Another step towards proving a conjecture by Plummer and Toft*

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Abstract. A cyclic colouring of a graph G embedded in a surface is a vertex colouring of G in which any two distinct vertices sharing a face receive distinct colours. The cyclic chromatic number $\chi_{\rm c}(G)$ of G is the smallest number of colours in a cyclic colouring of G. Plummer and Toft in 1987 conjectured that $\chi_{\rm c}(G) \le \Delta^* + 2$ for any 3-connected plane graph G with maximum face degree Δ^*. It is known that the conjecture holds true for $\Delta^* \le 4$ and $\Delta^* \ge 24$. The validity of the conjecture is proved in the paper for $\Delta^* \ge 18$.

1 Introduction

Let $G = (V, E, F)$ be a cell-embedding of a 2-connected graph in a 2-manifold. The *degree* $\deg(x)$ of $x \in V \cup F$ is the number of edges incident with x. A vertex of degree k is a k-*vertex*, a face of degree k is a k-face. By $V(x)$ we denote the set of all vertices incident with $x \in E \cup F$; similarly, $F(y)$ is the set of all faces incident with $y \in V \cup E$. If $e \in E$, $F(e) = \{f_1, f_2\}$ and $\deg(f_1) \le \deg(f_2)$, the pair $(\deg(f_1), \deg(f_2))$ is called the *type* of e. A (d_1, d_2)-*neighbour* of a vertex x is a vertex y such that the edge xy is of type (d_1, d_2). Paths and cycles in G will be understood as vertex sequences in which any two vertices placed on neighbouring positions are adjacent in G. A cycle in G is *facial* if its vertex set is equal to $V(f)$ for some $f \in F$. Though graphs we are dealing with are nonoriented, sometimes it will be useful to equip certain edges with one of two possible orientations. A vertex x_1 is *cyclically adjacent* to a vertex $x_2 \ne x_1$ if there is a face f with $x_1, x_2 \in V(f)$. The *cyclic neighbourhood* $N_{\rm c}(x)$ of a vertex x is the set of all vertices that are cyclically adjacent to x and the *closed cyclic neighbourhood* of x is $\bar{N}_{\rm c}(x) := N_{\rm c}(x) \cup \{x\}$. (The usual neighbourhood of x is denoted by $N(x)$.) The *cyclic degree* of x is

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$\mathrm{cd}(x) := |N_c(x)|$. A *cyclic colouring* of G is a mapping $\varphi : V \to C$ in which $\varphi(x_1) \neq \varphi(x_2)$ whenever x_1 is cyclically adjacent to x_2 (elements of C are *colours* of φ). The cyclic chromatic number $\chi_c(G)$ of the graph G is the minimum number of colours in a cyclic colouring of G.

The invariant $\chi_c(G)$ was introduced by Ore and Plummer [8] for plane graphs (and in the dual form). Sanders and Zhao [10] proved that $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^*(G) \rceil$ for any 2-connected plane graph G, where $\Delta^*(G)$ is the *maximum face degree* of G. On the other hand, there is an infinite family of 2-connected plane graphs G satisfying $\chi_c(G) = \lceil \frac{3}{2}\Delta^*(G) \rceil$. It is conjectured that $\chi_c(G) \leq \lceil \frac{3}{2}\Delta^*(G) \rceil$ for *any* 2-connected plane graph G.

However, our interest is concentrated on 3-connected plane graphs. By a classical result of Whitney [11] all plane embeddings of a 3-connected planar graph are essentially the same. This means that $\chi_c(G_1) = \chi_c(G_2)$ if G_1, G_2 are plane embeddings of a fixed 3-connected planar graph G; thus, we can speak simply about the cyclic chromatic number of G. On the other hand, when analysing $\chi_c(G)$ for a 3-connected planar graph G, any edge of G can be chosen to be incident or not to be incident with the unbounded face of an embedding of G in the plane. Plummer and Toft in [9] proved that $\chi_c(G) \leq \Delta^*(G) + 9$ and conjectured that $\chi_c(G) \leq \Delta^*(G) + 2$ for any 3-connected plane graph G. Let $\mathrm{PTC}(d)$ denote that conjecture restricted to graphs with $\Delta^*(G) = d$. Because of Four Colour Theorem we know that for a triangulation G we have $\chi_c(G) \leq 4 = \Delta^*(G) + 1$. PTC(4) is known to be true due to Borodin [2]. Horňák and Jendroľ [6] proved $\mathrm{PTC}(d)$ for any $d \geq 24$. The bound was moved to 22 by Morita [7], but the proof was probably never published in an article. Enomoto et al. [4] obtained for $\Delta^*(G) \geq 60$ even a stronger result, namely that $\chi_c(G) \leq \Delta^*(G) + 1$. The example of the (graph of) d-sided prism with maximum face degree d and cyclic chromatic number $d+1$ shows that the bound is best possible. The best known general result (with no restriction on $\Delta^*(G)$) is the inequality $\chi_c(G) \leq \Delta^*(G) + 5$ of Enomoto and Horňák [3].

The conjecture is still open. This means that we do not know any G with $\chi_c(G) - \Delta^*(G) \geq 3$. On the other hand, all G's with $\chi_c(G) - \Delta^*(G) = 2$ we are aware of satisfy $\Delta^*(G) = 4$. Therefore, the conjecture could be strengthened so that $\chi_c(G) \leq \Delta^*(G) + 1$ for any 3-connected plane graph G with $\Delta^*(G) \neq 4$.

For $p, q \in \mathbb{Z}$ let $[p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : p \leq z\}$. The *concatenation* of finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB := (a_1, \dots, a_m, b_1, \dots, b_n)$. Because of the obvious associativity of concatenation we can use the symbol $\prod_{i=1}^{k} A_i$ for the concatenation of $k \in [0, \infty)$ finite sequences in the order given by the sequence $(A_1, \dots, A_k)$. If $A_i = A$ for all $i \in [1, k]$, $\prod_{i=1}^{k} A_i$ is replaced by A^k, where $A^0 = (\ )$ is the empty sequence.

Let $d \in [5, \infty)$ and $k \in [1, 5]$. A (d, k)*-minimal* graph is a 3-connected plane graph G that satisfies (i) $\Delta^*(G) = d$, (ii) $\chi_c(G) > d + k$ and (iii) $\chi_c(H) \leq d + k$ for any 3-connected plane graph H such that $\Delta^*(H) \leq d$ and the pair $(|V(H)|, |E(H)|)$ is lexicographically smaller then the pair $(|V(G)|, |E(G)|)$. A configuration $\mathcal{C}$ is said to be (d, k)*-reducible* if it does not appear in any (d, k)-minimal graph.

Let G be an embedding of a 2-connected graph and let v be its vertex of degree n. Consider a sequence $(f_1, \dots, f_n)$ of faces incident with v in a cyclic order around v

(there are altogether $2n$ such sequences) and the sequence $D = (d_1, \dots, d_n)$ in which $d_i = \deg(f_i)$ for $i \in [1, n]$. The sequence D is called the *type* of the vertex v provided it is the lexicographical minimum of the set of all such sequences corresponding to v, i.e., of the set $\bigcup_{i=1}^{n}(\{\prod_{j=0}^{n-1}(d_{i+j})\} \cup \{\prod_{j=0}^{n-1}(d_{i-j})\})$, where indices are taken modulo n in the interval $[1, n]$. It is easy to see that $\mathrm{cd}(v) = \sum_{i=1}^{n}(d_i - 2)$. The multiset $\mathrm{dm}(v) := \{d_1, \dots, d_n\}$ is the *degree multiset* of the vertex v. A *contraction* of an edge $xy \in E(G)$ consists in a continuous identification of the vertices x and y forming a new vertex $x \leftrightarrow y$ and the removal of the created loop together with all possibly created multiedges; if G/xy is the result of such a contraction, then, clearly, $\Delta^*(G/xy) \le \Delta^*(G)$. An edge xy of a 3-connected plane graph G is *contractible* if G/xy is again 3-connected.

2 Auxiliary results

The lexicographical minimum of $(|V(G)|, |E(G)|)$ over 3-connected plane graphs G with $\Delta^*(G) = d$ is $(d+1, 2d)$ and is attended by a plane embedding Π_d of the graph of d-sided pyramid. Since $\chi_c(\Pi_d) = d + 1 = \Delta^*(\Pi_d) + 1$, if there is a graph violating PTC (with maximum face degree $d \in [5, 23]$), there must be a 3-connected plane graph G that is $(d, 2)$-minimal. We are now going to prove that the structure of such a graph is quite restricted. For that purpose the following assertions will be useful:

Lemma 1 (Halin [5]) *Any 3-vertex of a 3-connected plane graph G with $|V(G)| \ge 5$ is incident with a contractible edge.* ■

Lemma 2 (a consequence of results of Ando et al. [1]) *If a vertex of degree at least four of a 3-connected plane graph G with $|V(G)| \ge 5$ is not incident with a contractible edge, it is adjacent to three 3-vertices.* ■

Lemma 3 *If $d \in [6, \infty)$, the following configurations are $(d, 2)$-reducible:*

1. *a 3-vertex x with $\mathrm{cd}(x) \le d + 1$;*
2. *a vertex x with $\deg(x) \ge 4$ and $\mathrm{cd}(x) \le d+1$ that is incident with a contractible edge;*
3. *a vertex x with $\deg(x) \ge 4$ and $\mathrm{cd}(x) \le d + 1$ that is adjacent to a 3-vertex y with $\mathrm{cd}(y) \le d + 2$;*
4. *a triangle t incident with exactly one 3-vertex such that the face adjacent to t along the edge joining vertices of degree at least four is of degree at most $d - 1$;*
5. *a separating 3-cycle;*
6. *an edge of type $(3, d_2)$ with $d_2 \in [3, 4]$;*
7. *the configuration $\mathcal{C}_i$ of Fig. i, $i \in [1, 7]$, where encircled numbers represent degrees of corresponding vertices, vertices without degree specification are of an arbitrary degree and dashed lines are parts of facial cycles.*

Proof. 1.–4. The statements have already been proved in [6] (Lemma 3.1(e), 3.3(i), 3.3(ii) and 3.4). For the rest of the proof suppose there is a $(d, 2)$-minimal graph G that contains a configuration $\mathcal{C}$ described in Lemma 3.5, 3.6 or 3.7.

