A problem on optimal transportation

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Abstract

Mathematical optimization problems are not typical in the classical curriculum of mathematics. In this paper we show how several generalisations of an easy problem on optimal transportation were solved by gifted pupils and in a correspondence mathematical seminar, how they can be used in university courses of linear programming and what the remaining open questions are.
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1 Introduction

Some years ago I was asked to prepare a series of problems for STROM, a correspondence competition in mathematical problems solving for gifted secondary school pupils, organized in East Slovakia (see in Slovak www.strom.sk). And since my specialization is Mathematical optimization and Operations research, I tried to prepare something from this area.

Optimization problems are not typical for school mathematics. Often, rigorous optimization methods are not suitable for teaching in secondary schools, as they are too involved and require deep mathematical background. In my opinion, teaching pupils to perform an algorithm without its understanding does not improve their mathematical abilities, just the opposite: contributes to deepening their formal attitude to mathematics. Nevertheless, optimization problems, if correctly posed for the targeted group of pupils, can stimulate their active approach to problem solving, invention and creativity, and informal search for appropriate methods from their pool of knowledge. And indeed, the subsequent development of the ideas, originaly centered around a funny problem for secondary school pupils, later served as a nice motivation example for linear programming courses at our faculty and still provides several open questions for experts in transport modelling and optimization. A brief account on the approach to the problem by participants of STROM is described in Slovak in [1], but we feel that the problem, the solution approaches for its various variants, hints to their possible use for different groups of pupils and open questions its leads to deserve a more detailed description accessible to a broader international audience.

The whole story started with an interesting paper [5] in a Sovjet journal Kvant. The paper dealt with optimization and the main characters of one of the problems were two brothers, let us call them Andrew and Bob.

2 Two people, one bicycle

Andrew and Bob have one bicycle and want to get home from their grandmother's house, that is in a village 30 km far away. Andrew walks 6 km/hour and rides 12 km/hour; the speeds of Bob are 4 km/hour and 24 km/hour, respectively. The bike can be left at the roadside without supervision. How should the boys organize their journey if they want to get home as quickly as possible, meaning that the last one should be at home as soon as possible?

The problem of Andrew and Bob admits a nice graphical solution. First of all, one has to realize that the total transportation time depends neither on the number of times boys switch walking with riding, nor on the order of the boys in their use of the bike, it is only derived from the total distance covered on foot and on the bike. Therefore it is possible to represent the organisation of the transport by a kind of schematics, like in Figure 1.

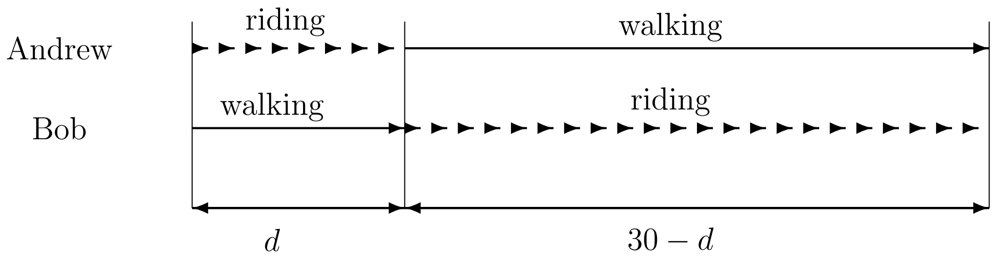

Figure 1: Transport organisation for two people and one bike

Here, variable d denotes the distance which Andrew rides and Bob walkes, so $30 - d$ is the distance walked by Andrew and ridden by Bob. Then, one can express

$$\begin{aligned} \text{Andrew's time: } t_A &= \frac{d}{12} + \frac{30-d}{6} = 5 - \frac{d}{12} \\ \text{Bob's time: } t_B &= \frac{30-d}{24} + \frac{d}{4} = \frac{5}{4} + \frac{5}{24}d \end{aligned}$$

