\frac{3}{7} + (3 \cdot \frac{1}{12} + \frac{1}{18}) < 0 \).

(212222) If \( k = 3 \), then \( f_6 = 2 \), \( c_0(w) \leq \gamma(3, 5, 6, 3, 6) = -\frac{3}{10} \), \( \sum_{i=1}^{5} a(v_i, w) \leq 2 \cdot \frac{1}{12} + \frac{20}{357} < \frac{3}{10} \) and \( c_4(w) < 0 \).

(212223) \( k = 2 \)

(2122231) If \( f_6 = 1 \), then \( c_0(w) \leq \gamma(3, 5, 5, 3, 6) = -\frac{4}{15} \), \( \sum_{i=1}^{5} a(v_i, w) \leq \frac{1}{12} + \frac{1}{18} + 2 \cdot \frac{20}{357} < \frac{4}{15} \) and \( c_4(w) < 0 \).

(2122232) If \( f_6 = 2 \), then \( c_0(w) \leq \gamma(3, 5, 3, 6, 6) = -\frac{3}{10} \), \( \sum_{i=1}^{5} a(v_i, w) \leq 2 \cdot \frac{1}{12} + \frac{20}{357} \), \( c_1(w) < 0 \).

(212224) If \( k = 1 \), then \( c_0(w) \leq \gamma(3, 5, 5, 3, 6) = -\frac{4}{15} \), \( \sum_{i=1}^{5} a(v_i, w) \leq \frac{1}{12} + 3 \cdot \frac{20}{357} \) and \( c_4(w) < 0 \).

(22) \( \text{cd}(w) \geq d + 2 \)

(221) If \( n = 7 \), then, by Claim 1.1, \( f_{5+} \geq f_5, f_3 \leq 3 \), and so, by Lemma 4, \( c_0(w) \leq \gamma((3) f_3 (5) f_3 (4)^{6-2f_3} (d - 8)) = -1 + \frac{f_3}{30} + \frac{1}{d-8} \leq -\frac{4}{5} \). On the other hand, \( \sum_{i=1}^{7} a(v_i, w) \leq 7 \cdot \frac{41}{246} < \frac{4}{5} \) and \( c_4(w) < 0 \).

(222) \( n = 6 \)

(2221) If \( f_3 \leq 2 \), using Claim 1.1 and the assumption \( \text{cd}(w) \geq d + 2 \) we see that \( f_{3+} \geq f_3 + 1 \), and so, by Lemma 4, \( c_0(w) \leq \gamma((3) f_3 (5) f_3 (4)^{6-2f_3} (d - 6)) = -\frac{2}{3} + \frac{f_3}{60} + \frac{1}{d-6} \leq -\frac{2}{3} + \frac{1}{18} \). On the other hand, Table 2 yields \( \sum_{i=1}^{6} a(v_i, w) \leq 2 f_3 \cdot \frac{41}{408} + (6 - 2 f_3) \cdot \frac{1}{18} \). Therefore, \( c_4(w) \leq \frac{377 f_3}{3060} - \frac{1}{3} \leq \frac{377}{1530} - \frac{1}{3} < 0 \).

(2222) If \( f_3 = 3 \), then, by Claim 1.1, \( w \) is of type \((3, 6, 3, 4, 3, 6)\) and, by Lemma 4, \( c_0(w) \leq \gamma(3, 5, 3, 5, 3, d - 5) = -\frac{3}{5} + \frac{2}{d-3} \leq -\frac{3}{5} + \frac{1}{13} = -\frac{34}{65} \). So, it is sufficient to show that \( \sum_{i=1}^{6} a(v_i, w) \leq \frac{34}{65} \).

(22221) If there is \( i \in [1, 6] \) with \( \text{deg}(v_i) \geq 4 \), then \( \sum_{i=1}^{6} a(v_i, w) \leq 5 \cdot \frac{41}{246} < \frac{34}{65} \).