5. If $\mathcal{C}$ is a separating 3-cycle $x_1x_2x_3$, let G_1 and G_2 be components of the graph $G - \{x_1, x_2, x_3\}$. It is easy to see that the subgraph H_i of G induced by $V(G_i) \cup \{x_1, x_2, x_3\}$ is a 3-connected plane graph with $\Delta^*(H_i) \le d$ and $|V(H_i)| < |V(G)|$, hence there is a cyclic colouring $\varphi_i : V(H_i) \to C$, $i = 1, 2$, where $|C| = d+2$. Without loss of generality we may suppose that $\varphi_1(x_i) = \varphi_2(x_i)$, $i = 1, 2, 3$. Then $\psi : V(G) \to C$ determined by $\psi(x) := \varphi_i(x) \overset{df.}{\Leftrightarrow} x \in V(H_i)$, $i = 1, 2$, is a cyclic colouring of G in contradiction with $\chi_c(G) > d + 2$.

6. Now let G contain a triangle xy_1y_2 adjacent to a quadrangle $y_1y_2z_2z_1$. Without loss of generality we may suppose that neither of the two faces incident with y_1y_2 is unbounded. By Lemma 3.1 we have $\deg(y_i) \ge 4$, $i = 1, 2$, and consequently, by Lemma 3.4, $\deg(x) \ge 4$. If the graph $G' := G - y_1y_2$ is 3-connected, it has a cyclic colouring using at most $d + 2$ colours which is also a cyclic colouring of G, a contradiction. Therefore, G' has to be 2-connected. Consider a cutset $\{v_1, v_2\}$ of G'. Clearly, $\{v_1, v_2\} \cap \{y_1, y_2\} = \emptyset$, so there is a component $C(y_i)$ of the graph $G'' := G' - \{v_1, v_2\}$ containing the vertex y_i, $i = 1, 2$. From 3-connectedness of G it follows that any vertex of G'' belongs either to $C(y_1)$ or to $C(y_2)$, hence $C(y_1) \ne C(y_2)$, $x \in \{v_1, v_2\}$ and $\{v_1, v_2\} \subseteq \{x, z_1, z_2\}$ (otherwise there is a path joining y_1 to y_2 in G'').Thus we may suppose without loss of generality that $v_1 = x$ and $v_2 = z_j$ for some $j \in [1, 2]$. Then both x and z_j are incident with the unbounded face f of G. Because of Lemma 3.5 the vertices x and z_j are not adjacent in G, otherwise (x, y_j, z_j, x) would be a separating 3-cycle of G. Therefore, the facial cycle of the unbounded face of G is of the form $(x)P^1(z_j)P^2(x)$, where both paths P^1 and P^2 are nonempty. For $i = 1, 2$ consider the cycle $C^i := (x)P^i(z_j, y_j, x)$, the plane subgraph G^i of G induced by all vertices lying in the closed disc bounded by the closed Jordan curve corresponding to C^i, and join vertices x and z_j of G^i by an arc lying in the unbounded face of G^i. It is easy to see that we obtain a 3-connected plane graph H^i with $\Delta^*(H^i) \le d$ and $|V(H^i)| < |V(G)|$, hence there is a cyclic colouring $\varphi^i : V(H^i) \to C$; if f^i is the unbounded face of H^i, then $V(f^1) \cup V(f^2) = V(f)$ has at most d vertices, and so we may suppose without loss of generality that $\varphi^1(v) = \varphi^2(v)$ for any $v \in \{x, y_j, z_j\}$ (note that xy_jz_j is a 3-face of both H^1 and H^2) and $\varphi^1(V(f^1) - \{x, z_j\}) \cap \varphi^2(V(f^2) - \{x, z_j\}) = \emptyset$. As in Lemma 3.5, the colouring $\psi : V(G) \to C$ with $\psi(x) := \varphi_i(x) \overset{df.}{\Leftrightarrow} x \in V(H_i)$, $i = 1, 2$, yields a contradiction.

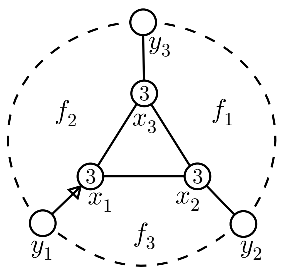

Fig. 1: $\mathrm{cd}(x_1)$=d+2

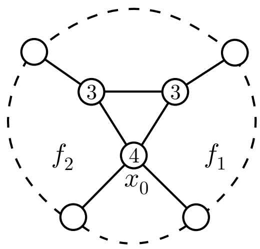

Fig. 2: cd(x_0)≤d+1

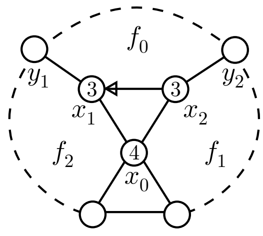

Fig. 3: cd(x_1)=d+2

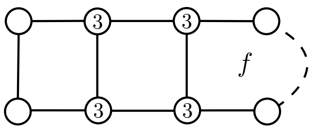

Fig. 4: deg(f)∈[4,5]

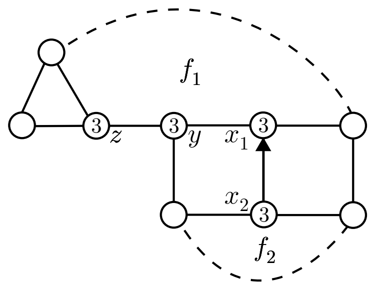

Fig. 5

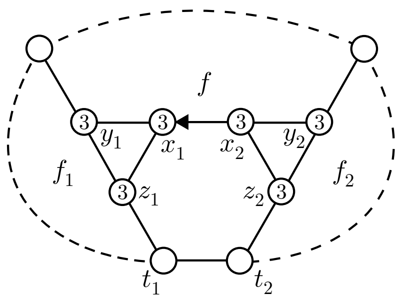

Fig. 6

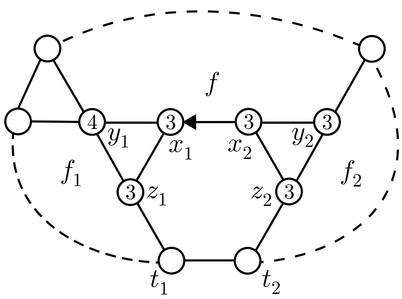

Fig. 7

7. If $\mathcal{C} = \mathcal{C}_i$, $i \in \{1, 3, 5, 6, 7\}$, the configuration $\mathcal{C}$ contains a 3-vertex x_1 incident with a contractible edge $u_i x_1$; the oriented edge (u_i, x_1) is indicated by an arrow. The graph $G' := G/u_i x_1$ is a 3-connected plane graph satisfying $\Delta^*(G') \leq d$ and $|V(G')| = |V(G)| - 1$, hence there is a cyclic colouring $\varphi : V(G') \to C$. This colouring will be used to find a cyclic colouring $\psi : V(G) \to C$ to obtain a contradiction with $\chi_c(G) > d + 2$. If not stated explicitly otherwise, we put $\psi(u) := \varphi(u)$ for any $u \in V(G) - \{u_i, x_1\}$ and $\psi(u_i) := \varphi(u_i \leftrightarrow x_1)$ (so that we have to determine only $\psi(x_1)$).

$i = 1$: If there is a colour that appears twice on vertices of $N_c(x_1)$ (under φ), from $\mathrm{cd}(x_1) = d + 2$ we see that at least one colour is available as $\psi(x_1)$. Henceforth suppose that $|\varphi(N_c(x_1))| = d + 2$. Put $W := \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $C_j := \varphi(V(f_j) - W)$, $j = 1, 2, 3$, then $C_2 \cap C_3 = \emptyset$. If there is $j \in [2, 3]$ such that $C_j - C_1 \neq \emptyset$, we take $\psi(x_j) \in C_j - C_1$ and define $\psi(x_1) := \varphi(x_j)$. To conclude this case notice that $C_2 - C_1$ and $C_3 - C_1$ cannot be both empty, since then $C_j \subseteq C_1$, $j = 2, 3$, and $\deg(f_1) = |C_1| + 4 \geq |C_2| + |C_3| + 4 = d + 1$, a contradiction.

$i = 2$: Since, by Lemma 3.6, $\deg(f_j) \geq 5$, the configuration $\mathcal{C}_2$ is $(d, 2)$-reducible by Lemma 3.2 of [6].

$i = 3$: As for $i = 1$ it is sufficient to analyse the case in which $|\varphi(N_c(x_1))| = d + 2$. Putting $W := \{x_0, x_1, x_2, y_1, y_2\}$ and $C_j := \varphi(V(f_j) - W)$, $j = 0, 1, 2$, we obtain $C_0 \cap C_2 = \emptyset$. If $C_2 - C_1 \neq \emptyset$, we are done by taking $\psi(x_2) \in C_2 - C_1$ and $\psi(x_1) := \varphi(x_2)$. On the other hand, $C_2 - C_1 = \emptyset$ implies $C_1 \subseteq C_2$, and so defining $\psi(x_1) := \varphi(x_0)$ leaves at least one colour available for $\psi(x_0)$.

$i = 4$: For the proof see Lemma 3.1(c) and 3.1(d) of [6].

$i = 5$: In this case $\varphi(x_2 \leftrightarrow x_1)$ can be used as either $\psi(x_1)$ or $\psi(x_2)$. By Lemma 3.1 we have $\deg(f_1) = \deg(f_2) = d$, and so we may suppose (similarly as for $i = 1$ or $i = 3$) that $|\varphi(N_c(x_1))| = d + 2$ and $|\varphi(N_c(x_2) - \{x_1\})| = d + 1$. Since $N_c(z) \subseteq \bar{N}_c(y)$, this allows us to define $\psi(x_1) := \varphi(x_1 \leftrightarrow x_2)$, $\psi(x_2) := \varphi(y)$, $\psi(y) := \varphi(z)$ and $\psi(z) := \varphi(y)$.

