On convexities of lattices

Judita Lihová

Abstract. In this paper, there are investigated some principal convexities of lattices and their mutual relations.

The notion of a convexity of lattices has been introduced by E. Fried (cf. [8]). He also proposed a problem concerning the "number" of convexities of lattices. J. Jakubík solved this problem in [3]; he showed that convexities of lattices form a proper class. In [3], there is also proved that the class of all convexities of lattices is a complete lattice (omitting the fact that it is a proper class) and the two–element chain generates an atom of this lattice.

Convexities can be defined also for various types of ordered algebraic structures. J. Jakubík defined and studied convexities of d-groups [4] and l-groups ([5],[6]). Some results concerning convexities of Riesz groups were derived in [7].

In the present paper we investigate the relation between some principal convexities. We also touch the problem of atoms in the lattice of all convexities and we prove that this lattice is distributive. Finally we propose some open questions.

1 Preliminaries

Let $\mathcal{L}$ be the class of all lattices. A subclass $\mathcal{K}$ of $\mathcal{L}$ is said to be a convexity of lattices (or simply a convexity), whenever $\mathcal{K}$ is closed under homomorphic images, convex sublattices and direct products (see [8]). Comparing this notion with that of a variety of lattices, we see that each variety is a convexity. The converse doesn't hold in general. E.g., the convexity $\mathcal{K}$ generated by a two–element chain is not a variety. Namely, in the opposite case, $\mathcal{K}$ would have to contain all distributive lattices. But this is not true, because there exist infinitely many convexities of distributive lattices, $\mathcal{K}$ being the least non–trivial one, as it follows from results of the section 3.

For a nonempty subclass $\mathcal{X}$ of the class $\mathcal{L}$ we denote by

$H\mathcal{X}$ the class of all homomorphic images of elements of $\mathcal{X}$;

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$C\mathcal{X}$ the class of all convex sublattices of elements of $\mathcal{X}$ and their isomorphic copies;

$P\mathcal{X}$ the class of all direct products of elements of $\mathcal{X}$ and their isomorphic copies.

We will use the following theorem (cf. [8] or [3]).

Theorem 1.1. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{L}$. Then $HCP\mathcal{X}$ is a convexity; moreover, it is the least one containing $\mathcal{X}$.*

If $\mathcal{X}$ is a one–element class, then the variety $HCP\mathcal{X}$ is said to be principal.

Let $\mathcal{C}$ be the class of all convexities of lattices. It is partially ordered by the class–theoretical inclusion. It is easy to verify that if $\{\mathcal{K}_i : i \in I\}$ is a nonempty subclass of $\mathcal{C}$, then $\bigcap_{i\in I} \mathcal{K}_i$ is a convexity, too. In view of this and the fact that $\mathcal{L}$ is the greatest element of $\mathcal{C}$, we will refer to $\mathcal{C}$ as a complete lattice (omitting the fact that $\mathcal{C}$ is a proper class). We will also apply the usual lattice-theoretical terminology and notation. So we will use the symbol $\bigvee_{i\in I} \mathcal{K}_i$ and $\bigwedge_{i\in I} \mathcal{K}_i$ for the least upper bound and the greatest lower bound of $\{\mathcal{K}_i : i \in I\} (\subseteq \mathcal{C})$, respectively. Evidently $\bigwedge_{i\in I} \mathcal{K}_i = \bigcap_{i\in I} \mathcal{K}_i$, $\bigvee_{i\in I} \mathcal{K}_i = HCP(\bigcup_{i\in I} \mathcal{K}_i)$.

As to the notation, N will be the chain of all positive integers, $N_0 = N \cup \{0\}$. Z will be the chain of all integers, while the symbol $\mathbb{Z}$ will be used for the additive group of all integers with the natural linear order. Analogously we will distinguish the chain R of all real numbers and the corresponding linearly ordered group $\mathbb{R}$. The n-element chain ($n \in N$) will be denoted by C_n.

2 Convexities generated by M_α

Let α be a cardinal, $\alpha \geq 3$. We denote by M_α the lattice consisting of elements u, v, $x_j (j \in J)$, where $\operatorname{card} J = \alpha$, $u < x_j < v$ and $x_{j(1)}$ is incomparable with $x_{j(2)}$ whenever $j(1)$ and $j(2)$ are distinct elements of J. J. Jakubík proved in [3] that if α, β are cardinals, $3 \leq \alpha \leq \beta$, then M_α does not belong to the convexity $HCP\{M_\beta\}$. We will show that if α, β are different finite cardinals, then $HCP\{M_\alpha\}$ and $HCP\{M_\beta\}$ are incomparable convexities, while for α, β infinite this is not the case, in general. To show this, we will use the notion of an f-subdirectly irreducible lattice and its connection with ultraproducts of lattices.

