NOTE
Rainbowness of cubic polyhedral graphs

Stanislav JENDROL'

[1)] Institute of Mathematics, P.J. Šafárik University
Jesenná 5, SK-041 54 Košice, Slovak Republic
e-mail: stanislav.jendrol@upjs.sk

Abstract. The *rainbowness*, $\mathrm{rb}(G)$, of a connected plane graph G is the minimum number k such that any colouring of vertices of the graph G using at least k colours involves a face all vertices of which have different colours. For a cubic polyhedral (i.e. 3-connected plane) graph G we prove that

$$\frac{n}{2} + \alpha_1^* - 1 \leq \mathrm{rb}(G) \leq n - \alpha_0^* + 1\,,$$

where α_0^* and α_1^* denote the independence number and the edge independence number, respectively, of the dual graph G^* of G. Moreover, we show that the lower bound is tight and that the upper bound cannot be less than $n - \alpha_0^*$ in general. We also prove that if the dual graph G^* of an n-vertex cubic polyhedral graph G has a perfect matching then

$$\mathrm{rb}(G) = \frac{3}{4}\,n\,.$$

AMS Subject Classification 2000: 05C15, 52B10
Keywords: vertex colouring, rainbowness, plane graph, cubic polyhedral graphs

1 Introduction

Colouring vertices of plane graphs under restrictions given by faces has recently attracted much attention, see e.g. [JKS], [K], [N], [RW] and references there. One natural problem of this kind is the following Ramsey type problem: Let us define the *rainbowness* of a connected plane graph G, $\mathrm{rb}(G)$, as the minimum number k such that any surjective colour assignment $\varphi : V(G) \rightarrow$

$\{1, 2, \ldots, k\}$ involves a face all vertices of which have different colours. Problem is to determine the rainbownes of the graph G.

We use the standard terminology according to [BM] except for few notation defined througout. However we recall some frequently used terms.

For a plane graph G let $\alpha_0(G)$ be the independence number of G and $\alpha_1(G)$ be the edge independence number of G. Let G^* be the dual graph to the plane graph G. Then we let $\alpha_0^*(G) = \alpha_0(G^*)$ and $\alpha_1^*(G) = \alpha_1(G^*)$.

The rainbowness, $\mathrm{rb}(T)$, of plane triangulations T has been recently studied (under the name looseness) by Negami [N]. He proved that for any triangulation T

$$\alpha_0(T) + 2 \leq \mathrm{rb}(T) \leq 2\alpha_0(T) + 1\,,$$

where $\alpha_0(T)$ is the independence number of T. Ramamurthi and West [RW] observed that the following inequality relating $\mathrm{rb}(G)$ to the independence number $\alpha_0(G)$ and the chromatic number $\chi_0(G)$ holds

$$\mathrm{rb}(G) \geq \alpha_0(G) + 2 \geq \left\lceil \frac{n}{\chi_0(G)} \right\rceil + 2\,,$$

where $n = |V(G)|$, the number of vertices of a plane graph G.

For an n-vertex plane graph G, the Four Colour Theorem yields $\mathrm{rb}(G) \geq \lceil \frac{n}{4} \rceil + 2$. If G is triangles-free, then Grötzsch's theorem (see [G], [T]) gives $\mathrm{rb}(G) \geq \lceil \frac{n}{3} \rceil + 2$. In [RW] Ramamurthi and West showed that the above lower bound is tight for a fixed n when $\chi(G) = 2, 3$ and it is within one of being tight for $\chi(G) = 4$. They conjectured the following bound for triangle-free plane graphs.

Conjecture 1.1. *If G is n-vertex triangles-free plane graph, $n \geq 4$, then* $\mathrm{rb}(G) \geq \lceil \frac{n}{2} \rceil + 2$.

Ramamurthi and West proved their conjecture for plane graphs with girth at least six. Jungić, Král' and Škrekovski [JKS] answered the conjecture in affirmative. Moreover, they proved for plane graphs G with girth $g \geq 5$ that the rainbowness $\mathrm{rb}(G)$ is at least $\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \rceil + 1$ if g is odd and $\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \rceil + 1$ if g is even. The bounds are tight for all pairs n and g with $g \geq 4$ and $n \geq \frac{5g}{2} - 3$.