$i = 6, 7$: By Lemma 3.7.1 and 3.7.3 (for $i = 7$) we have $\deg(f_1) = \deg(f_2) = \deg(f) = d$ and $\mathrm{cd}(v) = d + 3$ for any $v \in \{x_1, x_2, z_1, z_2\}$. If there is a colour (of C) not present in $\varphi(\bar{N}_c(x_2) - \{x_1\}) = \varphi(N_c(x_1))$, we use it as $\psi(x_1)$. Henceforth we suppose that the vertex x_2 is *saturated* – all colours of C appear on vertices of its closed cyclic neighbourhood; as x_1 is not coloured under φ, on vertices of the cyclic neighbourhood of x_2 one colour appears twice and d colours appear once. If $\varphi(z_j) \notin \varphi(V(f))$ and $c \in C - \varphi(N_c(z_j) - \{x_1\})$, then we are done (i.e., we obtain a contradiction) by putting $\varphi(z_j) := c$, $\psi(x_j) := \varphi(z_j)$ and $\psi(x_{3-j}) := \varphi(x_2 \leftrightarrow x_1)$. Therefore, we assume that $\varphi(z_j) \notin \varphi(V(f))$ implies the vertex x_j is saturated, $j = 1, 2$. There is $j \in [1, 2]$ such that the x_2-*duplicated* colour, i.e., one that appears twice on vertices of $N_c(x_2)$, is either $\varphi(t_j)$ or $\varphi(z_j)$. If $\varphi(t_j)$ is x_2-duplicated, then obviously $\varphi(z_j) \notin \varphi(V(f))$, so z_j is saturated, at most one of $\varphi(t_{3-j})$ and $\varphi(z_{3-j})$ is z_j-duplicated and $\{\varphi(t_{3-j}), \varphi(z_{3-j})\} - \varphi(V(f_j)) \neq \emptyset$. If, say, $\varphi(t_{3-j}) \notin \varphi(V(f_j))$, then, having in mind that $\varphi(t_{3-j}) \notin \varphi(V(f))$, we can take $\psi(y_j) := \varphi(t_{3-j})$ and $\psi(x_1) := \varphi(y_j)$. Now let $\varphi(z_j)$ be x_2-duplicated; as a consequence, z_{3-j} is saturated. If one of $\varphi(t_{3-j}), \varphi(z_{3-j})$ is out of $\varphi(V(f_j))$, we use it as $\psi(y_j)$ and put $\psi(x_1) := \varphi(y_j)$. On the other hand, provided $\{\varphi(t_{3-j}), \varphi(z_{3-j})\} \subseteq \varphi(V(f_j))$, there is a colour $c \in C - \varphi(\bar{N}_c(z_j) - \{x_1\})$, which allows us to define $\psi(z_j) := c$ together with either $\psi(z_{3-j}) := \varphi(z_j)$ and $\psi(x_1) := \varphi(z_{3-j})$ (if $\varphi(t_j)$ is z_{3-j}-duplicated) or $\psi(y_{3-j}) := \varphi(t_j)$ and $\psi(x_1) := \varphi(y_{3-j})$ (otherwise). ■

Note that the configurations of Lemma 3, except for $\mathcal{C}_6$ and $\mathcal{C}_7$, are even $(5, 2)$-reducible.

Our main theorem will be proved by Discharging Method. Namely, we shall suppose that there is a $(d, 2)$-minimal graph $G = (V, E, F)$ for some $d \in [18, \infty)$. From Euler's Theorem $|V| - |E| + |F| = 2$ it is easy to derive that $\sum_{v \in V} c_0(v) = 2$ for the mapping $c_0 : V \to \mathbb{Q}$ (called the *initial* charge) with

$$c_0(v) := 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)}.$$

Putting $\Sigma(c_0, W) := \sum_{v \in W} c_0(v)$ for $W \subseteq V$ we have $\Sigma(c_0, V) = 2$. We are able to find consecutively in four phases charge mappings $c_i : V \to \mathbb{Q}$, $i = 1, 2, 3, 4$, such that $\Sigma(c_i, V) = 2$, which means that passing from c_{i-1} to c_i is simply a redistribution of charges of vertices that is governed by redistribution rules. The restriction on the structure of G yielded by Lemma 3 enables us to prove that $c_4(v) \le 0$ for any $v \in V$, which represents a contradiction with $\Sigma(c_4, V) = 2$.

If a vertex $v \in V$ is of type $(d_1, \dots, d_n)$, then

$$c_0(v) = \gamma(d_1, \dots, d_n) := 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i}.$$

Clearly, if π is a permutation of the set $[1, n]$, then $\gamma(d_{\pi(1)}, \dots, d_{\pi(n)}) = \gamma(d_1, \dots, d_n)$. Let the *weight* of a sequence $D = (d_1, \dots, d_n) \in \mathbb{Z}^n$ be defined by $\mathrm{wt}(D) := \sum_{i=1}^{n} d_i$. For $n \in [2, \infty)$, $q \in [0, n-2]$, $(d_1, \dots, d_{n-1}) \in [1, \infty)^{n-1}$ and $w \in [\sum_{i=1}^{n-1} d_i + 1, \infty)$ let $S_q(d_1, \dots, d_{n-1}; w)$ be the set of all sequences $D = (d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in \mathbb{Z}^n$ satisfying $d'_i \ge d_i$ for any $i \in [q+1, n-1]$ and $\mathrm{wt}(D) \ge w$. An analogue of the following statement has been proved as Lemma 4 in [6] (with a different definition of γ).

Lemma 4 *The maximum of* $\gamma(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n)$ *over all sequences* $(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in S_q(d_1, \dots, d_{n-1}; w)$ *is equal to* $\gamma(d_1, \dots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$.

Proof. Pick a sequence $(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in S_q(d_1, \dots, d_{n-1}; w)$. Decrease d'_i to d_i and increase d'_n by $d'_i - d_i$ successively for all $i \in [q+1, n-1]$. If $a_1, a_2, a_3, a_4 \in [1, \infty)$, $a_1 + a_2 = a_3 + a_4$ and $a_1 < \min(a_3, a_4)$, then $\frac{1}{a_3} + \frac{1}{a_4} < \frac{1}{a_1} + \frac{1}{a_2}$. Moreover, with $d''_n := d'_n + \sum_{i=q+1}^{n-1}(d'_i - d_i)$ we have $\sum_{i=1}^{n-1} d_i + d''_n = \mathrm{wt}(d_1, \dots, d_n, d''_n) = \mathrm{wt}(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \ge w$, hence $(d_1, \dots, d_{n-1}, d''_n) \in S_q(d_1, \dots, d_{n-1}; w)$ and $\gamma(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \le \gamma(d_1, \dots, d_{n-1}, d''_n) \le \gamma(d_1, \dots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$. Here equalities apply if and only if $d'_i = d_i$ for any $i \in [q+1, n-1]$ and $d'_n = d''_n = w - \sum_{i=1}^{n-1} d_i$. ■

3 Proof of Theorem

As already mentioned, for the proof by contradiction we suppose that $G = (V, E, F)$ is a $(d, 2)$-minimal graph with $\Delta^*(G) = d \in [18, \infty)$. A set $W \subseteq V$ is *positive* if $\Sigma(c_0, W) > 0$, otherwise it is nonpositive; similarly is defined a negative and a nonnegative set. If $W = \{w\}$ or $W = V(f)$, $f \in F$, we shall speak simply about a positive (nonpositive, negative, nonnegative) vertex w or face f, respectively. A triangle $t \in F$ is an *i-triangle* if the number of 3-vertices in $V(t)$ is i. For a vertex $v \in V$ let $N_{4+}(v)$ denote the set of all neighbours of v of degree at least four and put $n_{4+}(v) := |N_{4+}(v)|$. Now we are going to prove a series of claims concerning vertices of V and faces of F (which is implicitly assumed in those claims).

Claim 1. 1. If faces f_1 and f_2 are adjacent to each other, then $\deg(f_1) + \deg(f_2) \ge 8$.

2. If a vertex is of type (d_1, d_2, d_3), then $d_3 \ge d + 8 - d_1 - d_2$.

3. If a vertex is positive, it is of degree 3.

4. If a vertex of type (d_1, d_2, d_3) is positive, then either $d_1 = 3$ and $d_2 \in [5, 11]$ or $d_1 = 4$ and $d_2 \in [4, 5]$.

5. If a vertex of type $(3, d_2, d_3)$ is nonpositive, then $d_2 \geq 7$.

Proof. 1. The inequality follows from Lemma 3.6.

For the rest of the proof consider an n-vertex v of type $(d_1, \ldots, d_n)$ and put $d_{n+i} := d_i$ for $i \in [1, n]$.

2. If $\deg(v) = 3$, then $\mathrm{cd}(v) = d_1 + d_2 + d_3 - 6$. To obtain the desired inequality use Lemma 3.1.

3. Suppose that $n \geq 4$. By Claim 1.1 we have $d_i + d_{i+1} \geq 8$ and $\frac{1}{d_i} + \frac{1}{d_{i+1}} \leq \max\{\frac{1}{3}+\frac{1}{5}, \frac{1}{4}+\frac{1}{4}\} = \frac{8}{15}$ for any $i \in [1, 2n-1]$, hence $\sum_{i=1}^{n} \frac{1}{d_i} = \frac{1}{2} \sum_{i=1}^{n} (\frac{1}{d_{2i-1}} + \frac{1}{d_{2i}}) \leq \frac{4n}{15}$ and $c_0(v) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i} \leq 1 - \frac{7n}{30}$. If $n \geq 5$, then $c_0(v) \leq -\frac{1}{6}$. It remains to analyse the case $n = 4$. If $d_1 \geq 4$, then $c_0(v) \leq -1 + 4 \cdot \frac{1}{4} = 0$. If $d_3 \geq 4$, then $c_0(v) \leq -1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{4} + \frac{1}{5} = -\frac{1}{60}$. Finally, suppose that v is of type $(3, d_2, 3, d_4)$. If $d_2 \geq 6$, then $c_0(v) = -\frac{1}{3} + \frac{1}{d_2} + \frac{1}{d_4} \leq -\frac{1}{3} + 2 \cdot \frac{1}{6} = 0$. If $d_2 = 5$ and $d_2 \geq 8$, then $c_0(v) \leq -\frac{1}{3} + \frac{1}{5} + \frac{1}{8} < 0$. So, let $d_2 = 5$ and $d_4 \in [5, 7]$. If v has at least three neighbours of degree three, then, because of $\mathrm{cd}(v) \leq 10 \leq d + 1$, we obtain a contradiction with ($(d, 2)$-reducibility of) $\mathcal{C}_2$. On the other hand, if v has at least two neighbours of degree at least four, by Lemma 2 the vertex v is incident with a contractible edge. Since $\mathrm{cd}(v) \leq d + 1$, this contradicts Lemma 3.2.