Definition 2.1. *A lattice L is said to be f-subdirectly irreducible (finitely subdirectly irreducible) if $\operatorname{card} L > 1$ and the intersection of any two non–trivial congruence relations of L is also a non–trivial congruence relation.*

Let us notice that all M_α for $\alpha \geq 3$ are f-subdirectly irreducible, since they have only trivial congruence relations.

If L_i is a lattice for each $i \in I$, $I \neq \emptyset$ and $\mathcal{F}$ is a dual ideal of the lattice of all subsets of I, the symbol $\prod(L_i/i \in I)/\mathcal{F}$ will be used for the reduced product of $(L_i/i \in I)$. If $\mathcal{F}$ is an ultrafilter (also called prime dual ideal), then $\prod(L_i/i \in I)/\mathcal{F}$ will be also referred to as an ultraproduct of $(L_i/i \in I)$. The symbol $\theta(\mathcal{F})$ will be used for the congruence relation corresponding to $\mathcal{F}$, $[f]\mathcal{F}$ will mean the congruence class containing $f \in \prod(L_i/i \in I)$. (See e.g. [2]).

For a nonempty subclass $\mathcal{X}$ of the class $\mathcal{L}$ we denote by

$P_{\mathcal{U}}\mathcal{X}$ the class of all lattices that are isomorphic to an ultraproduct of members of $\mathcal{X}$.

The proof of the following theorem is a slight modification of that of the analogous theorem for varieties of lattices (cf. [2], p. 302, Theorem 9).

Theorem 2.2. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{L}$, $\mathcal{K} = HCP\mathcal{X}$. If $L \in \mathcal{K}$ and L is f-subdirectly irreducible, then $L \in HCP_{\mathcal{U}}\mathcal{X}$.*

Proof. Let L be an f-subdirectly irreducible lattice, $L \in \mathcal{K}$. Then there exist $L_i \in \mathcal{X}$, $i \in I$, a convex sublattice B of $\prod(L_i/i \in I)$, and a congruence relation ϕ on B such that L is isomorphic to B/ϕ. By the above mentioned result from [2], there exists an ultrafilter $\mathcal{F}$ over I such that the corresponding congruence relation $\theta(\mathcal{F})$ restricted to B is contained in ϕ. Consider the set $\{[b]\mathcal{F} : b \in B\}$. It is evidently a sublattice of $\prod(L_i/i \in I)/\mathcal{F}$. We will show that it is convex. Let $[b_1]\mathcal{F} \leq [f]\mathcal{F} \leq [b_2]\mathcal{F}$ for some $b_1, b_2 \in B$, $f \in \prod(L_i/i \in I)$. We can suppose that $b_1(i) \leq b_2(i)$ for all $i \in I$ (in the opposite case we would take $b_1 \wedge b_2$ instead of b_1 and $b_1 \vee b_2$ instead of b_2). Now let $I_1 = \{i \in I : b_1(i) \leq f(i)\}$, $I_2 = \{i \in I : f(i) \leq b_2(i)\}$. Then $I_1, I_2 \in \mathcal{F}$ and so $I_1 \cap I_2 \in \mathcal{F}$, too. If we define $g \in \prod(L_i/i \in I)$ by $g(i) = f(i)$ if $i \in I_1 \cap I_2$ and $g(i) = b_1(i)$ otherwise, we have $b_1 \leq g \leq b_2$ and $[g]\mathcal{F} = [f]\mathcal{F}$. But B is convex, so that $g \in B$. We have shown that $[f]\mathcal{F} \in \{[b]\mathcal{F} : b \in B\}$. Now the correspondence $[b]\mathcal{F} \longmapsto [b]\phi$ is a homomorphisms of $\{[b]\mathcal{F} : b \in B\}$ onto B/ϕ. Thus $L \in HCP_{\mathcal{U}}\mathcal{X}$. □

Corollary 2.3. *Let $\mathcal{X}$ be a finite set of finite lattices. If $L \in HCP\mathcal{X}$ and L is f-subdirectly irreducible, then $L \in HC\mathcal{X}$.*

Proof. Under our assumptions concerning $\mathcal{X}$, $P_{\mathcal{U}}\mathcal{X}$ is, up to isomorphic copies, $\mathcal{X}$. □

Applying this theorem to $\mathcal{X} = \{M_\alpha\}$ for any finite cardinal α, $\alpha \geq 3$, we obtain that M_α and the two–element chain are the only f-subdirectly irreducible members of $HCP\{M_\alpha\}$. This implies:

Corollary 2.4. *If α, β are any distinct finite cardinals, $\alpha, \beta \geq 3$, then the convexities $HCP\{M_\alpha\}$, $HCP\{M_\beta\}$ are incomparable.*

Further we will consider α to be an infinite cardinal number.