(22222) If \( \text{deg}(v_i) = 3 \) for any \( i \in [1, 6] \), consider the expression \( c_4(w) = \sum_{i=1}^{6} q_i \), where \( q_i := \frac{1}{6} \left( \frac{1}{2} + \frac{1}{6} + \frac{1}{5} \max(d, d_{i+1}) \right) + a(v_i, w) \leq -\frac{1}{6} + \frac{1}{5} \max(d, d_{i+1}) + \tilde{u}(3, \max(d, d_{i+1})) \) and \( \max\{d_i, d_{i+1}\} \in [5, d] \). Using Table 2 it is easy to check that three maximal values of \( f(s) := -\frac{1}{6} + \frac{1}{2x} + \tilde{u}(3, s) \) for \( s \in [5, d] \) are \( f(5) = \frac{23}{180} \), \( f(6) = 0 \) and \( f(7) = -\frac{2}{3} \). Notice that \( c_4(w) = \sum_{i=1}^{6} q_i \leq 2 \sum_{i=1}^{6} f(d_{2i}) \).

(222221) If \( d_2 \geq 6 \), then, as \( \min\{d_4, d_6\} \geq d_2 \), we obtain \( c_4(w) \leq 0 \).

(222222) \( d_2 = 5 \)

(2222221) If \( \min\{d_4, d_6\} \geq 7 \), then \( c_4(w) \leq 2 \cdot (\frac{23}{180} - 2 \cdot \frac{2}{51}) < 0 \).

(2222222) If there is \( j \in \{4, 6\} \) with \( d_j \in [5, 6] \), then \( d_{10-j} \geq d - d_j \). Let \( d' \) be the degree of the face adjacent to both \( f_j \) and \( f_{10-j} \). By Claim 1.2 we know that \( d' \geq d + 5 - d_j \). Therefore, by RR 3.2, the summand \( a(v_k, w) \) corresponding to the vertex \( v_k \) with \( \text{dm}(v_k) = \{3, d_{10-j}, d'\} \) is equal to \( \gamma(3, d_{10-j}, d') = \frac{1}{6} + \frac{1}{d_{10-j}} + \frac{1}{d'} \leq \frac{1}{6} + \frac{1}{d-6} + \frac{1}{d-1} \leq \frac{1}{6} + \frac{1}{d-12} + \frac{1}{d-17} < 0 \) and \( \sum_{i=1}^{6} a(v_i, w) \leq 5 \cdot \frac{41}{246} < \frac{34}{65} \).

(223) \( n = 5 \)

(2231) If \( f_3 = 0 \), then, due to Lemma 4, \( c_0(w) \leq \gamma((4)^4(d - 4)) \leq -\frac{3}{7} \), and so \( c_4(w) \leq -\frac{3}{7} + 5 \cdot \frac{1}{18} < 0 \).

(2232) If \( f_3 = 1 \), then \( c_4(w) \leq \gamma(3, 5, 4, 4, d - 4) = -\frac{7}{15} + \frac{1}{d-4} \leq -\frac{83}{210} \). \( \sum_{i=1}^{5} a(v_i, w) \leq 2 \cdot \frac{41}{246} + 3 \cdot \frac{1}{18} \leq \frac{83}{210} \) and \( c_4(w) < 0 \).

(2233) If \( f_3 = 2 \), then, by Claim 1.1, \( f_4 = 0 \). By Lemma 4 we have \( c_0(w) \leq \)
\[
\gamma(3, 5, 3, 5, d - 4) = -13 \frac{1}{30} + \frac{1}{d - 4} \leq -38 \frac{1}{105}, \text{ and so it is sufficient to prove that } \sum_{i=1}^{5} a(v_i, w) \leq 38 \frac{1}{105}.
\]

**22331** If there is \( i \in [1, 5] \) such that \( v_i \) is incident with a triangle and \( \deg(v_i) \geq 4 \), then \( \sum_{i=1}^{5} a(v_i, w) \leq 3 \cdot \frac{41}{400} + \frac{15}{476} < 38 \frac{1}{105} \).

**22332** Now suppose that all neighbours of \( w \) incident with a triangle are of degree three. Let \( f_j \) be the face adjacent to two triangles.