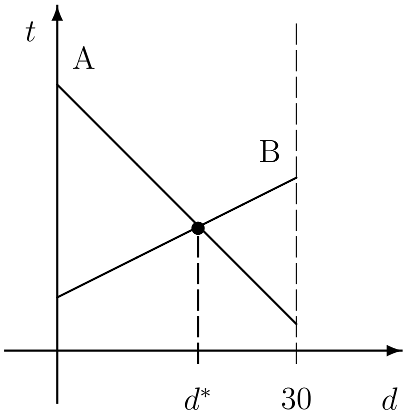

Figure 2: Two time functions in one graph

The transportation times are linear functions of variable d on the interval $\langle 0, 30\rangle$. One can draw the graphs of both functions into the same graph (Figure 2) with the horizontal axis corresponding to d and the vertical one to time t. The time-function of Andrew is labelled by A, that of Bob by B. It is easy to see that the time of the last arrival corresponds to the upper envelope of the two functions and its minimal possible value is achieved for that value d^*, that corresponds to the intersection of the two graphs.

There is an interesting and useful observation concerning this problem: in an optimal transport organization both boys must arrive simultaneously. One can argue for example in the following way: suppose that Andrew is the one who arrives later than Bob, see Figure 3 left. This means that the distance d_1 he was riding was too short, and Bob's was too long. Therefore in the next trial Bob will leave the bike for Andrew for a little longer distance, say d_2, and the time of the last arrival will be shorter. If, even when riding d_2, Andrew still arives later than Bob, as in Figure 3 middle, his time can further be improved when in the following run he will have the bike for an even longer distance. If it happens, that riding say d_3, Andrew arrives earlier than Bob (see Figure 3 right), next time Andrew will cut his riding distance shorter and so on. This is what people usually do in real life. After several practical 'iterations', the boys may converge to a transport organisation, in which they both arrive simultaneously, but this 'experimental' solution of the problem takes a lot of time and sweat. Instead, we can get the optimal transport organization in a few minutes, by solving the equation $t_A = t_B$. We get $d^* = \frac{90}{7}$, which is approximately 13 km by bike for Andrew, giving the arrival time $\frac{165}{42} \approx 4$ hours. (For simplicity, fractions of hours are not converted to minutes in this paper.)

The pictures and iterative arguments were quite suggestive that we really need simultaneous arrival for optimality. Later we shall return to the question, whether simultaneous arrival is always possible and whether each transport organization with simultaneous arrivals is optimal.

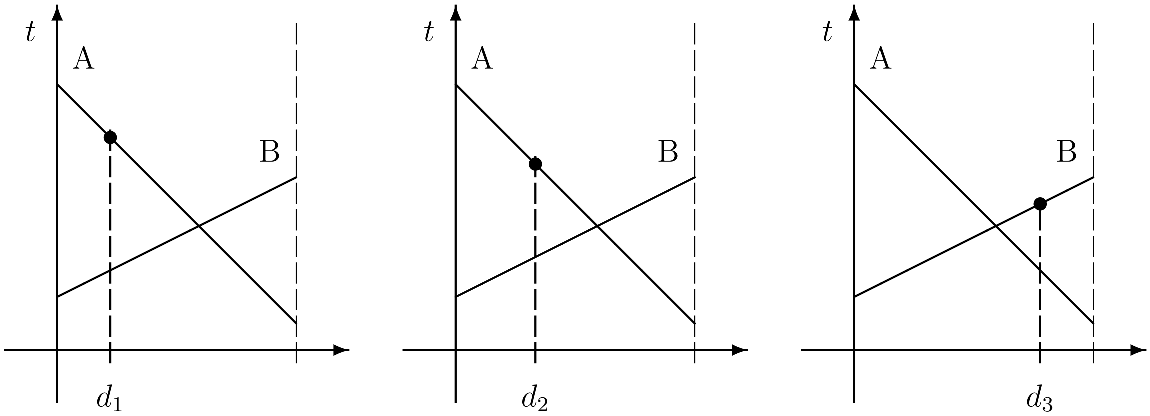

Figure 3: Iterating common arrivals

3 Three people, one bicycle

The problem of two brothers and its solution, although not very typical for the traditional school mathematics, is quite accessible for pupils at the age below 15 years. Except of a very general ability to formulate a simple mathematical model of a real situation and some kind of abstract thinking, the only necessary formal mathematical knowledge are working with fractions, solving linear equations and the basic idea of a linear function.