Lemma 2.5. *If $L \in P_{\mathcal{U}}\{M_\alpha\}$, then L is isomorphic to M_β for some $\beta \geq \alpha$.*

Proof. Let $L = M_\alpha^I/\mathcal{F}$ for a nonempty set I and an ultrafilter $\mathcal{F}$ over I. Take any $[f]\mathcal{F} \in L$ and let us denote $I_1 = \{i \in I : f(i) = u\}$, $I_2 = \{i \in I : f(i) = v\}$. Then just one of I_1, I_2, $I - (I_1 \cup I_2)$ belongs to $\mathcal{F}$. Now if $I_1 \in \mathcal{F}$, then $[f]\mathcal{F} = [\mathbf{u}]\mathcal{F}$, where $\mathbf{u}(i) = u$ for all $i \in I$. If $I_2 \in \mathcal{F}$, then $[f]\mathcal{F} = [\mathbf{v}]\mathcal{F}$ for the v-constant element $\mathbf{v}$ of M_α^I. If $I - (I_1 \cup I_2) \in \mathcal{F}$, then evidently $[\mathbf{u}]\mathcal{F} < [f]\mathcal{F} < [\mathbf{v}]\mathcal{F}$. Let us suppose that $[\mathbf{u}]\mathcal{F} < [f]\mathcal{F}$, $[g]\mathcal{F} < [\mathbf{v}]\mathcal{F}$, $[f]\mathcal{F} \neq [g]\mathcal{F}$. Then $\{i \in I : f(i) \parallel g(i)\} \supseteq \{i \in I : f(i) \neq g(i)\} \cap \{i \in I : f(i) \notin \{u, v\}\} \cap \{i \in I : g(i) \notin \{u, v\}\} \in \mathcal{F}$, so that $[f]\mathcal{F} \parallel [g]\mathcal{F}$. Thus L is isomorphic to M_β for a cardinal β. Moreover, $\beta \geq \alpha$, because if we define $f_j \in M_\alpha^I$ for each $j \in J$ by $f_j(i) = x_j$ for all $i \in I$, then $[f_j]\mathcal{F}$ are mutually different. $\square$

We will use the following assertion, which is a consequence of 6.1.14 and 6.3.21 of [1].

Theorem 2.6. *Let I be any infinite set of the cardinality λ, A a set of the cardinality α. Then there exists an ultrafilter $\mathcal{F}$ over I such that* $\operatorname{card} A^I/\mathcal{F} = \alpha^\lambda$.

As a consequence we obtain:

Theorem 2.7. *For each infinite cardinal α there exists a cardinal $\beta > \alpha$ with $M_\beta \in HCP\{M_\alpha\}$.*

Proof. Take any set I of the cardinality α and an ultrafilter $\mathcal{F}$ over I with $\operatorname{card} M_\alpha^I/\mathcal{F} = \alpha^\alpha$. Set $\beta = \alpha^\alpha$. Then evidently $\beta > \alpha$ and, in view of 2.5, $M_\alpha^I/\mathcal{F}$ is isomorphic to M_β. $\square$

Corollary 2.8. *For each infinite cardinal α there exists an increasing infinite sequence of cardinals $\alpha_0 < \alpha_1 < \ldots$ such that $\alpha_0 = \alpha$ and $HCP\{M_{\alpha_0}\} \supsetneqq HCP\{M_{\alpha_1}\} \supsetneqq HCP\{M_{\alpha_2}\} \supsetneqq \ldots$.*

3 Convexities generated by finite chains

We will consider principal convexities generated by finite and also by some infinite chains and study relations between them.

Theorem 3.1. *For each $n \in N$, $HCP\{C_n\} \subsetneqq HCP\{C_{n+1}\}$.*

Proof. Since C_{n+1} contains n-element chain as a convex sublattice, it holds $HCP\{C_n\} \subseteqq HCP\{C_{n+1}\}$. So we have only to show that $C_{n+1} \notin HCP\{C_n\}$ for each $n \in N$. By way of contradiction, let n_0 be the least positive integer with $C_{n_0+1} \in HCP\{C_{n_0}\}$. Evidently $n_0 \geq 3$, because $HCP\{C_1\}$ contains only one–element lattices and each $L \in HCP\{C_2\}$ is a relatively complemented lattice, while C_3 fails to have this property. The relation $C_{n_0+1} \in HCP\{C_{n_0}\}$ implies that there exist an index set I, a convex sublattice B of $C_{n_0}^I$ and a homomorphism φ of B onto C_{n_0+1}. As C_{n_0+1} is bounded, we can suppose that B is an interval, say $\langle f_0, f_1\rangle$ and $B = \prod(C_{k_i} : i \in J)$ with $J \subseteq I$, $1 < k_i \leq n_0$. Let us define $f \in B$ in such a way that, for $i \in J$, $f(i)$ is the least element of C_{k_i} if $k_i < n_0$ and the element covering the least one otherwise. Assume that C_{n_0+1} is the chain $c_0 < c_1 < \ldots < c_{n_0}$, $\varphi(f) = c_t$. Then we have $\varphi(\langle f_0, f\rangle) = \langle c_0, c_t\rangle$, $\varphi(\langle f, f_1\rangle) = \langle c_t, c_{n_0}\rangle$. As $\langle f_0, f\rangle$ is a convex sublattice of a product of two–element chains, it must be $t \leq 1$. On the other hand, $\langle f, f_1\rangle$ is a convex sublattice of a product of $(n_0 - 1)$-element chains, so that $t > 1$, by the choice of n_0. We have a contradiction. □