4. If $d_1 \geq 5$, then, by Lemma 4, $c_0(v) \leq -\frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{d-2} \leq -\frac{1}{10} + \frac{1}{16} < 0$. If $d_1 = 4$ and $d_2 \geq 6$, then, again by Lemma 4, $c_0(v) \leq -\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{d-2} \leq -\frac{1}{12} + \frac{1}{16} < 0$. If $d_1 = 3$, then $d_2 \geq 5$ (Claim 1.1) and with $d_3 \geq d_2 \geq 12$ we have $c_0(v) \leq -\frac{1}{6} + \frac{1}{12} + \frac{1}{12} = 0$.

5. If $d_1 = 3$ and $d_2 \leq 6$, then $c_0(v) = -\frac{1}{6} + \frac{1}{d_2} + \frac{1}{d_3} \geq \frac{1}{d_3} > 0$. ∎

By Claim 1.2 and Lemma 4, provided v is a vertex of type (d_1, d_2, d_3), we have $c_0(v) \leq \gamma(d_1, d_2, d + 8 - d_1 - d_2) \leq \gamma(d_1, d_2, 26 - d_1 - d_2) =: u(d_1, d_2)$. The positive upper bounds $u(d_1, d_2)$ are presented in Table 1.

d_1	3	3	3	3	3	3	3	4	4
d_2	5	6	7	8	9	10	11	4	5
$u(d_1, d_2)$	$\frac{4}{45}$	$\frac{1}{17}$	$\frac{13}{336}$	$\frac{1}{40}$	$\frac{1}{63}$	$\frac{2}{195}$	$\frac{1}{132}$	$\frac{1}{18}$	$\frac{3}{340}$

Table 1

A triangle is of *type* (d_1, d_2, d_3) if it is adjacent to three distinct faces f_1, f_2, f_3 with $\deg(f_1) = d_1 \leq \deg(f_2) = d_2 \leq \deg(f_3) = d_3$.

Claim 2. If a 3-triangle t of type (d_1, d_2, d_3) is positive, then $d_1 \in [6, 7]$, $d_2 \geq d + 6 - d_1$ and $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{d+6-d_1} =: \beta(d_1, d)$.

Proof. From Claim 1.1 and $\mathcal{C}_1$ it follows that $d_1 \geq 6$. Put $d_4 := d_1$. If $d_1 \geq 12$, then $\Sigma(c_0, V(t)) = \sum_{i=1}^{3} \gamma(3, d_i, d_{i+1}) = -\frac{1}{2} + 2 \sum_{i=1}^{3} \frac{1}{d_i} \leq -\frac{1}{2} + 2 \cdot \frac{3}{12} = 0$. Let $x \in V(t)$ be a vertex of type $(3, d_1, d_2)$. From $\mathcal{C}_1$ we obtain $d + 3 \leq \mathrm{cd}(x) = d_1 + d_2 - 3$, $d_3 \geq d_2 \geq d + 6 - d_1$, and so $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + 2(\frac{1}{d_1} + \frac{2}{d+6-d_1}) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{24-d_1}$. With $d_1 \in [8, 11]$ we have $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{8} + \frac{4}{16} = 0$, hence $d_1 \in [6, 7]$. ∎

Let us define *absorbing* vertices as follows: Any vertex of degree at least four is absorbing. A 3-vertex is absorbing if it is either of type $(5, d_2, d_3)$ with $d_2 \geq 11$ and $d_3 \geq d - 1$ or of type $(7, d_2, d_3)$ with $d_2 \geq 10$.

Claim 3. If a 5-face f is incident with a vertex of type $(4, 5, d_3)$, then f is incident with an absorbing vertex.

Proof. Let $C = (x_1, x_2, x_3, x_4, x_5, x_1)$ be a facial cycle of f and let f_i be the face adjacent to f along the edge $x_i x_{i+1}$ (with indices taken modulo 5). If $\deg(x_i) \geq 4$ for some $i \in [1, 5]$, then x_i is absorbing. If $\deg(x_i) = 3$ for any $i \in [1, 5]$, we may suppose without loss of generality that $\deg(f_3) = 4$. By Claim 1.2 then $\deg(f_i) \geq d - 1$ for $i = 2, 4$. By the same Claim we have $\max\{\deg(f_1), \deg(f_5)\} \geq 11$, and so at least one of the vertices x_2, x_5 is absorbing. ■

Claim 4. If a 7-face f is adjacent to a 3-triangle, then f is incident with an absorbing vertex.

Proof. Let $C = (x_1, x_2, \dots, x_7, x_1)$ be a facial cycle of f and let f_i be the face adjacent to f along the edge $x_i x_{i+1}$ (with indices taken modulo 7). If $\deg(x_i) \geq 4$ for some $i \in [1, 7]$, then x_i is absorbing. Henceforth assume that $\deg(x_i) = 3$ for any $i \in [1, 7]$. Since 3-triangles adjacent to f cover an even number of vertices of f, there is a subpath P of C of an odd order $k \in \{1, 3, 5\}$, without loss of generality $P = \prod_{i=1}^{k}(x_i)$, such that none of x_i with $i \in [1, k]$ is incident with a 3-triangle, but x_i is incident with a 3-triangle for any $i \in \{k+1\} \cup \{7\}$. By $\mathcal{C}_1$ then $\min\{\deg(f_k), \deg(f_7)\} \geq d-1$. If $k = 1$, then the vertex x_1 is absorbing. If $k \in \{3, 5\}$ and $\max\{\deg(f_1), \deg(f_{k-1})\} \geq 10$, at least one of the vertices x_1, x_k is absorbing; note that, by Claim 1.2, the inequality is certainly true if $k = 3$. Finally, if $k = 5$ and $\max\{\deg(f_1), \deg(f_4)\} \leq 9$, then, again by Claim 1.2, $\min\{\deg(f_2), \deg(f_3)\} \geq 10$, and hence the vertex x_3 is absorbing. ■

A *transition edge* of a vertex x of type $(4, 5, d_3)$ is an oriented edge (v, w) whose endvertex is an absorbing vertex of the 5-face f incident with x that is closest to x in one of two possible orientations of the cycle bounding f. Similarly, a *transition edge* of a 3-triangle t adjacent to a 7-face f is an oriented edge (v, w) whose endvertex is an absorbing vertex of f that is closest to (a vertex of) t in one of two possible orientations of the cycle bounding f. Finally, a *transition edge* of a 3-triangle t adjacent to a 6-face f is an oriented edge (v, w) with $v \in V(t)$ and $w \in V(f) - V(t)$. From Claims 1.1, 2, 3 and 4 it follows that any vertex of type $(4, 5, d_3)$ and any positive 3-triangle has exactly two transition edges. Moreover, the initial vertex of any transition edge is a 3-vertex.

Let us now present redistribution rules leading from c_0 to c_4. The first "co-ordinate" i of a rule RR $i.j$ means that RR $i.j$ is used when passing from c_{i-1} to c_i.

RR 1.1 If (v, w) is a transition edge of a vertex x of type $(4, 5, d_3)$, then x sends to w the amount $\frac{1}{2}c_0(x)$ through (v, w).

RR 1.2 If (v, w) is a transition edge of a positive 3-triangle t, then t sends to w the amount $\frac{1}{2}\Sigma(c_0, V(t))$ through (v, w) and $c_1(x) := 0$ for any $x \in V(t)$.

RR 1.3 If (v, w) is a transition edge involved in RR 1.1 or RR 1.2 and $c_0(v) < 0$, then v sends to w the amount $c_0(v)$ through (v, w).

RR 1.4 If t is a nonpositive 3-triangle, then $c_1(x) := \frac{1}{3}\Sigma(c_0, V(t))$ for any $x \in V(t)$.

RR 2.1 If v is a vertex of type $(4, d_2, d)$ with $c_1(v) < 0$ and $\tilde{N}(v) := \{w \in N(v) : c_1(w) > 0\} = \{w_i : i \in [1, \tilde{n}(v)]\} \neq \emptyset$, then v sends to w_i the amount $\frac{c_1(v)}{\tilde{n}(v)}$ for any $i \in [1, \tilde{n}(v)]$.

RR 3.1 A vertex v of type $(3, d_2, d_3)$ with $c_2(v) > 0$, that is incident with a 1-triangle, sends to its $(3, d_3)$-neighbour w the amount $c_2(v)$ through. (The rule is correct, since $c_2(v) > 0$ implies $c_0(v) > 0$, and so, by Claims 1.2 and 1.4, $d_3 > d_2$.)

RR 3.2 If t is a 2-triangle with $V(t) = \{v_1, v_2, w\}$, where v_1, v_2 are 3-vertices, then v_i sends to w the amount $c_2(v_i)$ through (v_i, w), $i = 1, 2$.

RR 3.3 If v is a vertex of type $(4, 4, d)$ satisfying $c_2(v) > 0$ and $n_{4+}(v) = 0$ and $n_{4+}(w) \geq 1$ for the $(4, 4)$-neighbour w of v, then v sends to w the amount $c_2(v)$.

RR 4.1 If v is a 3-vertex with $c_3(v) > 0$ and $N_{4+}(v) = \{w_i : i \in [1, n_{4+}(v)]\} \neq \emptyset$, then v sends to w_i the amount $\frac{c_3(v)}{n_{4+}(v)}$ through (v, w_i) for any $i \in [1, n_{4+}(v)]$.

Recall that our aim is to show that $c_4(w) \leq 0$ for any $w \in V$. The case $\deg(w) = 3$ will be treated separately at the end of our analysis. If $\deg(w) \geq 4$ and $v \in N(w)$, let $a(v, w)$ be the total amount received by w through the oriented edge (v, w) (according to one of RR 1.1, 1.2, 1.3, 3.1, 3.2 and 4.1). If $\deg(v) \geq 4$, then $a(v, w) = 0$. If $\deg(v) = 3$, then $a(v, w)$ depends among other things on the type of the edge vw. Let $\bar{u}(d_1', d_2')$ be a nonnegative upper bound for $a(v, w)$ provided vw is of type (d_1', d_2'). If $\bar{u}(d_1', d_2')$ is not mentioned at all, it is considered to be 0. We shall assume that $\mathrm{dm}(v) = \{d_1', d_2', d_3'\}$.