In [JS] the authors determined the precise values of the rainbowness for all, up to three, graphs of semiregular polyhedra.

In the present note we investigate cubic polyhedral (i.e. trivalent 3-connected plane) graphs. For this family of graphs we give better bounds than those mentioned above. The main result of this note is

Theorem 1.2. *Let G be an n-vertex cubic polyhedral graph. Let α_0^* and α_1^* be an independence number and edge independence number, respectively, of the dual G^* of the graph G. Then*

$$\frac{n}{2} + \alpha_1^* - 1 \leq \mathrm{rb}(G) \leq n - \alpha_0^* + 1\,. \tag{1}$$

2 Lower bound

Let G be a cubic polyhedral graph. Let $M^* = \{e_1^*, \dots, e_d^*\}$ be a maximum matching in G^*. Clearly $\alpha_1^*(G) = d = \alpha_1^*$. Every edge $e_1^* = xy$ of M^* is associated in G with a pair of two adjacent faces $f(x)$ and $f(y)$ which share an edge e_i in common.

Let $M = \{e_1, e_2, \dots, e_d\}$ be the set of such defined edges of G. Clearly M is a matching. Let $V(M)$ be the set of vertices incident with the edges of M (i.e. it is a set of end vertices of edges from M). Evidently $|V(M)| = 2d = 2\alpha_1^*$. Similarly, let $F(M)$ be the set of faces containg an edge from the set M. Observe that if $e_i \neq e_j$ then the pair of faces incident with the edge e_i is disjoint with the pair of faces incident with e_j. Hence $|F(M)| = 2\alpha_1^*$. The following observation is easy to see

Observation 1. Each face of G has at most one edge in the set M. □

Observation 2. If f_1 and f_2 are two distinct faces from $F(G) - F(M)$ then f_1 and f_2 do not share any common edge (and any vertex).

Proof. If f_1 and f_2 would share an edge h then it could be added to the set M and the edge $h^* = f_1^* f_2^*$ of G^* corresponding to h could extend maximum matching M^* of G^*, a contradiction. □

Observation 3. Every face $f \in F(G) - F(M)$ contains at least two vertices that are not in $V(M)$.

Proof. Let f be a face that is not in $F(M)$. It is easy to see that no two consecutive vertices of f belong to $V(M)$. Otherwise we have a contradiction with Observation 1. □

The following colouring does not involve any rainbow face. First we find the set $M = \{e_1, e_2, \dots, e_d\}$, $d = \alpha_1^*$. Next we find the set of faces $F(G) - F(M) = \{f_1, f_2, \dots, f_m\}$ where $m = |F(G)| - 2\alpha_1^*$. We color vertices of the edge e_i with colour i for any $i \in \{1, \dots, d\}$. For every face f_j, $j \in \{1, \dots, m\}$, choose two vertices that are not in $V(M)$ and colour them both with the colour $d + j$. The remaining not yet coloured vertices are colored with different colours from the set $\{d + m + 1, \dots, n - d - m\}$.

Because each face of G is adjacent with a monochromatic edge or with two vertices of the same colour, G does not contain any rainbow face. In this colouring we have used the following number of colours

$$n - d - m = n - \alpha_1^* - |F(G)| + 2\alpha_1^* = n + \alpha_1^* - |F(G)| \,. \tag{2}$$

Because G is a cubic polyhedral graph we have $3n = 2|E(G)|$. Using this fact and the Euler's polyhedral formula $n - |E(G)| + |F(G)| = 2$ we obtain

$$|F(G)| = \frac{n+4}{2} \,. \tag{3}$$

Substituting for $|F(G)|$ from (2) in (3) we find out that the number of used colours is $\frac{n}{2} + \alpha_1 - 2$. Hence we have proved that

$$\mathrm{rb}(G) \geq \frac{n}{2} + \alpha_1 - 1\,.$$

3 Upper bounds

Lemma 3.1. *Let G be an n-vertex plane graph and let $\{f_1, \ldots, f_k\}$ be a set of faces of G such that no two among them have a common vertex. Then*