Corollary 3.2. *If C is any infinite chain, the $HCP\{C_n\} \subsetneqq HCP\{C\}$ for each $n \in N$.*

Proof. Let $n \in N$, $c_1, c_2, \ldots, c_{n-1}$ be any elements of C with $c_1 < c_2 < \ldots < c_{n-1}$. Let θ be a binary relation on C defined by

$$c\,\theta\,c'(c, c' \in C) \iff c_i \leq c, c' < c_{i+1} \text{ for some } i \in \{1, \ldots, n-2\}$$
$$\text{or } c, c' < c_1 \text{ or } c, c' \geq c_{n-1}\,.$$

Then evidently θ is a congruence relation of C and C/θ is an n-element chain. This shows that $HCP\{C_n\} \subseteq HCP\{C\}$ for each $n \in N$. But since $HCP\{C_n\} \subsetneqq HCP\{C_{n+1}\} \subseteq HCP\{C\}$, the proof is complete. □

Theorem 3.3. *It is $HCP\{C_n : n \in N\} = HCP\{Z\}$.*

Proof. As the chain Z contains n-element chain for each $n \in N$ as its convex subset, we have $HCP\{C_n : n \in N\} \subseteq HCP\{Z\}$. To show the converse inclusion, take the set I of all odd positive integers and any ultrafilter $\mathcal{F}$ over I containing complements of all finite subsets of I. For each $n \in I$, $n = 2k + 1$, let C_n be the chain $-k < \ldots < -1 < 0 < 1 < \ldots < k$. Let us denote by L the product $\prod(C_n/n \in I)$. If $f \in L$, we will say that f is $\mathcal{F}$-constant whenever there exist $J \in \mathcal{F}$ and $t \in Z$ with $f(n) = t$ for all $n \in J$. Let L^* be the set of all $\mathcal{F}$-constant elements of L. Now consider the set $\{[f]\mathcal{F} : f \in L^*\}$. This set is a sublattice of the lattice $L/\mathcal{F}$, because the join and the meet of two $\mathcal{F}$-constant elements of L is evidently also $\mathcal{F}$-constant. Further, it is also convex. To see this, let $[f]\mathcal{F} < [h]\mathcal{F} < [g]\mathcal{F}$ for some $f, g \in L^*$, $h \in L$. Then there exist $I_1, I_2, I_3, I_4 \in \mathcal{F}$ such that $f(n) = t$ for all $n \in I_1$ and some $t \in Z$, $g(n) = s$ for all $n \in I_2$ and some $s \in Z$, $f(n) < h(n)$ for all $n \in I_3$ and $h(n) < g(n)$ for all $n \in I_4$. Denote $J = I_1 \cap I_2 \cap I_3 \cap I_4$. Evidently $J \in \mathcal{F}$ and $t = f(n) < h(n) < g(n) = s$ for all $n \in J$. Thus $h(n) \in \{t+1, t+2, \ldots, s-1\}$ for all $n \in J$. This implies that for some $p \in \{t+1, \ldots, s-1\}$, the set $\{n \in J : h(n) = p\}$ belongs to $\mathcal{F}$, hence h is $\mathcal{F}$-constant. We have proved that the set $\{[f]\mathcal{F} : f \in L^*\}$ is a convex sublattice of the lattice $L/\mathcal{F}$. Since $L/\mathcal{F}$ belongs to $HCP\{C_n : n \in N\}$, the same holds for

$\{[f]\mathcal{F} : f \in L^*\}$. Finally we are going to show that the chain Z is isomorphic to $\{[f]\mathcal{F} : f \in L^*\}$. For any $t \in Z$, let us define $f_t \in L$ by

$$f_t(n) = \begin{cases} t & \text{if} \quad n \geq 2|t| + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the mapping $t(\in Z) \longmapsto [f_t]\mathcal{F}$. It is evidently onto $\{[f]\mathcal{F} : f \in L^*\}$ and $t < s(t, s \in Z)$ implies $[f_t]\mathcal{F} < [f_s]\mathcal{F}$, because the set $\{n \in I : f_t(n) < f_s(n)\}$ has a finite complement. The proof is complete. $\square$