**223321** If \( d_j \in [5, 7] \), there is \( k \in [1, 5] \) such that \( d_k \geq 9 \). The face \( \tilde{f} \) adjacent to both \( f_j \) and \( f_k \) is of degree \( d' \geq d - 2 \) (Claim 1.2), hence for the vertex \( v_i \) incident with \( f_k \) and \( \tilde{f} \) we have \( a(v_i, w) = -\frac{1}{6} + \frac{1}{d_k} + \frac{1}{d} \leq \frac{1}{144} \) and, by Table 2,

\[
\sum_{i=1}^{5} a(v_i, w) \leq 3 \cdot \frac{41}{400} + \frac{1}{144} + \frac{15}{476} < 38 \frac{1}{105}.
\]

**223322** If \( d_j \in [8, d - 3] \), then \( \sum_{i=1}^{5} a(v_i, w) \leq 2 \cdot \frac{41}{400} + \frac{1}{105} + \frac{15}{476} < 38 \frac{1}{105} \).

**223323** If \( d_j \in [d - 2, d] \), notice that from Table 2 it follows that if \( \min\{d_i, d_{i+1}\} \geq 5 \), then \( p_i^5(w) + \overline{a}(d_i, d_{i+1}) < 0 \). Therefore, it suffices to show that if \( d_1 = 3 \), then \( \sum_{i=1}^{m} (p_i^5(w) + a(v_i, w)) \leq 0 \). Let \( d' \) be the degree of the face adjacent to \( f_{l-1}, f_l \) and \( f_{l+1} \). Claim 1.2 then yields \( d' \geq \max\{d+l-1, d+5-d_{i+1}\} \), and so, by RR 3.2,

\[
\sum_{i=1}^{m} (p_i^5(w) + a(v_i, w)) = -\frac{3}{5} + \frac{3}{2d_{i-1}} + \frac{3}{2d_{i+1}} + \frac{3}{2d_i} < \frac{3}{5} + \frac{3}{2d_{i-1}} + \frac{6}{2d_{i+1}} = 2d_i = \frac{3}{2d_i} < \frac{3}{5} + \frac{3}{32} + \frac{29}{60} < 0.
\]

(3) \( n = 4 \)

**31** If \( \deg(w) \leq d + 1 \), by Lemma 3.2 the vertex \( w \) is not incident with a contractible edge, hence, by Lemma 2, \( w \) has at least three neighbours of degree three. Since \( d_i < d \) for any \( i \in [1, 4] \), using Lemma 3.4 and \( C_2 \) we see that \( d_1 \geq 4 \). As in (21), \( d_1 = d_{i+1} = 4 \) implies \( a(v_i, w) = 0 \) and \( p_i^4(w) + a(v_i, w) = 0 \). Moreover, with help of Table 2 it is easy to check that \( p_i^4(w) + \overline{a}(d_i, d_{i+1}) \leq 0 \) whenever \( d_i + d_{i+1} \geq 9 \) (and \( \min\{d_i, d_{i+1}\} \geq 4 \)); as a consequence, \( c_4(w) \leq 0 \).

**32** If \( \deg(w) \geq d + 2 \), put \( q_i := p_i^4(w) + a(v_i, w) \) for \( i \in [1, \infty) \).

**321** If \( f_3 = 2 \), then, by Claim 1.1, \( w \) is of type \((3, d_2, 3, d_4)\), where \( d_2 + d_4 \geq d + 4 \). Since \( c_4(w) = (q_2 + q_3) + (q_4 + q_5) \), it is sufficient to show that \( q_i + q_{i+1} \leq 0 \) for any \( i \in \{2, 4\} \). So, in what follows we assume \( i \in \{2, 4\} \).

**3211** If \( \min\{\deg(v_i), \deg(v_{i+1})\} \geq 4 \), then \( q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d_4} \leq -\frac{1}{6} + \frac{1}{16} + \frac{1}{34} < 0 \).