As participants of STROM are in the age of 15-18 years, moreover, mathematically gifted pupils, the version of the transportation problem with two people and one bicycle seemed to be too trivial. Therefore I added a third person and now the story read like this:

Father, Mother and Son have one bicycle and want to travel the distance of 30 km to their home. It is not allowed to ride the bike by more than one person simultaneously, but the bike can be left at the roadside without supervision. How to organize the transport so as to get home as soon as possible?

The speeds of the members of the family are given in Figure 4.

Speed (km per hour)	walking W	riding R
Father F	10	40
Mother M	6	12
Son S	4	28

Figure 4: Speeds of Father, Mother and Son

STROM participants were instructed to use the necessary condition for the transport optimality – simultaneous arrival of all three people. We expected them to look for the optimal transport organized according to the schematics depicted in Figure 5. (Without loss of generality, it can be supposed that the bike is used by the members of the family in the order Father, Mother, Son.)

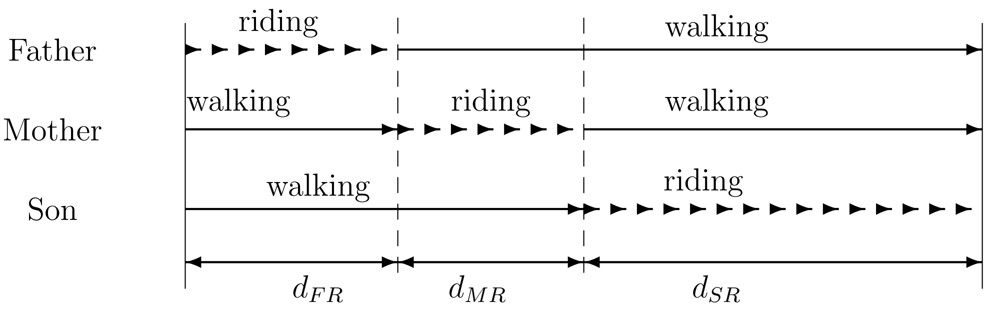

Figure 5: Transport organization for three people and one bicycle.

Let us denote by variable t the common arrival time, which is for each person equal to the sum of the times of walking and riding. These times will be denoted again by letter t, equipped with double subscripts, the first letter indicating the person and the second one the means of transport (W – walk, R – ride). Then, since each person has to travel the distance of 30 km, we have the following equations:

$$40t_{FR} + 10(t - t_{FR}) = 30 \qquad \text{Father} \tag{1}$$
$$12t_{MR} + 6(t - t_{MR}) = 30 \qquad \text{Mother} \tag{2}$$
$$28t_{SR} + 4(t - t_{SR}) = 30 \qquad \text{Son} \tag{3}$$

As we have 4 variables and only three equations, one more equation is needed. Let us add one for the bicycle:

$$t_{FR} + t_{MR} + t_{SR} = t \tag{4}$$

Now, with four equations for four unknowns, the system (1–4) has a unique solution given in Figure 6,

t_{FR}	t_{MR}	t_{SR}
2 min	126 min	46 min

Figure 6: The first solution.

leading to the common arrival time of 174 minutes.