In view of 3.2 and 3.3, if C is any infinite chain, then $HCP\{Z\} \subseteq HCP\{C\}$. Thus $HCP\{Z\} \subseteq HCP\{N\}$. But as N is a convex sublattice of Z, the converse inclusion holds, too. Hence $HCP\{Z\} = HCP\{N\}$ and evidently the equality holds also in the case that we take the chain of all negative integers instead of N. As more interesting can be regarded the fact that $HCP\{Z\} = HCP\{R\}$. This fact is a consequence of the following theorem, which can be found in [6] (see 2.5).

Theorem 3.4. *The linearly ordered group $\mathbb{R}$ belongs to the convexity of l-groups generated by the linearly ordered group $\mathbb{Z}$.*

Corollary 3.5. $HCP\{Z\} = HCP\{R\}$.

Proof. It is sufficient to show that $R \in HCP\{Z\}$. By 1.2 of [6], the assertion of 3.4 means that there exist an l-group $\mathbb{L}$ being a direct product $\prod(\mathbb{G}_i/i \in I)$ with $\mathbb{G}_i$ isomorphic to $\mathbb{Z}$, a convex l-subgroup B of $\mathbb{L}$ and an l-homomorphism φ of the l-group $\mathbb{B}$ onto $\mathbb{R}$. Ignoring the group structure of $\mathbb{L}$, $\mathbb{B}$ and $\mathbb{Z}$ we get immediately that $R \in HCP\{Z\}$. $\square$

Let us remark that, as it was shown in [6], the convexity of l-groups generated by $\mathbb{Z}$ is larger than that generated by $\mathbb{R}$.

4 An example

J. Jakubík proved in [3] that the convexity $HCP\{C_2\}$ is an atom in the lattice $\mathcal{C}$ of all convexities of lattices. He also formulated the question if there are other atoms in $\mathcal{C}$. This question remains open. We give here some results concerning this problem. Further we prove that the lattice $\mathcal{C}$ is distributive.

Let L be a lattice. Consider the following conditions concerning L:

(i) L contains a non–trivial distributive interval;

(ii) L has a non–trivial distributive homomorphic image.

Theorem 4.1. *Let L be a lattice satisfying any of the conditions* (i), (ii). *Then* $HCP\{L\} \supseteq HCP\{C_2\}$.

Proof. Without regard to which of the conditions (i), (ii) is fulfilled, the convexity $HCP\{L\}$ contains a distributive lattice L_1 with $\operatorname{card} L_1 > 1$. By a well–known theorem each distributive lattice containing more than one element can be homomorphically mapped onto C_2. So $HCP\{C_2\} \subseteq HCP\{L_1\} \subseteq HCP\{L\}$. □

Let us remark that the condition (i) is fulfilled, e.g., by each finite lattice or, more generally, by each lattice containing a prime interval. We are going to show that the converse assertion to 4.1 doesn't hold, in general.

If P is any partially ordered set and S is a bounded partially ordered set, we will use the denotation $(S \to P)$ for the partially ordered set obtained in such a way that each prime interval of P is replaced by S. It is easy to see that if S, P are lattices, then so is $(S \to P)$ and P can be regarded as its sublattice.

Now take the lattice M_3 and define $L_i (i \in N_0)$ as follows:
$L_0 = M_3$;
if L_i is defined for some $i \in N_0$, then $L_{i+1} = (M_3 \to L_i)$.

We have an ascending chain of lattices $L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots$ with L_i being a sublattice of L_{i+1} for each $i \in N_0$. Let L be the join of all L_i. Evidently L is a lattice and each L_i is its sublattice. It is easy to see that L is not even modular. The following lemma shows that L doesn't satisfy (ii).

Lemma 4.2. *The lattice L has only trivial congruence relations.*

Proof. It can be easily shown, by induction on i, that all L_i have only trivial congruence relations. Now if θ is a congruence relation of L, $a, b \in L$, $a \neq b$, $(a, b) \in \theta$, we take $i \in N_0$ such that both a and b belong to L_i. The restriction of θ to L_i is the greatest congruence relation, so θ glues together the least element u and the greatest element v of L_i. But since u is the least element and v is the greatest element also in L, θ must be the greatest congruence relation. □

To show that L doesn't satisfy (i), let us look at the intervals of L.