**3212** If there is \( j \in [i, i+1] \) such that \( \deg(v_j) = 3 \) and \( \deg(v_{i+1-j}) \geq 4 \), then, by Lemma 3.4, \( d_4 = d \) and \( q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d} + a(v_j, w) \leq -\frac{1}{6} + \frac{1}{10} + \frac{1}{36} + a(v_j, w) = -\frac{7}{180} + a(v_j, w) \).

**32121** If \( a(v_j, w) \leq 0 \), then \( q_i + q_{i+1} < 0 \).

**32122** If \( a(v_j, w) > 0 \), then, by RR 3.1, \( v_j \) is of type \((3, d', d_2)\) (where \( d_2 \) appears either without loss of generality, namely if \( w \) is of type \((3, d, 3, d)\), or due to Lemma 3.4). By Claim 1.4 we obtain \( d' \in [5, 11] \), and so, by Claim 1.2, \( d_2 \geq d + 5 - d' \geq d - 6 \). Therefore, \( q_i + q_{i+1} \leq -\frac{1}{6} + \frac{1}{2d-6} + \frac{1}{24} + \frac{4}{45} \leq -\frac{1}{6} + \frac{1}{24} + \frac{1}{36} + \frac{4}{45} < 0 \).

**3213** If \( \deg(v_i) = \deg(v_{i+1}) = 3 \), then, by \( C_3 \), \( \min\{\deg(v_i), \deg(v_{i+1})\} \geq 4 \). Therefore, Claim 1.2 yields \( \min\{d_2, d_4\} \geq 6 \). Let \( d' \) be the degree of the face adjacent to the triangle \( v_i w v_{i+1} \) along the edge \( v_i v_{i+1} \). Then \( d_2 + d' - 3 = \min\{\deg(v_i), \deg(v_{i+1})\} \geq d + 3 \), hence \( d' \geq d + 6 - d_2 \).
(32131) If $d_2 \leq 8$, then $q_i = -\frac{1}{12} + \frac{3}{2d_2} + \bar{u}(3, d_2)$ and $q_{i+1} = -\frac{1}{3} + \frac{3}{2d_2} + \frac{1}{d^2} \leq -\frac{1}{2} + \frac{3}{18} + \frac{3}{26} + \frac{2}{10} < 0$.

(32131(1)) If $d_2 = 6$, then $q_i + q_{i+1} \leq -\frac{1}{12} - \frac{1}{3} + \frac{3}{32} + \frac{1}{18} < 0$.

(32131(2)) If $d_2 \in [7, 8]$ then $q_i + q_{i+1} \leq -\frac{1}{12} + \frac{1}{14} + \frac{3}{357} - \frac{1}{4} + \frac{3}{28} + \frac{1}{16} < 0$.

(32131(3)) If $d_2 \in [9, 14]$, then $d' \geq 10$ and $q_i + q_{i+1} = -\frac{1}{2} + \frac{1}{2d_2} + \frac{3}{2d_2} + \frac{2}{d^2} \leq -\frac{1}{2} + \frac{3}{18} + \frac{3}{26} + \frac{2}{10} < 0$.

(32133) If $d_2 \in [15, d - 2]$, then $q_i + q_{i+1} \leq -\frac{1}{2} + \frac{3}{4} + \frac{3}{20} + \frac{2}{8} < 0$.

(32134) If $d_2 = d - 1$, then $q_i + q_{i+1} \leq -\frac{1}{2} + \frac{3}{4} + \frac{3}{7} < 0$.

(32135) If $d_2 = d$, then $q_i + q_{i+1} \leq -\frac{1}{2} + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} = 0$.

(322) If $f_3 = 1$, consider the inequalities $q_i \leq -\frac{1}{d} + \frac{1}{2d_2} + \frac{1}{2d_1} + \bar{u}(d_i, d_{i+1}) \leq \bar{u}(d_i, d_{i+1})$, where $i \in [1, \infty)$, $\bar{u}(d_i', d_i^2)$ with $d_i' \leq d_i^2$ is an upper bound for $-\frac{1}{d} + \frac{1}{2d_2} + \frac{1}{2d_1} + \bar{u}(d_i', d_{i+1})$ presented in Table 3 (that is created using Table 2) and, provided $d_i' > d_i^2$, $\bar{u}(d_i', d_i^2) := \bar{u}(d_i', d_i^2)$. Since $d_1 = 3$, by Claim 1.2 we have $d_4 \geq d_2 \geq 5$; as $d_3 \geq 4$, from Table 3 we see that $q_i < 0$, $i = 2, 3$.

