FROM BINOMIAL TO BLACK-SCHOLES MODEL USING THE LIAPUNOV VERSION OF CENTRAL LIMIT THEOREM

Valéria Skřivánková* and Marek Kočan**

Faculty of Science, Šafárik University, Košice

In this paper a proof of the binomial formula for option pricing is presented using the method of mathematical induction. Weak convergence of this discrete model to the Black-Scholes model will be guaranteed by the Liapunov version of central limit theorem.

Key Words: option pricing, binomial model, continuous model, central limit theorem

1. Introduction

The inspiration for this work was provided by Cox, Ross and Rubinstein (1979). In their paper "Option Pricing: A Simplified Approach" the binomial option pricing model was derived by a recursive procedure and a hypothesis of a convergence of this discrete model to the well known continuous Black-Scholes model was presented.

In this paper we first formulate some conditions under which the binomial formula for the prices of European call options will be proved using the method of mathematical induction. Further we show, that using the Liapunov version of central limit theorem, weak convergence of discrete model to continuous model can be guaranteed.

European call option on stock is a financial derivative, which gives the owner the right (but not the obligation) to buy one unit of stock at the specified price *K* (the exercise price) at time *T* (expiration date). The main problem is to price this option on stock *n* periods before expiration date. For this purpose we consider the following **assumptions**:

1) the stock price follows a binomial process over discrete periods;
2) the stock pays no dividends;
3) the riskless interest rate is constant through time;
4) it is possible to borrow and lend any amount by the same interest rate;
5) it is possible to buy, borrow and lend any fraction of a security;
6) there are no taxes, transaction costs, margin requirements;
7) a short sale is possible;
8) an arbitrage is not possible;
9) options are of European type.

More precisely the first assumption means, that if the present stock price is *S*, then the stock price at the end of the first period will be *uS* with probability *q* and *dS* with probability *1-q*, where $0 < d < 1 < u$ and we put $d = 1/u$.

Under assumptions 1 to 9 for the price of a stock option *n* periods before the expiration by Cox *et al.* (1979) the following formula holds:

$$C(n) = \frac{1}{r^n}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}p^j(1-p)^{n-j}\max\left\{0, u^j d^{n-j} S - K\right\}, \tag{1}$$

where $p = (r-d)/(u-d)$, $r = 1 + r_f$, r_f is riskless interest rate.

* Department of Mathematics, Faculty of Science, Šafárik University, Košice, Slovakia, skrivan@kosice.upjs.sk

** Department of Mathematics, Faculty of Science, Šafárik University, Košice, Slovakia, kocan@science.upjs.sk

Black and Scholes (1973) derived the continuous option pricing formula under similar assumptions, but instead of 1) they assumed

10) the stock price S_t follows a random walk process in continuous time and it is lognormally distributed with expected return μ and variance σ^2.

Under assumptions 2 to 10 for the option price at time *t*, $t \in (0,T)$ by the following formula holds:

$$C(t,S_t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2), \tag{2}$$

where $d_1 = \dfrac{\ln\dfrac{S_t}{K} + \left(r + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = \dfrac{\ln\dfrac{S_t}{K} + \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, and $\Phi(x)$ is standard normal distribution function.

2. Binomial model

Now we prove the binomial formula (1) by the method of mathematical induction.

Theorem 1: Let the assumptions 1 to 9 in the introduction section hold. Then the value of a call option on stock *n* periods before the expiration is given by (1).

Proof: We will prove the formula (1) using mathematical induction with regard to *n* in two steps.

Step 1. We prove (1) for $n=1$.

Let $C(1)=C$ be the value of a call option at the beginning of a period. Let C_u be the value of a call option at the end of a period, if the stock price is *uS* and C_d, if the stock price is *dS*. So $C_u = \max\{0, uS-K\}$ with probability *q* and $C_d = \max\{0, dS-K\}$ with probability *1-q*.