Most STROM participants who obtained this solution, were quite happy with it. However, after a deeper thought one has to realize the following: if Father is the first one to use the bike and if he rides for just 2 minutes, he will travel the distance of 1.33 km. However, Mother can walk in two minutes only 0.2 km. (We arrive at a similar discrepancy if somebody else will be the first one to take the bike.) So this solution cannot be realized in practice, as the time needed by Mother to get home will be greater that the sum of her riding and walking times.

Really, equation (4) for the bike was misleading: the total time of each person really equals the time the person walks plus the time the person rides, if he/she does not have to wait. But to avoid waiting of people during the transportation, the bicycle will necessarily spend some time waiting at the roadside. This idle time, however, is difficult to estimate.

Other group of STROM participants used as unknowns the distances (with the same interpretation of the double subscripts). Their equations expressed that everybody, including the bike, has to travel the distance of 30 km:

$$\begin{aligned} d_{FW} + d_{FR} &= 30 \qquad \text{Father} && (5)\\ d_{MW} + d_{MR} &= 30 \qquad \text{Mother} && (6)\\ d_{SW} + d_{SR} &= 30 \qquad \text{Son} && (7)\\ d_{FR} + d_{MR} + d_{SR} &= 30 \qquad \text{Bicycle} && (8) \end{aligned}$$

Simultaneous arrival of all three people is expressed by two more equations:

$$\begin{aligned} \frac{d_{FW}}{10} + \frac{d_{FR}}{40} &= \frac{d_{MW}}{6} + \frac{d_{MR}}{12} && (9)\\ \frac{d_{FW}}{10} + \frac{d_{FR}}{40} &= \frac{d_{SW}}{4} + \frac{d_{SR}}{28} && (10) \end{aligned}$$

Now, although the model is already correct and system (5–10) has a unique solution, it is negative! So where was the mistake now?

Some participants of STROM have observed that Father can get home in 3 hours without riding, but Mother and Son cannot manage within this time even with the bike! Really, if Mother wants to get home in at most 3 hours, we get the inequality

$$\frac{d_{MR}}{12} + \frac{30 - d_{MR}}{6} \leq 3$$

which implies $d_{MR} \geq 24$. This means that Son has to walk at least 24 km, and for that he needs more than 6 hours!

The previous mistake was in an incorrect use of the intuitive argument asking the fastest person to ride the bike a little less to 'lend' it to the slowest person. In this argument we implicitly assumed that the fastest person rides a nonzero distance, which, obviously, was not the case with our example.

Therefore it seemed necesary to abandon the requirement of simultaneous arrival of all three persons. If we just want that Mother and Son arrive simultaneously, we get the system of equations:

$$
\begin{aligned}
d_{MW} + d_{MR} &= 30 && \text{Mother} && (11)\\
d_{SW} + d_{SR} &= 30 && \text{Son} && (12)\\
d_{MR} + d_{SR} &= 30 && \text{Bicycle} && (13)\\
\frac{d_{MW}}{6} + \frac{d_{MR}}{12} &= \frac{d_{SW}}{4} + \frac{d_{SR}}{28} && \text{Simultaneous arrival} && (14)
\end{aligned}
$$

with the unique solution given in Figure 7.

$d_{MW} = d_{SR}$	$d_{MR} = d_{SW}$
16,8 km	13,2 km

Figure 7: The second solution.

and the common arrival time of Mother and Son equal to 3,9 hours.

Is this really the fastest possible transport organization? No! Two STROM participants made an unexpected breakthrough: Somebody must ride in the opposite direction to ensure the fastest arrival of the family. Since Father is the fastest one riding as well as walking, he will be the one, who in fact brings the bike closer to his Son, so as he has not to walk too far. A new organization of the transport is depicted in Figure 8 and here we have the new system of equations:

$$
\begin{aligned}
d_{FW} - d_{FR} &= 30 && \text{Father} && (15)\\
d_{MW} + d_{MR} &= 30 && \text{Mother} && (16)\\
d_{SW} + d_{SR} &= 30 && \text{Son} && (17)\\
-d_{FR} + d_{MR} + d_{SR} &= 30 && \text{Bicycle} && (18)\\
\frac{d_{FW}}{10} + \frac{d_{FR}}{40} &= \frac{d_{MW}}{6} + \frac{d_{MR}}{12} && && (19)\\
\frac{d_{FW}}{10} + \frac{d_{FR}}{40} &= \frac{d_{SW}}{4} + \frac{d_{SR}}{28} && && (20)
\end{aligned}
$$