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<th>3</th>
<th>3</th>
<th>3</th>
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<tr>
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<td>7</td>
<td>8</td>
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<td>$d - 1, d$</td>
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<td>5</td>
</tr>
<tr>
<td>$\bar{u}(d_i', d_i^2)$</td>
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<td>$\frac{12}{12}$</td>
<td>$\frac{3}{5}$</td>
<td>$\frac{1}{20}$</td>
<td>$-\frac{1}{3}$</td>
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<tbody>
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<td>$d$</td>
<td>$\in [5, d - 5]$</td>
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<td>$\in [d - 2, d]$</td>
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<tr>
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<td>$-\frac{1}{24}$</td>
<td>$-\frac{1}{20}$</td>
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<th>7</th>
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<tr>
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<td>$\in [d - 5, d - 1]$</td>
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<td>$\in [d - 6, d] \in [d', d]$</td>
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<td>$-\frac{1}{9}$</td>
<td>$-\frac{3}{28}$</td>
<td>$-\frac{2}{17}$</td>
</tr>
</tbody>
</table>

Table 3

(3221) If $q_i \leq 0$, $i = 1, 4$, then $c_4(w) = \sum_{i=1}^{4} q_i < 0$.

(3222) $\max\{q_1, q_3\} > 0$

(32221) If $q_j + q_{j+2} \leq 0$ for $j = 1, 4$, then $c_4(w) = (q_1 + q_3) + (q_4 + q_6) \leq 0$.

(32222) Let $i \in \{1, 4\}$ be such that $q_i + q_{i+2} \geq q_{5-i} + q_{7-i}$ and $q_i + q_{i+2} > 0$ (so that $q_{i+2} < 0$ implies $q_i > 0$).

(322221) If $a(v_i, w) = 0$, then $q_i = -\frac{1}{12} + \frac{1}{2\max\{d_i, d_{i+1}\}}$, and so $\max\{d_i, d_{i+1}\} = 5$ and $q_i = \frac{1}{5}$ (for otherwise $q_i \leq 0$). Then, however, $d_i + d_{i+1} = cd(w) \geq d + 2$ and $\min\{d_i + d_{i+1}, d_i + d_{i+3}\} \geq 4$, so that Table 3 yields $q_i \leq -\frac{3}{32}$ and $q_i + q_{i+2} < 0$, a contradiction.

(322222) If $a(v_i, w) \neq 0$, then $\deg(v_i) = 3$ and $\mathrm{dm}(v_i) = \{3, s, d'\}$, where $s := \max\{d_i, d_{i+1}\}$.

(3222221) If $v_i$ is incident with a 1-triangle, then $s > d'$ (we are using RR 3.1), and so, by Claim 1.2, $s \geq 12$; then, by Table 3, $s \geq d - 1$ and $q_i \leq \frac{1}{35}$. 
Moreover, \( a(v_{5-i}, w) = 0 \) and, by Lemma 3.4, the edge \( v_{5-i}w \) is of type \((3, d)\) so that \( q_{5-i} = \frac{1}{12} + \frac{1}{36} \leq \frac{1}{12} + \frac{1}{36} = \frac{1}{18} \) and \( \sum_{j=1}^{4} q_j < q_i + q_4 \leq \frac{1}{30} - \frac{1}{18} < 0 \).

\[(322222)\] Now suppose that \( v_i \) is incident with a 2-triangle (which means that \( \deg(v_1) = \deg(v_4) = 3 \)). From Table 3 it follows that \( s \in \{5, 8\} \cup [d-1, d] \). We have \( s + d_{i+2} + d_{i+3} - 5 = cd(w) \geq d + 2 \), hence \( d_{i+2} + d_{i+3} \geq d + 7 - s \).