Cox *et al.*(1979) used for finding the price of an option a portfolio constructed from stocks and a risk free asset $\Delta S + B$, where Δ is the number of stocks and *B* is risk free asset. At the end of the period the value of the portfolio is $uS\Delta + rB$ with probability *q*, or $dS\Delta + rB$ with probability *1-q*. The values for Δ and *B* are chosen so, that at the end of the period the value of the portfolio will be equal to the value of a call option, e.g.

$$uS\Delta + rB = C_u, \qquad dS\Delta + rB = C_d.$$

Solving these two equations we have

$$\Delta = \frac{C_u - C_d}{(u-d)S}, \qquad B = \frac{dC_u - uC_d}{r(d-u)}. \tag{3}$$

By assumption 8 arbitrage is not possible. So the value of the portfolio at the beginning of the period with Δ and *B* from (3) has to be equal to the value of a call option at the beginning:

$$C(1) = S\Delta + B = \frac{C_u - C_d}{u-d} + \frac{uC_d - dC_u}{(u-d)r} = \frac{1}{r}\left(\frac{r-d}{u-d}C_u + \frac{u-r}{u-d}C_d\right). \tag{4}$$

Substituting $p = (r-d)/(u-d)$ and $1-p = (u-r)/(u-d)$, we can write (4) as:

$$C(1) = \frac{1}{r}\left(pC_u + (1-p)C_d\right) = \frac{1}{r}\left(p\max\{0,uS-K\} + (1-p)\max\{0,dS-K\}\right),$$

what is (1) for $n=1$.

Step 2. Let the induction assumption holds for $n=k$, e.g.

$$C(k)=\frac{1}{r^k}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}p^j(1-p)^{k-j}\max\left\{0,u^jd^{k-j}S-K\right\}. \qquad (5)$$

We will prove that (5) implies (1) for $n=k+1$.

Let $n=k+1$. We are $k+1$ periods before the expiration of a call option. If the value of the stock after $k+1$ periods is $u^jd^{k+1-j}S$, similarly the value of a call option after $k+1$ periods is

$$C_{u^jd^{k+1-j}}=\max\left\{0,u^jd^{k+1-j}S-K\right\} \quad \forall j=0,...,k+1.$$

The value of a call option, when we are in k-th period and we have one period to the expiration is

$$C_{u^jd^{k-j}}(1)=\frac{1}{r}\left(pC_{u^{j+1}d^{k-j}}+(1-p)C_{u^jd^{k-j+1}}\right) \quad \forall j=0,...,k. \qquad (6)$$

By the induction assumption (5) and after substituting formula (6) we have

$$C(k+1)=\frac{1}{r^{k+1}}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}p^j(1-p)^{k-j}\left[pC_{u^{j+1}d^{k-j}}+(1-p)C_{u^jd^{k-j+1}}\right]=$$

$$=\frac{1}{r^{k+1}}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}p^{j+1}(1-p)^{k-j}C_{u^{j+1}d^{k-j}}+\frac{1}{r^{k+1}}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}p^j(1-p)^{k+1-j}C_{u^jd^{k-j+1}}$$

If we now substitute in the first sum $m=j+1$, we have

$$C(k+1)=\frac{1}{r^{k+1}}\sum_{m=1}^{k+1}\frac{k!}{(m-1)!(k-m+1)!}p^m(1-p)^{k-m+1}C_{u^md^{k-m+1}}+$$

$$+\frac{1}{r^{k+1}}\left(\sum_{j=1}^{k}\frac{k!}{j!(k-j)!}p^j(1-p)^{k+1-j}C_{u^jd^{k-j+1}}+(1-p)^{k+1}C_{u^0d^{k+1}}\right)=$$

$$=\frac{1}{r^{k+1}}p^{k+1}C_{u^{k+1}d^0}+\frac{1}{r^{k+1}}\sum_{m=1}^{k}\frac{k!}{(m-1)!(k-m+1)!}p^m(1-p)^{k-m+1}C_{u^md^{k-m+1}}+$$

$$+\frac{1}{r^{k+1}}\sum_{j=1}^{k}\frac{k!}{j!(k-j)!}p^j(1-p)^{k+1-j}C_{u^jd^{k-j+1}}+\frac{1}{r^{k+1}}(1-p)^{k+1}C_{u^0d^{k+1}}.$$