Lemma 4.3. *Each non–trivial interval of L is isomorphic to $(L \to C_n)$ for some $n \in N$, $n \geq 2$.*

Proof. Let $a, b \in L$, $a < b$, $I = \{x \in L : a \leq x \leq b\}$. There exists $n \in N_0$ with $a, b \in L_n$. We will proceed by induction on n. If $n = 0$, then evidently I is isomorphic to L, which is isomorphic to $(L \to C_2)$. Let $k \in N_0$ and suppose that the assertion holds whenever $a, b \in L_k$. Now assume that $a, b \in L_{k+1}$. Distinguish the following cases:

1. $a, b \in L_k$,

2. $a \in L_{k+1} - L_k$, $b \in L_k$,

3. $a \in L_k$, $b \in L_{k+1} - L_k$,

4. $a, b \in L_{k+1} - L_k$.

In the first case the assertion follows immediately from the induction hypothesis. In the second case there exists a unique element $c \in L_k$ such that c covers a in L_{k+1}. It holds $c \leq b$. Let $I_1 = \{x \in L : a \leq x \leq c\}$, $I_2 = \{x \in L : c \leq x \leq b\}$. Evidently $I = I_1 \cup I_2$, I_1 is isomorphic to L, I_2 is isomorphic to $(L \to C_m)$ for some $m \in N$, $m \geq 2$. Thus I is isomorphic to $(L \to C_{m+1})$. The case 3. is analogous to the preceding one. Finally, let $a, b \in L_{k+1} - L_k$. Then there exists a unique couple of elements $c, d \in L_k$ such that c covers a and d is covered by b in L_{k+1}. Evidently $c \leq d$. If $c = d$, then I is isomorphic to $(L \to C_2)$. If $c < d$, then using the induction hypothesis we obtain that $\{x \in L : c \leq x \leq d\}$ is isomorphic to $(L \to C_m)$ for some $m \in N$, $m \geq 2$. Since $I = \{x \in L : a \leq x \leq c\} \cup \{x \in L : c \leq x \leq d\} \cup \{x \in L : d \leq x \leq b\}$, we have that I is isomorphic to $(L \to C_{m+2})$. □

Using 4.3 we obtain immediately that L doesn't satisfy (i), because L is not distributive, as we have already remarked. In spite of the fact that L satisfies neither (i) nor (ii), we will show that $HCP\{L\} \supseteq HCP\{C_2\}$.

Let us define a sequence $a_0, a_1, a_2, \dots$ of elements of L as follows: a_0 will be the least element of L_0. Take any $b \in L_0$ covering a_0 in L_0 and denote by a_1 any element of L_1 which lies between a_0 and b. Further denote by a_2 any element of L_2 which lies between a_1 and b, and so on.

We have $a_0 < a_1 < \dots < b$, and for each $i \in N$, $a_i \in L_i - L_{i-1}$, a_i covers a_{i-1} in L_i. Introduce the following denotations: for each $i \in N$, I_i will be the set $\{x \in L : a_0 \leq x \leq a_i\}$, $M = \prod(I_i/i \in N)$. It is easy to see that I_i is isomorphic to $(L \to C_{i+1})$ (see Figure 1).

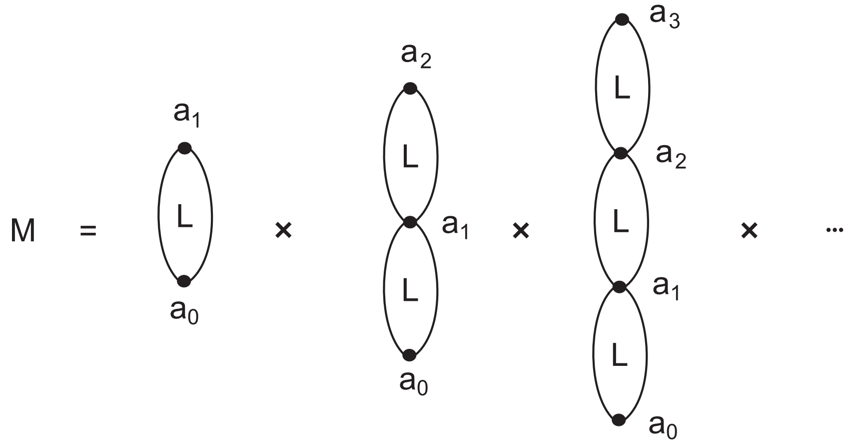

Figure 1

The aim is to show that there exists a prime ideal P of M, $P \neq M$. Then the chain C_2 is a homomorphic image of M and so $C_2 \in HCP\{L\}$.

For each $i \in N$, let C_i be the chain $0 < 1 < \ldots < i-1$, $T = \prod(C_{i+1}/i \in N)$ (see Figure 2). For each $f \in T$ let us define $f^{+1} \in T$ in such a way that $f^{+1}(i) = \min\{i, f(i)+1\}$ for each $i \in N$.