First suppose that $d_1' = 3$. If $d_2' = 5$, then v is of type $(3, 5, d)$ (Claim 1.2), and so, because of RR 1.1 and RR 3.2, we have $a(v, w) \leq \gamma(3, 5, d) + \frac{1}{2}\gamma(4, 5, d) + \gamma(4, 5, d-1) = -\frac{1}{24} + \frac{1}{d-1} + \frac{3}{2d} \leq \frac{41}{408}$. Let $d_2' = 6$. If $c_2(v) \neq c_0(v)$, it is because of RR 1.2; in such a case, by $\mathcal{C}_1$, $d_3' = d$, and so, by Claim 2, $a(v, w) = c_2(v) \leq \gamma(3, 6, d) + \frac{1}{2}\beta(6, d) = \frac{3}{d} - \frac{1}{12} \leq \frac{1}{12}$. If $c_2(v) = c_0(v)$, Claim 1.2 yields $d_3' \geq d - 1$ and $a(v, w) = c_0(v) = \frac{1}{d_3'} \leq \frac{1}{17}$. Thus, we can take $\bar{u}(3, 6) := \frac{1}{12}$. Similarly, we can define $\bar{u}(3, 7) := \gamma(3, 7, 17) + \beta(7, 18)$. If $d_2' \in [8, d]$, then $c_2(v) = c_0(v)$, $\mathrm{cd}(v) = d_2' + d_3' - 3 \geq d + 2$ and $d_3' \geq d + 5 - d_2'$. Therefore, because of RR 3.1 or RR 3.2, $a(v, w) \leq \gamma(3, d_2', 23 - d_2')$. Moreover, $\gamma(3, d_2', 23 - d_2') \leq \gamma(3, 8, 15) =: \bar{u}(3, d_2')$ for any $d_2' \in [12, d-3]$; for $d_2' \in [8, 11] \cup [d-2, d]$ we put $\bar{u}(3, d_2') := \gamma(3, d_2', 23 - d_2')$.

Now consider the case $d_1' = 4$. If $d_2' = 4$, RR 4.1 yields $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) =: \bar{u}(4, 4)$. If $d_2' = 5$, then, by RR 1.1, $a(v, w) \leq 2\gamma(4, 5, 17) =: \bar{u}(4, 5)$. If $d_2' = 6$ and $\deg(v) = 3$, then, by RR 1.2 and Claim 2, $a(v, w) \leq \gamma(4, 6, d) + \frac{1}{2}\beta(6, d) = \frac{3}{d} - \frac{1}{6} \leq 0$ and we can take $\bar{u}(4, 6) := 0$. If $d_2' = 7$ and $\deg(v) = 3$, then, by RR 1.2 with Claim 2 and by RR 1.3 with Claim 1.2, $a(v, w) \leq \beta(7, 18) + \gamma(4, 7, 17) < 0$; therefore, we take again $\bar{u}(4, 7) := 0$. If $(d_1', d_2') = (4, d)$, then, using $\mathcal{C}_4, \mathcal{C}_5$, RR 2.1 and RR 3.3 we can obtain $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) = \bar{u}(4, d)$.

With $d_1' \in [5, 7]$ the following bounds are easily derived: $\bar{u}(5, d_2') := 2\gamma(4, 5, 17)$ for $d_2' \in [d-1, d]$, $\bar{u}(6, d) := \frac{1}{2}\beta(6, 18)$, $\bar{u}(7, d-2) := \beta(7, 18)$, and $\bar{u}(7, d_2') := \frac{3}{2}\beta(7, 18)$ for $d_2' \in [d-1, d]$. The (positive) upper bounds $\bar{u}(d_1', d_2')$ are summarised in Table 2; for our analysis it is helpful to have them ordered in a decreasing sequence $(\frac{41}{408}, \frac{4}{45}, \frac{1}{12}, \frac{1}{17}, \frac{20}{357}, \frac{1}{18}, \frac{13}{336}, \frac{15}{476}, \frac{1}{36}, \frac{1}{40}, \frac{5}{238}, \frac{3}{170}, \frac{1}{63}, \frac{2}{195}, \frac{1}{132})$. Finally, for $d_1' > d_2'$ we put

$\bar{u}(d_1', d_2') := \bar{u}(d_2', d_1')$.

d_1'	3	3	3	3	3	3	3	3	3	3	3
d_2'	5	6	7	8	9	10	11	$\in [12, d-3]$	$d-2$	$d-1$	d
$\bar{u}(d_1', d_2')$	$\frac{41}{408}$	$\frac{1}{12}$	$\frac{20}{357}$	$\frac{1}{40}$	$\frac{1}{63}$	$\frac{2}{195}$	$\frac{1}{132}$	$\frac{1}{40}$	$\frac{13}{336}$	$\frac{1}{17}$	$\frac{4}{45}$

d_1'	4	4	4	5	6	7	7
d_2'	4	5	d	$d-1, d$	d	$d-2$	$d-1, d$
$\bar{u}(d_1', d_2')$	$\frac{1}{18}$	$\frac{3}{170}$	$\frac{1}{18}$	$\frac{3}{170}$	$\frac{1}{36}$	$\frac{5}{238}$	$\frac{15}{476}$

Table 2

Now consider an n-vertex w of type $D = (d_1, \dots, d_n)$ and let $(v_1, \dots, v_n)$ be a sequence of neighbours of w in a cyclic order around w such that the edge $v_i w$ is incident with faces f_i of degree d_i and f_{i+1} of degree d_{i+1} (if $i \in [n+1, \infty)$, the index i in v_i, f_i or d_i is taken modulo n so as to belong to $[1, n]$). Then $c_0(w) = 1 - \frac{n}{2} + \sum_{i-1}^{n} \frac{1}{d_i} = \sum_{i=1}^{n} p_i^n(w)$, where $p_i^n(w) := \frac{1}{n} - \frac{1}{2} + \frac{1}{2d_i} + \frac{1}{2d_{i+1}}$ is the ith *partial* charge of the vertex w (corresponding to the edge $v_i w$). If $n \ge 4$, we have $c_4(w) = c_0(w) + \sum_{i=1}^{n} a(v_i, w) = \sum_{i=1}^{n} (p_i^n(w) + a(v_i, w)) \le \sum_{i=1}^{n} (p_i^n(w) + \bar{u}(d_i, d_{i+1}))$. To bound $p_i^n(w)$ we use the following inequality yielded by Claim 1.1: $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} \le \max\{\frac{1}{6} + \frac{1}{10}, \frac{1}{8} + \frac{1}{8}\} = \frac{4}{15}$ for any $i \in [1, n]$. By $f_k := |\{i \in [1, n] : d_i = k\}|$ we denote the *frequency* of k in D; we put $f_{k+} := \sum_{l=k}^{d} f_l$.

(1) If $n \ge 8$, using Table 2 we see that $p_i^n(w) + \bar{u}(d_i, d_{i+1}) \le \frac{1}{8} - \frac{1}{2} + \frac{4}{15} + \frac{41}{408} < 0$ for any $i \in [1, n]$, and so $c_4(w) < 0$.

(2) $n \in [5, 7]$

(21) If $\mathrm{cd}(w) \le d + 1$, then, by Claim 1.1, $d_i \le d - 5$ for any $i \in [1, n]$. Further, by Lemma 3.3, $\deg(v_i) = 3$ implies $\mathrm{cd}(v_i) \ge d + 3$, and so from $d_i + d_{i+1} = 8$ it follows that $a(v_i, w) = 0$ and $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{10} = \frac{4}{15}$. Using Table 2 it is easy to check that $d_i + d_{i+1} \ge 9$ yields $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$; moreover, if $\{d_i, d_{i+1}\} \ne \{3, 6\}$, then $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{14} + \frac{20}{357} = \frac{5}{17}$.

(211) If $n \in [6, 7]$, then $p_i^n(w) + a(v_i, w) \le \frac{1}{n} - \frac{1}{2} + \max\{\frac{4}{15}, \frac{1}{3}\} \le 0$ for any $i \in [1, n]$ and $c_4(w) \le 0$.

(212) If $n = 5$, then, since $\frac{1}{5} - \frac{1}{2} + \max\{\frac{4}{15}, \frac{5}{17}\} < 0$, $p_i^5(w) + a(v_i, w)$ can be positive only if $\{d_i, d_{i+1}\} = \{3, 6\}$. Let $k := |\{i \in [1, 5] : \{d_i, d_{i+1}\} = \{3, 6\}\}|$.

(2121) If $k = 0$, then $c_4(w) < 0$ as a sum of five negative summands.

(2122) If $k \ge 1$, then, by Claim 1.1, $f_3 \in [1, 2]$. If $\deg(v_i) = 3$, $v_i w$ is of type $(3, 6)$ and v_i is not involved in RR 1.2, then $a(v_i, w) \le \gamma(3, 6, d) \le \frac{1}{18}$; notice that the number of i's such that $\deg(v_i) = 3$, $v_i w$ is of type $(3, 6)$ and v_i is involved in RR 1.2 is at most f_6.

(21221) If $f_3 = 1$, then, by Claim 1.1 and Table 2, $c_0(w) + \sum_{i=1}^{5} a(v_i, w) \le (-\frac{3}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + 2 \cdot \frac{1}{4}) + 2 \cdot \frac{1}{12} + 3 \cdot \frac{3}{170} < 0$.

(21222) If $f_3 = 2$, then, by Claim 1.1, $f_4 = 0$. In such a case $a(v_i, w) = 0$ for (the unique) $i \in [1, 5]$ satisfying $\min\{d_i, d_{i+1}\} \ge 5$.

(212221) If $k \ge 4$, then w is of type $(3,6,3,6,6)$ and $c_0(w) + \sum_{i=1}^5 a(v_i, w) \le -\frac{1}{3} + (3 \cdot \frac{1}{12} + \frac{1}{18}) < 0$.

(212222) If $k = 3$, then $f_6 = 2$, $c_0(w) \le \gamma(3,5,6,3,6) = -\frac{3}{10}$, $\sum_{i=1}^5 a(v_i, w) \le 2 \cdot \frac{1}{12} + \frac{1}{18} + \frac{20}{357} < \frac{3}{10}$ and $c_4(w) < 0$.

(212223) $k = 2$

(2122231) If $f_6 = 1$, then $c_0(w) \le \gamma(3,5,5,3,6) = -\frac{4}{15}$, $\sum_{i=1}^5 a(v_i, w) \le \frac{1}{12} + \frac{1}{18} + 2 \cdot \frac{20}{357} < \frac{4}{15}$ and $c_4(w) < 0$.

(2122232) If $f_6 = 2$, then $c_0(w) \le \gamma(3,5,3,6,6) = -\frac{3}{10}$, $\sum_{i=1}^5 a(v_i, w) \le 2 \cdot \frac{1}{12} + 2 \cdot \frac{20}{357} < \frac{3}{10}$ and $c_4(w) < 0$.