Now the unique (and optimal) solution is given in Figure 9, giving the best possible arrival time of 3,61 hours.[1]

Let us mention here, that the obtained solution is unique as far as the distances walked and riden are concerned. However, this does not mean that the optimal

[1]My, at that time 5 years old son, was present during a lecture I was giving on this transportation problem at a meeting of the Mathematics Teachers Club. Later he was asked whether he had understood what his mother had been talking about. Without any hesitation he replied: ‘Yes. If a family wants to get home before dark, father has to ride the bike very quickly.’

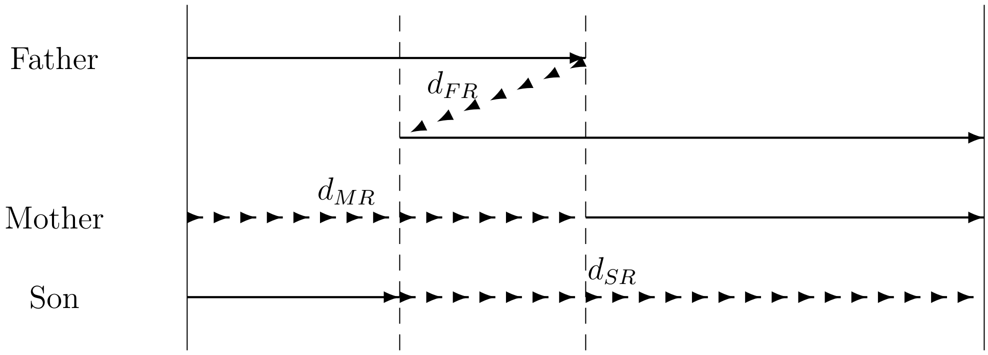

Figure 8: New transport organisation

Solution (km)	walking	riding
Father	34,86	4,86 (in opposite direction)
Mother	13,3	16,7
Son	11,84	18,16

Figure 9: Optimal transport organisation

transportation can be organized in one way only. The reader is invited to label the individual segments of Figure 8 by the computed distances and also to draw a schematics with corresponding distances of an optimal transport organization in which Son will be the first person riding the bicycle.

4 Three people, two vehicles

When I was talking about the transportation problem of Father, Mother and Son in a seminar, somebody in the audience asked: can you generalize this problem to n people and m vehicles? This, at the first glance funny question of a crazy mathematician who wants to generalize everything and everywhere, is in fact of a great practical importance. Similar important problems arise in various logistic processes: imagine you have an army and it is necessary to move it from point A to point B as quickly as possible with a restricted pool of vehicles, or you have to organize an evacuation of an area, etc. An interested reader might have a look at the formulation of a transportation problem of a similar flavour and deep mathematical results used in its solution in [4].

So we had a look at the transportation problem with three people, one bicycle and one motorcycle. The speeds are recorded in Figure 10.

In the system of equations (21–28) the first five equations express that Father, Mother, Son, the bike and the motorcycle have to travel the whole distance of 30 km. The last three equations are used just to denote the common arrival time

Velocity (km/hour)	walking W	riding R	driving D
Father F	10	40	80
Mother M	6	12	60
Son S	4	28	100

Figure 10: Speeds of Father, Mother and Son for two vehicles

by a new unknown t.