$$\mathrm{rb}(G) \leq n - k + 1\,.$$

Proof. Let $V(f_i)$ be the set of vertices incident with the face f_i. Suppose there is a $(n-k+1)$-colouring of G that has no rainbow face. Then there are at most $|V(f_i)| - 1$ colours at f_i and at most $n - \sum_{i=1}^{k} |V(f_i)|$ colours at the vertices outside of the set $\bigcup_{i=1}^{k} V(f_i)$. This means that there are at most

$$n - \sum_{i=1}^{k} |V(f_i)| + \sum_{i=1}^{k} (|V(f_i)| - 1) = n - k$$

colours used at the vertices of G; a contradiction. □

Observe that maximum number of faces in a cubic polyhedral graph G that no two among them have a vertex in common is $\alpha_0^* = \alpha_0(G^*)$, the independence number of G^*, the dual of G. This observation together with Lemma 3.1 yield

Lemma 3.2. *For any n-vertex cubic polyhedral graph G there is*

$$\mathrm{rb}(G) \leq n - \alpha_0^* + 1\,.$$

□

The upper bound in Lemma 3.2 can be improved if $\alpha_0^* < \frac{|F(G)|}{2}$.

Theorem 3.3. *Let G be an n-vertex cubic polyhedral graph with m faces. If $\alpha_0^* < \frac{m}{2}$ then*

$$\mathrm{rb}(G) \leq n - \alpha_0^*\,.$$

Moreover, this bound is tight.

Proof. Suppose there is a $(n - \alpha_0^*)$-colouring φ of a cubic polyhedral graph G that has no rainbow face. Let $V(j)$ be the set of vertices coloured with colour j and let $F(j)$ be the set of faces that are not rainbow because they contain at least two vertices from the set $V(j)$. Let us estimate the number of pairs (v, f) with a vertex v from $V(j)$ and a face f from $F(j)$.
Let $|V(j)| = a_j$ and $|F(j)| = d_j$. Each face of $F(j)$ contains at least two vertices from $V(j)$, hence there are at least $2d_j$ such pairs. On the other hand each vertex can be incident with at most three faces of $F(j)$ therefore there are at most $3a_j$ such pairs. Altogether we have

$$2d_j \le 3a_j\,. \tag{4}$$

It is easy to see that if $a_j = 1$ then $d_j = 0$, if $a_j = 2$ then $d_j \le 2$, and for $a_j \ge 3$ we have from (4) that

$$d_j \le \left\lfloor \frac{3a_j}{2} \right\rfloor \le 2(a_j - 1)\,.$$

The number of non-rainbow faces in G is at most

$$\sum_{j=1}^{n-\alpha_0^*} d_j \le \sum_{j=1}^{n-\alpha_0^*} 2(a_j - 1) = 2\sum_{j=1}^{n-\alpha_0^*} a_j - 2(n - \alpha_0^*) = 2\alpha_0^*\,.$$

Because $2\alpha_0 < m$ there is a rainbow face in G; a contradiction. For tightness of the bound see the last section of this paper. □

Theorem 3.4. *Let G be an n-vertex cubic polyhedral graph. Then*

$$\mathrm{rb}(G) \le \frac{3}{4}n\,.$$

Moreover, the bound is tight.

Proof. Suppose there is a $\lfloor \frac{3n}{4} \rfloor$-colouring φ of G without any rainbow face. Analogously as in the proof of Theorem 3.3 we can show that for $r = \lfloor \frac{3n}{4} \rfloor$

$$\sum_{i=1}^{r} d_i \le 2\sum_{i=1}^{r}(a_i - 1) = 2(n - r) = 2\left(n - \left\lfloor \frac{3n}{4} \right\rfloor\right) = 2\left\lceil \frac{n}{4} \right\rceil.$$

If m is the number of faces of G then, because G does not contain rainbow faces, $2\lceil \frac{n}{4} \rceil \ge m$. If e is the number of edges of G then $2e = 3n$. Using these two relations and Euler's polyhedral formula we obtain the inequality

$$4\left\lceil \frac{n}{4} \right\rceil \ge n + 4$$

which immediately yields a contradiction. For tightness of the bound $\frac{3}{4}n$ see below. □