(212224) If $k = 1$, then $c_0(w) \le \gamma(3,5,3,5,6) = -\frac{4}{15}$, $\sum_{i=1}^5 a(v_i, w) \le \frac{1}{12} + 3 \cdot \frac{20}{357} < \frac{4}{15}$ and $c_4(w) < 0$.

(22) $\mathrm{cd}(w) \ge d + 2$

(221) If $n = 7$, then, by Claim 1.1, $f_{5+} \ge f_3$, $f_3 \le 3$, and so, by Lemma 4, $c_0(w) \le \gamma((3)^{f_3}(5)^{f_3}(4)^{6-2f_3}(d-8)) = -1 + \frac{f_3}{30} + \frac{1}{d-8} \le -\frac{4}{5}$. On the other hand, $\sum_{i=1}^7 a(v_i, w) \le 7 \cdot \frac{41}{408} < \frac{4}{5}$ and $c_4(w) < 0$.

(222) $n = 6$

(2221) If $f_3 \le 2$, using Claim 1.1 and the assumption $\mathrm{cd}(w) \ge d + 2$ we see that $f_{5+} \ge f_3 + 1$, and so, by Lemma 4, $c_0(w) \le \gamma((3)^{f_3}(5)^{f_3}(4)^{5-2f_3}(d-6)) = -\frac{3}{4} + \frac{f_3}{30} + \frac{1}{d-6} \le -\frac{2}{3} + \frac{f_3}{30}$. On the other hand, Table 2 yields $\sum_{i=1}^6 a(v_i, w) \le 2f_3 \cdot \frac{41}{408} + (6 - 2f_3) \cdot \frac{1}{18}$. Therefore, $c_4(w) \le \frac{377 f_3}{3060} - \frac{1}{3} \le \frac{377}{1530} - \frac{1}{3} < 0$.

(2222) If $f_3 = 3$, then, by Claim 1.1, w is of type $(3, d_2, 3, d_4, 3, d_6)$ and, by Lemma 4, $c_0(w) \le \gamma(3,5,3,5,3,d-5) = -\frac{3}{5} + \frac{1}{d-5} \le -\frac{3}{5} + \frac{1}{13} = -\frac{34}{65}$. So, it is sufficient to show that $\sum_{i=1}^6 a(v_i, w) \le \frac{34}{65}$.

(22221) If there is $i \in [1,6]$ with $\deg(v_i) \ge 4$, then $\sum_{i=1}^6 a(v_i, w) \le 5 \cdot \frac{41}{408} < \frac{34}{65}$.

(22222) If $\deg(v_i) = 3$ for any $i \in [1,6]$, consider the expression $c_4(w) = \sum_{i=1}^6 q_i$, where $q_i := \frac{1}{6} - \frac{1}{2} + \frac{1}{6} + \frac{1}{2\max\{d_i, d_{i+1}\}} + a(v_i, w) \le -\frac{1}{6} + \frac{1}{2\max\{d_i, d_{i+1}\}} + \bar{u}(3, \max\{d_i, d_{i+1}\})$ and $\max\{d_i, d_{i+1}\} \in [5, d]$. Using Table 2 it is easy to check that three maximal values of $f(s) := -\frac{1}{6} + \frac{1}{2s} + \bar{u}(3, s)$ for $s \in [5, d]$ are $f(5) = \frac{23}{680}$, $f(6) = 0$ and $f(7) = -\frac{2}{51}$. Notice that $c_4(w) = \sum_{i=1}^3 (q_{2i-1} + q_{2i}) \le 2 \sum_{i=1}^3 f(d_{2i})$.

(222221) If $d_2 \ge 6$, then, as $\min\{d_4, d_6\} \ge d_2$, we obtain $c_4(w) \le 0$.

(222222) $d_2 = 5$

(2222221) If $\min\{d_4, d_6\} \ge 7$, then $c_4(w) \le 2 \cdot (\frac{23}{680} - 2 \cdot \frac{2}{51}) < 0$.

(2222222) If there is $j \in \{4, 6\}$ with $d_j \in [5, 6]$, then $d_{10-j} \ge d - d_j$. Let d' be the degree of the face adjacent to both f_j and f_{10-j}. By Claim 1.2 we know that $d' \ge d + 5 - d_j$. Therefore, by RR 3.2, the summand $a(v_k, w)$ corresponding to the vertex v_k with $\mathrm{dm}(v_k) = \{3, d_{10-j}, d'\}$ is equal to $\gamma(3, d_{10-j}, d') = -\frac{1}{6} + \frac{1}{d_{10-j}} + \frac{1}{d'} \le -\frac{1}{6} + \frac{1}{d-6} + \frac{1}{d-1} \le -\frac{1}{6} + \frac{1}{12} + \frac{1}{17} < 0$ and $\sum_{i=1}^6 a(v_i, w) < 5 \cdot \frac{41}{408} < \frac{34}{65}$.

(223) $n = 5$

(2231) If $f_3 = 0$, then, due to Lemma 4, $c_0(w) \le \gamma((4)^4(d-4)) \le -\frac{3}{7}$, and so $c_4(w) \le -\frac{3}{7} + 5 \cdot \frac{1}{18} < 0$.

(2232) If $f_3 = 1$, then $c_4(w) \le \gamma(3,5,4,4,d-4) = -\frac{7}{15} + \frac{1}{d-4} \le -\frac{83}{210}$, $\sum_{i=1}^5 a(v_i, w) \le 2 \cdot \frac{41}{408} + 3 \cdot \frac{1}{18} < \frac{83}{210}$ and $c_4(w) < 0$.

(2233) If $f_3 = 2$, then, by Claim 1.1, $f_4 = 0$. By Lemma 4 we have $c_0(w) \le$

$\gamma(3,5,3,5,d-4) = -\frac{13}{30} + \frac{1}{d-4} \le -\frac{38}{105}$, and so it is sufficient to prove that $\sum_{i=1}^{5} a(v_i, w) \le \frac{38}{105}$.

(22331) If there is $i \in [1,5]$ such that v_i is incident with a triangle and $\deg(v_i) \ge 4$, then $\sum_{i=1}^{5} a(v_i,w) \le 3 \cdot \frac{41}{408} + \frac{15}{476} < \frac{38}{105}$.

(22332) Now suppose that all neighbours of w incident with a triangle are of degree three. Let f_j be the face adjacent to two triangles.

(223321) If $d_j \in [5,7]$, there is $k \in [1,5]$ such that $d_k \ge 9$. The face $\tilde{f}$ adjacent to both f_j and f_k is of degree $d' \ge d-2$ (Claim 1.2), hence for the vertex v_l incident with f_k and $\tilde{f}$ we have $a(v_l,w) = -\frac{1}{6} + \frac{1}{d_k} + \frac{1}{d'} \le \frac{1}{144}$ and, by Table 2, $\sum_{i=1}^{5} a(v_i,w) \le 3 \cdot \frac{41}{408} + \frac{1}{144} + \frac{15}{476} < \frac{38}{105}$.

(223322) If $d_j \in [8, d-3]$, then $\sum_{i=1}^{5} a(v_i,w) \le 2 \cdot \frac{41}{408} + 2 \cdot \frac{1}{40} + \frac{15}{476} < \frac{38}{105}$.

(223323) If $d_j \in [d-2, d]$, notice that from Table 2 it follows that if $\min\{d_i, d_{i+1}\} \ge 5$, then $p_i^5(w) + \bar{u}(d_i, d_{i+1}) < 0$. Therefore, it suffices to show that if $d_l = 3$, then $\sum_{i=l-1}^{l} (p_i^5(w) + a(v_i,w)) \le 0$. Let d' be the degree of the face adjacent to f_{l-1}, f_l and f_{l+1}. Claim 1.2 then yields $d' \ge \max\{d+5-d_{l-1}, d+5-d_{l+1}\}$, and so, by RR 3.2, $\sum_{i=l-1}^{l}(p_i^5(w) + a(v_i,w)) = -\frac{3}{5} + \frac{3}{2d_{l-1}} + \frac{3}{2d_{l+1}} + \frac{2}{d'}$. If $m \in \{-1, 1\}$, then $\frac{3}{2d_{i+m}} + \frac{2}{d'} \le \frac{3}{2d_{i+m}} + \frac{2}{23-d_{i+m}} \le \frac{3}{36} + \frac{2}{5} = \frac{29}{60}$, and so, as $j \in \{m-1, m+1\}$, we have $\sum_{i=m-1}^{m}(p_i^5(w) + a(v_i,w)) \le -\frac{3}{5} + \frac{3}{32} + \frac{29}{60} < 0$.

(3) $n = 4$

(31) If $\mathrm{cd}(w) \le d+1$, by Lemma 3.2 the vertex w is not incident with a contractible edge, hence, by Lemma 2, w has at least three neighbours of degree three. Since $d_i < d$ for any $i \in [1,4]$, using Lemma 3.4 and $\mathcal{C}_2$ we see that $d_1 \ge 4$. As in (21), $d_i = d_{i+1} = 4$ implies $a(v_i,w) = 0$ and $p_i^4(w) + a(v_i,w) = 0$. Moreover, with help of Table 2 it is easy to check that $p_i^4(w) + \bar{u}(d_i, d_{i+1}) \le 0$ whenever $d_i + d_{i+1} \ge 9$ (and $\min\{d_i, d_{i+1}\} \ge 4$); as a consequence, $c_4(w) \le 0$.

(32) If $\mathrm{cd}(w) \ge d+2$, put $q_i := p_i^4(w) + a(v_i,w)$ for $i \in [1,\infty)$.

(321) If $f_3 = 2$, then, by Claim 1.1, w is of type $(3, d_2, 3, d_4)$, where $d_2 + d_4 \ge d+4$. Since $c_4(w) = (q_2+q_3) + (q_4+q_5)$, it is sufficient to show that $q_i + q_{i+1} \le 0$ for any $i \in \{2,4\}$. So, in what follows we assume $i \in \{2,4\}$.

(3211) If $\min\{\deg(v_i), \deg(v_{i+1})\} \ge 4$, then $q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d_4} \le -\frac{1}{6} + \frac{1}{10} + \frac{1}{34} < 0$.

(3212) If there is $j \in [i, i+1]$ such that $\deg(v_j) = 3$ and $\deg(v_{2i+1-j}) \ge 4$, then, by Lemma 3.4, $d_4 = d$ and $q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d} + a(v_j,w) \le -\frac{1}{6} + \frac{1}{10} + \frac{1}{36} + a(v_j,w) = -\frac{7}{180} + a(v_j,w)$.