$$d_{FW} + d_{FR} + d_{FD} = 30 \tag{21}$$
$$d_{MW} + d_{MR} + d_{MD} = 30 \tag{22}$$
$$d_{SW} + d_{SR} + d_{SD} = 30 \tag{23}$$
$$d_{FR} + d_{MR} + d_{SR} = 30 \tag{24}$$
$$d_{FD} + d_{MD} + d_{SD} = 30 \tag{25}$$
$$\frac{d_{FW}}{10} + \frac{d_{FR}}{40} + \frac{d_{FD}}{80} = t \qquad \text{Father's time} \tag{26}$$
$$\frac{d_{MW}}{6} + \frac{d_{MR}}{12} + \frac{d_{MD}}{60} = t \qquad \text{Mother's time} \tag{27}$$
$$\frac{d_{SW}}{4} + \frac{d_{SR}}{28} + \frac{d_{SD}}{100} = t \qquad \text{Son's time} \tag{28}$$

Now, this system has 10 variables and only 8 equations and it is impossible to add another one that will not be too restrictive. Hence, if system (21–28) is solvable, basic linear algebra implies that it will necessarily have infinitely many solutions. Those correspond to various transport organizations ensuring simultaneous arrival of all three members of the family. And of them we want to get one such solution, that minimizes the arrival time. So we add to the system of equations (21–28) the so-called objective function

$$t \rightarrow min \tag{29}$$

and we get, after adding nonnegativity constraints for each variable, a standard linear program. The knowledge required for its solving is already beyond the material covered in pre-university education, but the transportation problems of this kind could provide an interesting subject for operations research courses in various faculties.

We have solved the above linear program by our educational software for linear programming called CASSIM (Computer ASsisted SImplex Method) [2]. With real practical problems, which were not primarily designed for exercises in simples method, it is often the case that manual computations are almost impossible. The first feasible solution (i.e. one in which all three people arrive simultaneously) has been obtained only after 19 iterations, and Figure 11 summarizes the subsequent iterations, leading to the final optimum with the total time equal to 1,89 hour.

d_{FW}	d_{FR}	d_{FD}	d_{MW}	d_{MR}	d_{MD}	d_{SW}	d_{SR}	d_{SD}	t
21,64	2,92	5,44	0	27,08	2,92	8,36	0	21,64	2,31
19,34	10,66	0	2,74	19,34	7,92	7,92	0	22,08	2,20
15,51	14,49	0	9,42	0	20,58	5,07	15,51	9,42	1,91
16,84	3,85	9,31	9,31	0	20,69	3,85	26,15	0	1,89

Figure 11: The sequence of feasible solutions of the linear program (21–29)

Again, one should realize that unique travel times still may lead to different real transport organizations and the reader is advised to represent them by a chart. Let us remark here, that in this case, drawing a chart is more than a simple illustration of the transport organization. Just the opposite, it serves as a verification of the possibility to really implement the obtained solution. Let us look more closely at the optimal solution given in Figure 11 and suppose that Father will first drive, then ride and finally walk. This means, that for the first segment of the journey of the length of 9,31 km (driving) he will need 0,12 hours and immediately he has to start riding. However, according to the obtained solution, this first segment will be ridden by Son, who needs for 9,31 km on bike 0,33 hours. Hence, necessarily, Fathers journey will contain some idle time and he will not be able to get home in the computed time.

5 Conclusion and further questions

We already know, that the fastest transportation schedule might require some person to travel in the opposited direction, but we have not included this option into the solution of our last problem. We do not know a simple method, enabling us to guess in advance, when travel speeds are arbitrary, who this person might be and which vehicle he or she would use. However, it is quite a simple exercise to formulate a linear program (using a construction with variables t^+ and t^-) for n people and m vehicles (each vehicle can simultaneously carry only one person) if speeds v_{ij} for each person i and each vehicle j are given that counts with the possibility that anybody can travel in the opposite direction by any vehicle. However, we have no idea about a possible formulation of a similar problem in which each vehicle can carry more than one, but not more than c_j persons.

I will be happy to receive any comments on the open questions resulting from these transportation problems as well as on experience with teaching optimization problems.

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