Theorem 3.5. *Let G be an n-vertex polyhedral graph and let G^* have a perfect matching. Then*

$$\mathrm{rb}(G) = \frac{3n}{4}\,.$$

Proof. If G^* has a perfect matching then $\alpha_1^* = \frac{|F(G)|}{2}$ and, by (3), $\alpha_1^* = \frac{n+4}{4}$. This together with the lower bound of Theorem 1.2 and Theorem 3.4 yields our equality. □

4 Quality of the bounds

Consider a d-sided prism D_d. It is a $2d$-vertex cubic polyhedral map which is in fact a cartesian product $P_2 \times C_d$ of a path P_2 on two vertices and a cycle C_d on d-vertices. It is easy to see that $\alpha_0^*(D_d) = \lfloor \frac{d}{2} \rfloor$ and $\alpha_1^*(D_d) = \lceil \frac{d+1}{2} \rceil$. In [JS] there is proved that $\mathrm{rb}(D_d) = \lceil \frac{3d-1}{2} \rceil$ for $d \geq 3$. Because for the prism D_d there is $\frac{n}{2} + \alpha_1^* - 1 = d + \lceil \frac{d+1}{2} \rceil - 1 = \lceil \frac{3d-1}{2} \rceil$ the lower bound in Theorem 1.2 is tight.

Let the dual graph G^* of an n-vertex cubic polyhedral graph has an almost perfect matching, i.e. let $\alpha_1^*(G) = \frac{|F(G)|-1}{2}$. In this case $n = 4k + 2$ for some $k \geq 1$. Then $\alpha_1(G^*) = \frac{n+2}{4} = k + 1$. By the lower bound in Theorem 2.1 we have

$$\mathrm{rb}(G) \geq 2k + 1 + k + 1 - 1 = 3k + 1\,.$$

On the other hand, by Theorem 3.3, there is

$$\mathrm{rb}(G) \leq \frac{3}{4}n = \left\lfloor \frac{3}{4}(4k+2) \right\rfloor = 3k + \frac{6}{4}\,.$$

Hence $\mathrm{rb}(G) = 3k + 1$. So we have proved that the rainbowness of such graphs equals to the lower bound in Theorem 2.1.

We believe that the following is true.

Conjecture 4.1. *For every n-vertex cubic polyhedral graph G there is*

$$\mathrm{rb}(G) = \frac{n}{2} + \alpha_1^*(G) - 1\,.$$

We do not know any example of a cubic polyhedral graph G with $\mathrm{rb}(G) = n - \alpha_0^*(G) + 1$. For the d-sided prism with d even there is $\mathrm{rb}(D_d) = n - \alpha_0^*(G) = 2d - \frac{d}{2} = \frac{3d}{2}$. This means that the upper bound of Theorem 3.3 is sharp. We believe that the following is true.

Conjecture 4.2. *Let G be an n-vertex polyhedral graph. Then*

$$\mathrm{rb}(G) \leq \frac{3n}{4}\,.$$

Acknowledgement.

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104. Support of Slovak VEGA Grant 1/3004/06 is acknowledged as well.

References

[BM] J.A. Bondy and U.S.R. Murty, *Graph Theory with application*, North-Holland, New York (1976).

[G] H. Grötzsch, *Dreifarbensatz für dreikreisfreie Netze of der Kugel*, Wiss. Z. Martin-Luther-U., Halle-Wittenberg, Math.-Nath. Reihe **8** (1959), 109-120.

[JS] S. Jendrol' and Š. Schrötter, *On rainbowness of semiregular polyhedra*, (submitted).

[JKS] V. Jungić, D. Kráľ and R. Škrekovski, *Coloring of plane graphs with no rainbow faces*, Combinatorica (to appear).

[K] D. Kráľ, *On maximum face-constrained coloring of plane graphs with no short face cycles*, Discrete Math. **277** (2004), 301-307.

[N] S. Negami, *Looseness and independence number of maximal planar graphs*, Talk to the Japan Workshop on Graph Theory and Combinatorics 2005, Keio University, Yokohama, June 20-24 (2005), Japan.

[RW] R. Ramamurthi and D.B. West, *Maximum face-constrained coloring of plane graphs*, Discrete Math. **274** (2004), 233-240.

[T] C. Thomassen, *Grötzsch's 3-color Theorem*, J. Combin. Theorey B **62** (1994), 268-279.