(32121) If $a(v_j,w) \le 0$, then $q_i + q_{i+1} < 0$.

(32122) If $a(v_j,w) > 0$, then, by RR 3.1, v_j is of type $(3, d', d_2)$ (where d_2 appears either without loss of generality, namely if w is of type $(3,d,3,d)$, or due to Lemma 3.4). By Claim 1.4 we obtain $d' \in [5,11]$, and so, by Claim 1.2, $d_2 \ge d+5-d' \ge d-6$. Therefore, $q_i + q_{i+1} \le -\frac{1}{6} + \frac{1}{2(d-6)} + \frac{1}{2d} + \frac{4}{45} \le -\frac{1}{6} + \frac{1}{24} + \frac{1}{36} + \frac{4}{45} < 0$.

(3213) If $\deg(v_i) = \deg(v_{i+1}) = 3$, then, by $\mathcal{C}_3$, $\min\{\mathrm{cd}(v_i), \mathrm{cd}(v_{i+1})\} \ge d+3$. Therefore, Claim 1.2 yields $\min\{d_2, d_4\} \ge 6$. Let d' be the degree of the face adjacent to the triangle $v_i w v_{i+1}$ along the edge $v_i v_{i+1}$. Then $d_2 + d' - 3 = \min\{\mathrm{cd}(v_i), \mathrm{cd}(v_{i+1})\} \ge d+3$, hence $d' \ge d+6-d_2$.

(32131) If $d_2 \le 8$, then $q_i \le -\frac{1}{12} + \frac{1}{2d_2} + \bar{u}(3, d_2)$ and $q_{i+1} = -\frac{1}{4} + \frac{3}{2d_4} + \frac{1}{d'} \le -\frac{1}{4} + \frac{3}{2(d+4-d_2)} + \frac{1}{d+6-d_2}$.

(321311) If $d_2 = 6$, then $q_i + q_{i+1} \le \frac{1}{12} - \frac{1}{4} + \frac{3}{32} + \frac{1}{18} < 0$.

(321312) If $d_2 \in [7, 8]$ then $q_i + q_{i+1} \le -\frac{1}{12} + \frac{1}{14} + \frac{20}{357} - \frac{1}{4} + \frac{3}{28} + \frac{1}{16} < 0$.

(32132) If $d_2 \in [9, 14]$, then $d' \ge 10$ and $q_i + q_{i+1} = -\frac{1}{2} + \frac{3}{2d_2} + \frac{3}{2d_4} + \frac{2}{d'} \le -\frac{1}{2} + \frac{3}{18} + \frac{3}{26} + \frac{2}{10} < 0$.

(32133) If $d_2 \in [15, d-2]$, then $q_i + q_{i+1} \le -\frac{1}{2} + 2 \cdot \frac{3}{30} + \frac{2}{8} < 0$.

(32134) If $d_2 = d - 1$, then $q_i + q_{i+1} \le -\frac{1}{2} + 2 \cdot \frac{3}{34} + \frac{2}{7} < 0$.

(32135) If $d_2 = d$, then $q_i + q_{i+1} \le -\frac{1}{2} + 2 \cdot \frac{3}{36} + \frac{2}{6} = 0$.

(322) If $f_3 = 1$, consider the inequalities $q_i \le -\frac{1}{4} + \frac{1}{2d_i} + \frac{1}{2d_{i+1}} + \bar{u}(d_i, d_{i+1}) \le \bar{\bar{u}}(d_i, d_{i+1})$, where $i \in [1, \infty)$, $\bar{\bar{u}}(d'_1, d'_2)$ with $d'_1 \le d'_2$ is an upper bound for $-\frac{1}{4} + \frac{1}{2d'_1} + \frac{1}{2d'_2} + \bar{u}(d'_1, d'_2)$ presented in Table 3 (that is created using Table 2) and, provided $d'_1 > d'_2$, $\bar{\bar{u}}(d'_1, d'_2) := \bar{\bar{u}}(d'_2, d'_1)$. Since $d_1 = 3$, by Claim 1.2 we have $d_4 \ge d_2 \ge 5$; as $d_3 \ge 4$, from Table 3 we see that $q_i < 0$, $i = 2, 3$.

d'_1	3	3	3	3	3	3	4	4	4
d'_2	5	6	7	8	$\in [9, d-2]$	$d-1, d$	4	5	$\in [6, d-5]$
$\bar{\bar{u}}(d'_1, d'_2)$	$\frac{239}{2040}$	$\frac{1}{12}$	$\frac{3}{68}$	$\frac{1}{240}$	$-\frac{1}{84}$	$\frac{1}{30}$	$\frac{1}{18}$	$-\frac{1}{136}$	$-\frac{1}{24}$

d'_1	4	4	5	5	5
d'_2	$\in [d-4, d-1]$	d	$\in [5, d-5]$	$d-4, d-3$	$\in [d-2, d]$
$\bar{\bar{u}}(d'_1, d'_2)$	$-\frac{5}{56}$	$-\frac{1}{24}$	$-\frac{1}{20}$	$-\frac{4}{35}$	$-\frac{7}{68}$

d'_1	6	6	6	7	7	$\in [8, d]$
d'_2	$\in [6, d-6]$	$[d-5, d-1]$	d	$\in [7, d-7]$	$\in [d-6, d]$	$\in [d'_1, d]$
$\bar{\bar{u}}(d'_1, d'_2)$	$-\frac{1}{12}$	$-\frac{5}{39}$	$-\frac{1}{9}$	$-\frac{3}{28}$	$-\frac{2}{17}$	$-\frac{1}{8}$

Table 3

(3221) If $q_i \le 0$, $i = 1, 4$, then $c_4(w) = \sum_{i=1}^{4} q_i < 0$.

(3222) $\max\{q_1, q_4\} > 0$

(32221) If $q_j + q_{j+2} \le 0$ for $j = 1, 4$, then $c_4(w) = (q_1 + q_3) + (q_4 + q_6) \le 0$.

(32222) Let $i \in \{1, 4\}$ be such that $q_i + q_{i+2} \ge q_{5-i} + q_{7-i}$ and $q_i + q_{i+2} > 0$ (so that $q_{i+2} < 0$ implies $q_i > 0$).

(322221) If $a(v_i, w) = 0$, then $q_i = -\frac{1}{12} + \frac{1}{2\max\{d_i, d_{i+1}\}}$, and so $\max\{d_i, d_{i+1}\} = 5$ and $q_i = \frac{1}{60}$ (for otherwise $q_i \le 0$). Then, however, $d_{i+2} + d_{i+3} = \mathrm{cd}(w) \ge d + 2$ and $\min\{d_{i+2}, d_{i+3}\} \ge 4$, so that Table 3 yields $q_{i+2} \le -\frac{3}{32}$ and $q_i + q_{i+2} < 0$, a contradiction.

(322222) If $a(v_i, w) \ne 0$, then $\deg(v_i) = 3$ and $\mathrm{dm}(v_i) = \{3, s, d'\}$, where $s := \max\{d_i, d_{i+1}\}$.

(3222221) If v_i is incident with a 1-triangle, then $s > d'$ (we are using RR 3.1), and so, by Claim 1.2, $s \ge 12$; then, by Table 3, $s \ge d - 1$ and $q_i \le \frac{1}{30}$.

Moreover, $a(v_{5-i}, w) = 0$ and, by Lemma 3.4, the edge $v_{5-i}w$ is of type $(3, d)$ so that $q_{5-i} = -\frac{1}{12} + \frac{1}{2d} \le -\frac{1}{12} + \frac{1}{36} = -\frac{1}{18}$ and $\sum_{j=1}^{4} q_j < q_1 + q_4 \le \frac{1}{30} - \frac{1}{18} < 0$.

(3222222) Now suppose that v_i is incident with a 2-triangle (which means that $\deg(v_1) = \deg(v_4) = 3$). From Table 3 it follows that $s \in [5, 8] \cup [d-1, d]$. We have $s + d_{i+2} + d_{i+3} - 5 = \mathrm{cd}(w) \ge d + 2$, hence $d_{i+2} + d_{i+3} \ge d + 7 - s$.

(32222221) If $s = 5$, then $d' = d$ (by Claim 1.2) and either $\min\{d_{i+2}, d_{i+3}\} \in [4, 5]$ or $\{d_{i+2}, d_{i+3}\} = \{6, d\}$, since otherwise $q_{i+2} \le -\frac{2}{17}$ and $q_i + q_{i+2} \le \frac{239}{2040} - \frac{2}{17} < 0$. Thus, w is of one of types $(3, 5, 4, d_4)$, $(3, 5, 5, d_4)$, $(3, 5, 6, d)$, $(3, 5, d, 6)$ and $(3, 5, d_3, 5)$; in the first four cases we have immediately $i = 1$ and in the last case we may suppose without loss of generality that $i = 1$.

(322222211) If $d_3 = 4$, then $d_4 \ge d - 2$, $q_3 \le \bar{\bar{u}}(4, d_4)$ and $q_4 = -\frac{1}{4} + \frac{1}{d} + \frac{3}{2d_4} \le -\frac{7}{36} + \frac{3}{2d_4}$. Since $\bar{\bar{u}}(4, d_4) + \frac{3}{2d_4} \le \max\{-\frac{5}{56} + \frac{3}{32}, -\frac{1}{24} + \frac{3}{36}\} = \frac{1}{24}$, we obtain $c_4(w) \le \frac{239}{2040} - \frac{1}{136} - \frac{7}{36} + \frac{1}{24} < 0$.

(322222212) If w is of type $(3, 5, 5, d_4)$, then $d_4 \ge d-3$, $a(v_4, w) = -\frac{1}{6} + \frac{1}{d_4} + \frac{1}{d} \le -\frac{1}{6} + \frac{1}{15} + \frac{1}{18} = -\frac{2}{45}$, $q_4 \le -\frac{1}{12} + \frac{1}{30} - \frac{2}{45} = -\frac{17}{180}$ and $c_4(w) \le \frac{239}{2040} - \frac{1}{20} - \frac{7}{68} - \frac{17}{180} < 0$.