Using the combinatorial formula $\binom{k}{j}+\binom{k}{j-1}=\binom{k+1}{j}$, where $\binom{k}{j}=\frac{k!}{j!(k-j)!}$ we get

$$C(k+1)=\frac{1}{r^{k+1}}\frac{(k+1)!}{(k+1)!(k+1-(k+1))!}p^{k+1}(1-p)^0C_{u^{k+1}d^0}+\frac{1}{r^{k+1}}\sum_{j=1}^{k}\binom{k+1}{j}p^j(1-p)^{k+1-j}C_{u^jd^{k-j+1}}+$$

$$+\frac{1}{r^{k+1}}\frac{(k+1)!}{0!(k+1-0)!}p^0(1-p)^{k+1}C_{u^0d^{k+1}}=\frac{1}{r^{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}p^j(1-p)^{k+1-j}C_{u^jd^{k+1-j}},$$

what is (1) for $n=k+1$.

□

To prove the convergence from (1) to (2) we need modify formula (1). Let a be the smallest non-negative integer satisfying inequality $u^ad^{n-a}S>K$, e.g. $a>\ln\frac{K}{d^nS}\Big/\ln\frac{u}{d}$. Then we can rewrite formula (1) to:

$$C(n)=\frac{1}{r^n}\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}\left(u^jd^{n-j}S-K\right)=$$

$$= S\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}\frac{u^j d^{n-j}}{r^n}-\frac{K}{r^n}\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}.$$

The latter sum represents the complementary binomial distribution function with parameters *(n,p)* in a

$$\bar{B}^{n,p}(a)=\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}=1-\sum_{j=0}^{a-1}\binom{n}{j}p^j(1-p)^{n-j}=1-B^{n,p}(a).$$

The first sum can be also expressed in similar way if we substitute $p^*=\frac{u}{r}p$, where $p=(r-d)/(u-d)$.

So we can write formula (1) as

$$C(n)=S\bar{B}^{n,p^*}(a)-Kr^{-n}\bar{B}^{n,p}(a),\text{ for } a\le n, \tag{7}$$

$$C(n)=0 \text{ for } a>n.$$

Variable p here is the value, the probability q would have, when the investor was risk neutral. Since in formula (7) we have not variables that say something about investor's preferences, we can assume, that investor is risk neutral. We will use this assumption further in this paper.

2. Limiting model

By deriving the binomial option pricing formula (1) there is no assumption what one period is. It could be a minute, second, year. So by increasing the number of periods (decreasing the length of the periods) we can converge to some continuous model. Cox *et al.* (1979) showed that it will be the Black-Scholes model (2). In this paper we will use Liapunov version of central limit theorem (see Neuts, 1973) to prove this hypothesis precisely.

Theorem 2: (Liapunov central limit theorem)

Let $\{X_k\}_{k=1}^{\infty}$ be the sequence of independent random variables with finite means $E(X_k)$ and variances $D(X_k)$, let $S_n=\sum_{k=1}^{n}X_k$, $\hat{S}_n=(S_n-E(S_n))\big/\sqrt{D(S_n)}$ and moreover

$$\lim_{n\to\infty}\frac{1}{\left(\sum_{k=1}^{n}D(X_k)\right)^{3/2}}\sum_{k=1}^{n}E\left(\left|X_k-E(X_k)\right|^3\right)=0. \tag{8}$$

Then

$$\lim_{n\to\infty}P\left(\hat{S}_n<x\right)=\Phi(x),\qquad x\in R.$$

Theorem 3: Let the assumptions 1 to 9 hold. Then

$$\lim_{n\to\infty}\bar{B}^{n,p^*}(a)=\Phi(x), \tag{9}$$

$$\lim_{n\to\infty}\bar{B}^{n,p}(a)=\Phi\left(x-\sigma\sqrt{t}\right), \tag{10}$$

where $x=\dfrac{\ln\dfrac{S}{Kr^{-t}}}{\sigma\sqrt{t}}+\dfrac{1}{2}\sigma\sqrt{t}$.

Proof:

Let us have n periods before expiration and direction u occurs j times. Then the stock price after n periods is $S^* = u^j d^{n-j} S$ and using logarithm we have

$$\ln\frac{S^*}{S} = j\ln\frac{u}{d} + n\ln d = j\ln u + (n-j)\ln d\,, \tag{11}$$

When we put $X_i = a_i \ln u + (1-a_i)\ln d$, $i = 1,\dots,n$, where $a_i \sim Bi(1,p)$, a_i are independent random variables, we have from (11)

$$\ln\frac{S^*}{S} = \sum_{i=1}^{n} X_i\,. \tag{12}$$

Now we have to show that the conditions of central limit theorem are fulfilled, e.g. $E(X_i)$, $D(X_i)$ are finite and the Liapunov condition (8) holds.