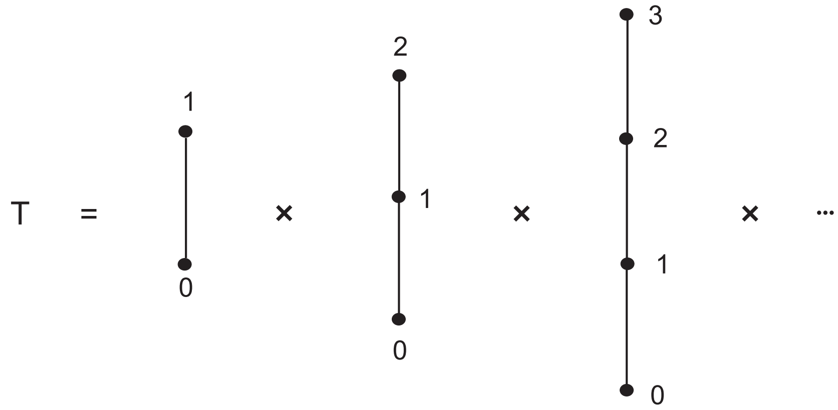

Figure 2

Lemma 4.4. *There exists a prime ideal I of T, $I \neq T$, such that $f^{+1} \in I$ whenever $f \in I$.*

Proof. Let $\mathcal{F}$ be any ultrafilter over N containing complements of all finite subsets of N. Let T^* be the set of all $\mathcal{F}$-constant elements of T. It is easy to verify that T^* is an ideal of T, $T^* \neq T$. To show that T^* is a prime ideal, let $f, g \in T$, $f \wedge g \in T^*$. Then there exist $p \in N_0$ and $J \in \mathcal{F}$ such that $(f \wedge g)(i) = p$ for all $i \in J$. Now let $J_1 = \{i \in J : f(i) \leq g(i)\}$, $J_2 = \{i \in J : f(i) \geq g(i)\}$. As $J_1 \cup J_2 = J \in \mathcal{F}$, at least one of J_1, J_2 belongs to $\mathcal{F}$. If, e.g., $J_1 \in \mathcal{F}$, then $f \in T^*$, because $f(i) = f(i) \wedge g(i) = (f \wedge g)(i) = p$ for all $i \in J_1$. Finally let $f \in T^*$, $f(i) = p (\in N_0)$ for all $i \in J (\in \mathcal{F})$. Evidently $J' = J - \{p\}$ also belongs to $\mathcal{F}$ and $f^{+1}(i) = p + 1 \leq i$ for all $i \in J'$. Thus $f^{+1} \in T^*$. $\square$

Lemma 4.5. *There exists a prime ideal P of M, $P \neq M$.*

Proof. If $f \in M$, let $\bar{f}$ be the element of T defined as follows:

$\bar{f}(i) = k$ iff k is the least nonnegative integer with $f(i) \leq a_k$.

Set $P = \{f \in M : \bar{f} \in I\}$, where I is as in 4.4. If $g \leq f \in P$, $g \in M$, then $\bar{g} \leq \bar{f} \in I$ and so $\bar{g} \in I$, $g \in P$. Let $f, g \in P$. Then $\bar{f}, \bar{g} \in I$, so that $\bar{f} \vee \bar{g} \in I$. It is easy to verify that $\bar{f} \vee \bar{g} = \overline{f \vee g}$. Hence $\overline{f \vee g} \in I$, which implies $f \vee g \in P$. We have proved that P is an ideal of M. Since $(1, 2, 3, \ldots) \in T - I$, it is $(a_1, a_2, a_3, \ldots) \in M - P$. Hence $P \neq M$. Finally we are going to show that the ideal P is prime. Let $f \wedge g \in P$ for some $f, g \in M$. Then $\overline{f \wedge g} \in I$, which implies $\overline{f \wedge g}^{+1} \in I$. We will show that $\bar{f} \wedge \bar{g} \leq \overline{f \wedge g}^{+1}$. This will imply $\bar{f} \wedge \bar{g} \in I$. Using that I is a prime ideal, we will obtain $\bar{f} \in I$ or $\bar{g} \in I$, which will conclude $f \in P$ or $g \in P$. Take any $i \in N$ and suppose that $\bar{f}(i) = k$, $\bar{g}(i) = l$. Distinguish two cases:

1. $k = l$,
2. $k \neq l$.