(322222213) If w is of type $(3, 5, d_3, 5)$, then $d_3 \ge d - 3$ and $c_0(w) \le \gamma(3, 5, d-3, 5) = -\frac{4}{15} + \frac{1}{d-3} \le -\frac{4}{15} + \frac{1}{15} = -\frac{1}{5}$. It is easy to see that if a face f_j with $j \in \{2, 4\}$ is incident with a vertex of type $(4, 5, \hat{d})$, then the number of such vertices is at most two and besides w there is at least one other absorbing vertex incident with f_j. Therefore, the total amount received by w due to RR 1.1 is bounded from above by $2\gamma(4, 5, 17)$, $\sum_{j=1}^{4} a(v_j, w) \le 2\gamma(3, 5, 18) + 2\gamma(4, 5, 17) = \frac{299}{1530}$ and $c_4(w) \le -\frac{1}{5} + \frac{299}{1530} < 0$.

(322222214) If $\{d_3, d_4\} = \{6, d\}$, then $c_0(w) = \gamma(3, 5, 6, d) = -\frac{3}{10} + \frac{1}{d} \le -\frac{3}{10} + \frac{1}{18} = -\frac{11}{45}$, $\sum_{j=1}^{4} a(v_j, w) \le \frac{41}{408} + \frac{1}{36} + \max\{\frac{3}{170} + \frac{1}{12}, 0 + \frac{4}{45}\} < \frac{11}{45}$, and so $c_4(w) < 0$.

(32222222) If $s \in [6, 8]$, then $q_i \le \bar{\bar{u}}(3, s)$ and $q_{i+2} \le \max\{\bar{\bar{u}}(d_1', d_2') : d_1' \ge 4, d_1' + d_2' \ge d + 7 - s\}$. From Table 3 it follows that $i = 1$, $d_3 = 4$ and $d_4 = d$ (for otherwise $q_i + q_{i+2} < 0$, a contradiction). Claim 1.2 yields $d' \ge d + 5 - s$, hence $q_4 = -\frac{1}{12} + \frac{1}{2d} + (-\frac{1}{6} + \frac{1}{d} + \frac{1}{d'}) \le -\frac{1}{4} + \frac{3}{36} + \frac{1}{15} = -\frac{1}{10}$ and, by Table 3, $\sum_{j=1}^{4} q_j \le \frac{1}{12} - \frac{1}{24} - \frac{1}{24} - \frac{1}{10} < 0$.

(32222223) If $s \in [d-1, d]$, then $\{d_{i+2}, d_{i+3}\} = [4, 5]$, for otherwise $q_i + q_{i+2} \le \frac{1}{30} - \frac{1}{24} < 0$. By Claim 1.1 then w is of type $(3, 5, 4, d_4)$, hence $i = 4$ and $d' = d$ (by Claim 1.2). Therefore, $q_4 = -\frac{1}{4} + \frac{1}{d} + \frac{3}{2d_4} \le -\frac{1}{4} + \frac{1}{18} + \frac{3}{34} < 0$, a contradiction.

(323) $f_3 = 0$

(3231) If $q_i \le 0$ or $q_i + q_{i+2} \le 0$ for every $i \in [1, 4]$, then $c_4(w) \le 0$.

(3232) Let $i \in [1, 4]$ be such that $q_i > 0$ and $q_i + q_{i+2} > 0$. From Table 3 it follows that $d_i = d_{i+1} = 4$ and $q_i \le \frac{1}{18}$. Since $d_{i+2} + d_{i+3} = \mathrm{cd}(w) \ge d + 2$, Table 3 yields also $\{d_{i+2}, d_{i+3}\} = \{4, d\}$. Thus, w is of type $(4, 4, 4, d)$, we may suppose without loss of generality that $i = 1$ and $c_0(w) = \gamma(4, 4, 4, d) = -\frac{1}{4} + \frac{1}{d} \le -\frac{7}{36}$.

(32321) If $\max\{\deg(v_j) : j \in [1, 4]\} \ge 4$, then $c_4(w) \le -\frac{7}{36} + 3 \cdot \frac{1}{18} < 0$.

(32322) If $\deg(v_j) = 3$ for any $j \in [1, 4]$, consider the quadrangle v_1wv_2x.

(323221) If $\deg(x) = 3$, then x is of type $(4, d, d)$ and, by RR 2.1, $c_2(v_1) = \gamma(4, 4, d) + \frac{1}{2}\gamma(4, d, d) = -\frac{1}{8} + \frac{2}{d} \le -\frac{1}{72}$, hence $q_1 = a(v_1, w) = 0$, which contradicts $q_i > 0$.

(323222) If $\deg(x) \ge 4$, then, by RR 4.1, $q_1 = a(v_1, w) \le \frac{1}{2}c_3(v_1) \le \frac{1}{2}\gamma(4, 4, d) =$

$\frac{1}{2d} \leq \frac{1}{36}$ and $q_1 + q_3 \leq \frac{1}{36} - \frac{1}{24} < 0$ in contradiction with $q_i + q_{i+2} > 0$.

(4) $n = 3$

(41) If $d_1 = 3$, then w belongs to an i-triangle t, $i \in [1, 3]$.

(411) $i = 1$

(4111) If $c_0(w) \leq 0$, then $d_2 \geq 9$ (Claim 1.5), hence $c_4(w) = c_0(w) \leq 0$.

(4112) If $c_0(w) > 0$, then $c_2(w) \geq c_0(w) > 0$, and so, by RR 3.1, $c_4(w) = 0$.

(412) If $i = 2$, then applying RR 3.2 yields $c_4(w) = 0$.

(413) $i = 3$

(4131) If t is positive, then, by RR 1.2 and $\mathcal{C}_6$, we have $c_4(w) = 0$.

(4132) If t is nonpositive, then, by RR 1.4, $c_4(w) = \frac{1}{3}\Sigma(c_0, V(t)) \leq 0$.

(42) $d_1 = 4$

(421) $d_2 = 4$

(4211) If $c_3(w) \leq 0$, then $c_4(w) = c_3(w) \leq 0$.

(4212) If $c_3(w) > 0$, then necessarily also $c_2(w) > 0$.

(42121) If $n_{4+}(w) \geq 1$, then, by RR 4.1, $c_4(w) = 0$.

(42122) $n_{4+}(w) = 0$

(421221) If $n_{4+}(v_1) \geq 1$, then, by RR 3.3, $c_4(w) = 0$.

(421222) If $n_{4+}(v_1) = 0$, then, by $\mathcal{C}_4$, for any $i \in [2, 3]$ the type $(4, d'_i, d)$ of the vertex v_i is such that $d'_i \geq 6$. Therefore, by $\mathcal{C}_5$ and RR 2.1, $c_3(w) = \gamma(4, 4, d) + \gamma(4, d'_2, d) + \gamma(4, d'_3, d) = -\frac{1}{2} + \frac{3}{d} + \frac{1}{d'_2} + \frac{1}{d'_3} \leq -\frac{1}{2} + \frac{3}{18} + 2 \cdot \frac{1}{6} = 0$, a contradiction.

(422) If $d_2 = 5$, then, by RR 1.1, $c_4(w) = 0$.

(423) If $d_2 \geq 6$, then $c_0(w) \leq 0$ (Claim 1.4).

(4231) If w has not received any amount, then $c_0(w) \leq c_4(w) \leq 0$.

(4232) If w has received an amount, then $d_2 = 6$ and the rule RR 1.2 has been applied; then, by Claim 2, $c_1(w) \leq \gamma(4, 6, d) + \frac{1}{2}\beta(6, d) = -\frac{1}{6} + \frac{3}{d} \leq 0$, and so $c_1(w) \leq c_4(w) \leq 0$.

(43) If $d_1 \geq 5$, then, by Claim 1.4, $c_0(w) \leq 0$.

(431) If w has not received any amount, then $c_0(w) \leq c_4(w) \leq 0$.

(432) If w has received an amount, then either $d_1 = 5$ and RR 1.1 has been applied or $[6, 7] \cap \mathrm{dm}(w) \neq \emptyset$ and RR 1.2 has been applied.

(4321) If $d_1 = 5$, then $d_2 \geq 11$, $d_3 \geq d - 1$ and $c_4(w) \leq \gamma(5, 11, d-1) + 4\gamma(4, 5, d-1) \leq -\frac{9}{22} + \frac{5}{17} < 0$.

(4322) If $6 \in \mathrm{dm}(w)$, then $\mathrm{dm}(w) = \{6, s, d\}$ with $s \in [5, d]$ and $c_4(w) \leq \gamma(6, 5, d) + \frac{1}{2}\beta(6, d) = -\frac{13}{60} + \frac{3}{d} \leq -\frac{13}{60} + \frac{3}{18} < 0$.

(4323) If $7 \in \mathrm{dm}(w)$, then $d_1 = 7$, $d_2 \geq 10$ and $c_4(w) \leq \gamma(7, 10, 10) + 3\beta(7, d) \leq -\frac{4}{5} + \frac{12}{17} < 0$.

Since $c_4(w) \leq 0$ for any $w \in V$, the proof is complete. ∎

References

[1] K. Ando, H. Enomoto and A. Saito, Contractible edges in 3-connected graphs, J. Combin. Theory (Ser. B) **42** (1987) 87–93

[2] O.V. Borodin, Solution of Ringel's problem on vertex-face coloring of plane graphs and coloring of 1-planar graphs (Russian), Met. Diskr. Anal. **41** (1984) 12–26

[3] H. Enomoto and M. Horňák, A general upper bound for the cyclic chromatic number of 3-connected plane graphs, submitted.

[4] H. Enomoto, M. Horňák and S. Jendrol', Cyclic chromatic number of 3-connected plane graphs, SIAM J. Discrete Math. **14** (2001) 121–137

[5] R. Halin, Zur Theorie der n-fach zusammenhäng,enden Graphen, Abh. Math. Sem. Univ. Hamburg, **33** (1969) 133–164

[6] M. Horňák and S. Jendrol', On a conjecture by Plummer and Toft, J. Graph Theory **30** (1999) 177–189

[7] A. Morita, Cyclic chromatic number of 3-connected plane graphs (Japanese, M. S. Thesis), Keio University, Yokohama 1998

[8] O. Ore and M.D. Plummer, Cyclic coloration of plane graphs, in: Recent Progress in Combinatorics (Proceedings of the Third Waterloo Conference on Combinatorics, Academic Press, New York 1969) 287–293

[9] M.D. Plummer and B. Toft, Cyclic coloration of 3-polytopes, J. Graph Theory **11** (1987) 507–515

[10] D.P. Sanders and Y. Zhao, A new bound on the cyclic chromatic number, J. Combin. Theory Ser. B **83** (2001) 102–111

[11] H. Whitney, Congruent graphs and the connectivity of graphs, Am. J. Math. **54** (1932) 150–168