\[(322222)\] If \( s = 3 \), then \( d' = d \) (by Claim 1.2) and either \( \min\{d_{i+2}, d_{i+3}\} \in [4, 5] \) or \( \{d_{i+2}, d_{i+3}\} = \{6, d\} \), since otherwise \( q_{i+2} \leq -\frac{1}{12} \) and \( q_i + q_{i+2} \leq \frac{239}{2040} - \frac{5}{17} < 0 \). Thus, \( w \) is of one of types \((3, 5, 4, d_4), (3, 5, 5, d_4), (3, 5, 6, d), (3, 5, d, 6) \) and \((3, 5, d_3, 5)\); in the first four cases we have immediately \( i = 1 \) and in the last case we may suppose without loss of generality that \( i = 1 \).

\[(322222)\] If \( d_3 = 4 \), then \( d_4 \geq d - 2 \), \( q_3 \leq \tilde{u}(4, d_4) \) and \( q_4 = -\frac{1}{4} + \frac{1}{d} + \frac{3}{2d_4} \leq -\frac{7}{36} + \frac{3}{2d_4} \). Since \( \tilde{u}(4, d_4) + \frac{3}{2d_4} \leq \max\{\frac{5}{56} + \frac{3}{32}, \frac{1}{24} + \frac{3}{36}\} = \frac{1}{24} \), we obtain \( c_3(w) \leq \frac{239}{2040} - \frac{1}{36} - \frac{7}{36} + \frac{1}{24} < 0 \).

\[(322222)\] If \( w \) is of type \((3, 5, 5, d_4) \), then \( d_4 \geq d - 3 \), \( a(v_4, w) = -\frac{1}{5} + \frac{1}{d} + \frac{1}{2d_4} \leq -\frac{1}{6} + \frac{1}{15} + \frac{1}{18} = -\frac{2}{45} = -\frac{1}{45} \), \( q_4 \leq -\frac{1}{12} + \frac{1}{30} - \frac{2}{45} = -\frac{17}{180} \) and \( c_4(w) \leq \frac{239}{2040} - \frac{1}{20} - \frac{7}{68} - \frac{17}{180} < 0 \).

\[(322222)\] If \( w \) is of type \((3, 5, d_3, 5) \), then \( d_3 \geq d - 3 \) and \( c_4(w) \leq \gamma(3, 5, d - 3, 5) = -\frac{4}{15} + \frac{1}{d_3} \leq -\frac{4}{15} + \frac{1}{15} = -\frac{1}{5} \). It is easy to see that if a face \( f_j \) with \( j \in \{2, 4\} \) is incident with a vertex of type \((4, 5, d) \), then the number of such vertices is at most two and besides \( w \) there is at least one other absorbing vertex incident with \( f_j \). Therefore, the total amount received by \( w \) due to RR. 1.1 is bounded from above by \( 2\gamma(4, 5, 17) \), \( \sum_{j=1}^{4} a(v_j, w) \leq 2\gamma(3, 5, 18) + 2\gamma(4, 5, 17) \leq \frac{239}{1530} \) and \( c_4(w) \leq -\frac{1}{5} + \frac{239}{1530} < 0 \).

\[(322222)\] If \( \{d_3, d_4\} = \{6, d\} \), then \( c_0(w) = \gamma(3, 5, 6, d) = -\frac{3}{10} + \frac{1}{d} \leq -\frac{3}{10} + \frac{1}{18} = -\frac{11}{36} \), \( \sum_{j=1}^{4} a(v_j, w) \leq \max\{\frac{3}{170}, \frac{1}{12}, 0, \frac{4}{3}\} \leq \frac{11}{18} \), and so \( c_4(w) < 0 \).