If $k = l = 0$, then $(\bar{f} \wedge \bar{g})(i) = \bar{f}(i) \wedge \bar{g}(i) = 0 \leq \overline{f \wedge g}^{+1}(i)$. If $k = l > 0$, then $a_{k-1} < f(i),\ g(i) \leq a_k$, which implies $a_{k-1} \leq f(i) \wedge g(i) \leq a_k$. Thus $\overline{f \wedge g}(i) \in \{k-1, k\}$, which yields $\overline{f \wedge g}^{+1}(i) \geq k$. So we have $(\bar{f} \wedge \bar{g})(i) = \bar{f}(i) \wedge \bar{g}(i) = k \wedge l = k \leq \overline{f \wedge g}^{+1}(i)$. We are ready with the case 1. As to the case 2, we can suppose, without loss of generality, that $k < l$. Then $f(i) \leq a_k \leq a_{l-1} < g(i) \leq a_l$, which implies $\overline{f \wedge g}(i) = \bar{f}(i) = k$ and $\overline{f \wedge g}^{+1}(i) = k+1$. So we have $(\bar{f} \wedge \bar{g})(i) = \bar{f}(i) \wedge \bar{g}(i) = k \wedge l = k < k+1 = \overline{f \wedge g}^{+1}(i)$, completing the proof. □

Theorem 4.6. *Let L be the lattice defined before 4.2. Then $HCP\{C_2\} \subsetneqq HCP\{L\}$.*

Proof. $C_2 \in H\{M\}$ by 4.5, $M \in PC\{L\}$, hence $C_2 \in HCP\{L\}$. Further, $HCP\{C_2\}$ contains only distributive lattices, while L fails to be distributive. So $HCP\{C_2\} \subsetneqq HCP\{L\}$. □

J. Jakubík proved in [6] that the lattice of all convexities of l-groups is distributive. This result was extended to the lattice of all convexities of Riesz groups in [7]. Now we are going to prove the distributivity of the lattice $\mathcal{C}$ of all convexities of lattices.

Lemma 4.7. *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be convexities of lattices. Then $\mathcal{K} = \{L \times M : L \in \mathcal{K}_1, M \in \mathcal{K}_2\}$ is a convexity and it holds $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2$.*

Proof. The inclusion $\mathcal{K} \subseteq \mathcal{K}_1 \vee \mathcal{K}_2$ is trivial. Let $A \in \mathcal{K}_1 \vee \mathcal{K}_2 = HCP(\mathcal{K}_1 \cup \mathcal{K}_2)$. Then there exist lattices $L_i \in \mathcal{K}_1 \cup \mathcal{K}_2 (i \in I, I \neq \emptyset)$, a convex sublattice B of $\prod(L_i/i \in I)$ and a congruence relation ϕ of B such that A is isomorphic to B/ϕ. Let $I_1 = \{i \in I : L_i \in \mathcal{K}_1\}$, $U = \prod(L_i/i \in I_1)$, $V = \prod(L_i/i \in I - I_1)$. If some of the sets I_1, $I - I_1$ is empty, the corresponding direct product is regarded as a one–element lattice. We can suppose that B is a convex sublattice of $U \times V$. Let us denote by B_1 and B_2 the projection of B into U and V, respectively. It is easy to verify that B_1 and B_2 is a convex sublattice of U and V, respectively, and $B = B_1 \times B_2$. Now there exist congruence relations ϕ_1 of B_1 and ϕ_2 of B_2 with $\phi = \phi_1 \times \phi_2$. Then B/ϕ is isomorphic to $B_1/\phi_1 \times B_2/\phi_2$ and $B_1/\phi_1 \in \mathcal{K}_1$, $B_2/\phi_2 \in \mathcal{K}_2$. Hence A, being isomorphic to B/ϕ, belongs to $\mathcal{K}$. □

Theorem 4.8. *The lattice $\mathcal{C}$ of all convexities of lattices is distributive.*

Proof. Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \in \mathcal{C}$. We are going to verify $\mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3) = (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. Clearly $\mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3) \supseteq (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. Now let $L \in \mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3)$. Then $L \in \mathcal{K}_1$ and L is isomorphic to $L_1 \times L_2$ for some $L_1 \in \mathcal{K}_2$, $L_2 \in \mathcal{K}_3$ by 4.7. We can suppose that L_1, L_2 are convex sublattices of L, so that $L_1, L_2 \in C\mathcal{K}_1 = \mathcal{K}_1$. Hence we have $L \in (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. □

We conclude with some open questions. The first of them has already been formulated in [3].

1. Is $HCP\{C_2\}$ the only atom in $\mathcal{C}$?
2. What are the necessary and sufficient conditions for a distributive relatively complemented lattice L to belong to $HCP\{C_2\}$?
3. What are the necessary and sufficient conditions for a distributive lattice L to belong to $HCP\{C_3\}$?
4. Does the convexity $HCP\{C_{n+1}\}$ cover the convexity $HCP\{C_n\}$ for $n \in N$?
5. Does the convexity $HCP\{M_n\}$ $(n \in N)$ cover the convexity $HCP\{C_2\}$?

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Judita Lihová
Institute of Mathematics
P. J. Šafárik University
Jesenná 5
SK-041 54 Košice
Slovakia
e-mail: `lihova@science.upjs.sk`

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK-040 01 Košice
Slovakia