\[(322222)\] If \( s \in [6, 8] \), then \( q_i \leq \tilde{u}(3, s) \) and \( q_{i+2} \leq \max\{\tilde{u}(d_i', d_2') : d_i' \geq 4, d_i' + d_2' \geq d + 7 - s\} \). From Table 3 it follows that \( i = 1 \), \( d_3 = 4 \) and \( d_4 = d \) (for otherwise \( q_i + q_{i+2} < 0 \), a contradiction). Claim 1.2 yields \( d' \geq d + 5 - s \), hence \( q_4 = -\frac{1}{12} + \frac{1}{15} + (\frac{1}{12} + \frac{1}{15}) \leq -\frac{4}{15} + \frac{3}{36} + \frac{1}{15} = -\frac{1}{10} \) and, by Table 3, \( \sum_{j=1}^{4} q_j \leq \frac{1}{15} - \frac{1}{25} - \frac{1}{25} - \frac{1}{10} < 0 \).

\[(322222)\] If \( s \in [d-1, d] \), then \( \{d_{i+2}, d_{i+3}\} = [4, 5] \), for otherwise \( q_i + q_{i+2} \leq \frac{1}{30} - \frac{1}{24} < 0 \). By Claim 1.1 \( w \) is of type \((3, 5, 4, d_4) \), hence \( i = 4 \) and \( d' = d \) (by Claim 1.2). Therefore, \( q_4 = -\frac{1}{4} + \frac{1}{2} + \frac{3}{2d_4} \leq -\frac{1}{4} + \frac{1}{18} + \frac{3}{36} < 0 \), a contradiction.

\[(323)\] \( f_3 = 0 \)

\[(3231)\] If \( q_i \leq 0 \) or \( q_i + q_{i+2} \leq 0 \) for every \( i \in [1, 4] \), then \( c_4(w) \leq 0 \).

\[(3232)\] Let \( i \in [1, 4] \) such that \( q_i > 0 \) and \( q_i + q_{i+2} > 0 \). From Table 3 it follows that \( d_i = d_{i+1} = 4 \) and \( q_i \leq \frac{1}{18} \). Since \( d_{i+2} + d_{i+3} = cd(w) \geq d + 2 \), Table 3 yields also \( \{d_{i+2}, d_{i+3}\} = [4, d] \). Thus, \( w \) is of type \((4, 4, 4, d) \), we may suppose without loss of generality that \( i = 1 \) and \( c_0(w) = \gamma(4, 4, 4, d) = -\frac{1}{3} + \frac{1}{d} \leq -\frac{7}{36} \).

\[(32321)\] If \( \max\{\deg(v_j) : j \in [1, 4]\} \geq 4 \), then \( c_4(w) \leq -\frac{5}{36} + 3 \cdot \frac{1}{18} < 0 \).

\[(32322)\] If \( \deg(v_3) = 3 \) for any \( j \in [1, 4] \), consider the quadrangle \( v_1wv_2x \).

\[(323221)\] If \( \deg(x) = 3 \), then \( x \) is of type \((4, 4, d) \) and, by RR 2.1, \( c_2(v_1) = \gamma(4, 4, d) + \frac{1}{2}\gamma(4, 4, d) = -\frac{1}{6} + \frac{3}{4} \leq -\frac{1}{12} \), hence \( q_1 = a(v_1, w) = 0 \), which contradicts \( q_1 > 0 \).

\[(323222)\] If \( \deg(x) \geq 4 \), then, by RR 4.1, \( q_1 = a(v_1, w) \leq \frac{1}{2}c_3(v_1) \leq \frac{1}{2}\gamma(4, 4, d) =
\[ \frac{1}{22} \leq \frac{1}{36} \quad \text{and} \quad q_1 + q_3 \leq \frac{1}{36} - \frac{1}{24} < 0 \quad \text{in contradiction with} \; q_i + q_{i+2} > 0. \]

(4) \( n = 3 \)

(41) If \( d_1 = 3 \), then \( w \) belongs to an \( i \)-triangle \( t \), \( i \in [1, 3] \).

(411) \( i = 1 \)

(4111) If \( c_0(w) \leq 0 \), then \( d_4 \geq 9 \) (Claim 1.5), hence \( c_4(w) = c_0(w) \leq 0 \).

(4112) If \( c_0(w) > 0 \), then \( c_2(w) \geq c_0(w) > 0 \), and so, by RR 3.1, \( c_4(w) = 0 \).

(412) If \( i = 2 \), then applying RR 3.2 yields \( c_4(w) = 0 \).

(413) \( i = 3 \)

(4131) If \( t \) is positive, then, by RR 1.2 and \( C_6 \), we have \( c_4(w) = 0 \).

(4132) If \( t \) is nonpositive, then, by RR 1.4, \( c_4(w) = \frac{1}{2} \sum (c_0, V(t)) \leq 0 \).

(42) \( d_1 = 4 \)

(421) \( d_2 = 4 \)

(4211) If \( c_3(w) \leq 0 \), then \( c_4(w) = c_3(w) \leq 0 \).

(4212) If \( c_3(w) > 0 \), then necessarily also \( c_2(w) > 0 \).

(42121) If \( n_{++}(w) \geq 1 \), then, by RR 4.1, \( c_4(w) = 0 \).

(42122) \( n_{++}(w) = 0 \)

(421221) If \( n_{++}(v_1) \geq 1 \), then, by RR 3.3, \( c_4(w) = 0 \).

(421222) If \( n_{++}(v_1) = 0 \), then, by \( C_4 \), for any \( i \in [2, 3] \) the type \( (4, d'_i, d) \) of the vertex \( v_i \) is such that \( d'_i \geq 6 \). Therefore, by \( C_5 \) and RR 2.1, \( c_3(w) = \gamma(4, 4, d) + \gamma(4, d'_2, d) + \gamma(4, d'_3, d) = -\frac{1}{2} + \frac{3}{4} + \frac{1}{d'_2} + \frac{1}{d'_3} \leq -\frac{1}{2} + \frac{3}{18} + 2 \cdot \frac{1}{6} = 0 \), a contradiction.

(422) If \( d_2 = 5 \), then, by RR 1.1, \( c_4(w) = 0 \).

(423) If \( d_2 \geq 6 \), then \( c_0(w) \leq 0 \) (Claim 1.4).

(4231) If \( w \) has not received any amount, then \( c_0(w) \leq c_4(w) \leq 0 \).

(4232) If \( w \) has received an amount, then \( d_2 = 6 \) and the rule RR 1.2 has been applied; then, by Claim 2, \( c_1(w) \leq \gamma(4, 6, d) + \frac{1}{2} \beta(6, d) = -\frac{1}{6} + \frac{3}{4} \leq 0 \), and so \( c_3(w) \leq c_4(w) \leq 0 \).

(43) If \( d_1 \geq 5 \), then, by Claim 1.4, \( c_0(w) \leq 0 \).

(431) If \( w \) has not received any amount, then \( c_0(w) \leq c_4(w) \leq 0 \).

(432) If \( w \) has received an amount, then either \( d_1 = 5 \) and RR 1.1 has been applied or \( [6, 7] \cap \text{dm}(w) \neq \emptyset \) and RR 1.2 has been applied.

(4321) If \( d_1 = 5 \), then \( d_2 \geq 11, d_3 \geq d - 1 \) and \( c_4(w) \leq \gamma(5, 11, d - 1) + 4 \gamma(4, 5, d - 1) \leq -\frac{9}{22} + \frac{5}{17} < 0 \).

(4322) If \( 6 \in \text{dm}(w) \), then \( \text{dm}(w) = \{ 6, s, d \} \) with \( s \in [5, d] \) and \( c_4(w) \leq \gamma(6, 5, d) + \frac{1}{2} \beta(6, d) = -\frac{13}{60} + \frac{3}{7} \leq -\frac{13}{60} + \frac{3}{18} < 0 \).

(4323) If \( 7 \in \text{dm}(w) \), then \( d_1 = 7, d_2 \geq 10 \) and \( c_4(w) \leq \gamma(7, 10, 10) + 3 \beta(7, d) \leq -\frac{4}{5} + \frac{12}{17} < 0 \).

Since \( c_4(w) \leq 0 \) for any \( w \in V \), the proof is complete. \( \blacksquare \)

References


[7] A. Morita, Cyclic chromatic number of 3-connected plane graphs (Japanese, M. S. Thesis), Keio University, Yokohama 1998