We know, that $E(a_i) = p,\quad E(a_i^2) = p$ and $D(a_i) = p(1-p)\quad \forall i = 1,\dots,n$, so

$$E(X_i) = E(a_i\ln u + \ln d - a_i\ln d) = p\ln\frac{u}{d} + \ln d = \hat{\mu}_p < \infty\,. \tag{13}$$

$$D(X_i) = E(X_i^2) - (E(X_i))^2 = E\left(a_i^2(\ln u)^2 + 2a_i(1-a_i)\ln u\ln d + (1-a_i)^2(\ln d)^2\right) - (E(X_i))^2 =$$

$$= p\left((\ln u)^2 - (\ln d)^2\right) + (\ln d)^2 - \left(p(\ln u - \ln d) + \ln d\right)^2 =$$

$$= p\left((\ln u)^2 - (\ln d)^2\right) + (\ln d)^2 - p^2(\ln u - \ln d)^2 - 2p(\ln u - \ln d)\ln d - (\ln d)^2 =$$

$$= p\left((\ln u)^2 - (\ln d)^2 - p(\ln u)^2 + 2p\ln u\ln d - p(\ln d)^2 - 2\ln u\ln d + 2(\ln d)^2\right) =$$

$$= p(1-p)\left((\ln u)^2 - 2\ln u\ln d + (\ln d)^2\right) = p(1-p)\left(\ln\frac{u}{d}\right)^2 = \hat{\sigma}_p^2 < \infty\,. \tag{14}$$

We already know, that mean and variance of X_i are finite and now we have to show that Liapunov condition (8) holds. First we calculate the absolute central moment of third order. We know

$$X_i - E(X_i) = a_i\ln u + (1-a_i)\ln d - p\ln\frac{u}{d} - \ln d = (a_i - p)\ln u - (a_i - p)\ln d = (a_i - p)\ln\frac{u}{d}\,,$$

$$\left|X_i - E(X_i)\right| = \left|(a_i - p)\ln\frac{u}{d}\right| = |a_i - p|\ln\frac{u}{d}\,,$$

where $E|a_i - p|^3 = E|a_i - E(a_i)|^3 = p^3(1-p) + (1-p)^3 p = p(1-p)(1-2p(1-p))$. So

$$E\left|X_i - E(X_i)\right|^3 = \left|\ln\frac{u}{d}\right|^3 E|a_i - p|^3 = \left(\ln\frac{u}{d}\right)^3 p(1-p)(1-2p(1-p)).$$

From this and using (14):

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} E\left|X_i - E(X_i)\right|^3}{\left(\sum_{i=1}^{n} D(X_i)\right)^{3/2}} = \lim_{n\to\infty}\frac{n\left(\ln\frac{u}{d}\right)^3\left(p(1-p)^3 + p^3(1-p)\right)}{n\sqrt{n}\hat{\sigma}_p^3} =$$

$$= \lim_{n\to\infty}\frac{\left(\ln\frac{u}{d}\right)^3\left(p(1-p)^3 + p^3(1-p)\right)}{\sqrt{np^3(1-p)^3}\left(\ln\frac{u}{d}\right)^3} = \lim_{n\to\infty}\frac{(1-p)^2 + p^2}{\sqrt{np(1-p)}} = *\,.$$

Now consider fixed length of time to expiration *t* divided into *n* trading periods. Clearly, the interest rate $\hat{r}$ depends on the length of periods, $\hat{r} = r^{t/n}$. By Cox *et al.* (1979) also the variables *p, u, d* will depend on t/n as following:

$$u = e^{\sigma\sqrt{\frac{t}{n}}},\ d = e^{-\sigma\sqrt{\frac{t}{n}}},\ p = p(n) = \frac{1}{2} + \frac{1}{2}\frac{\ln r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{t/n}\,.$$

So $\lim_{n\to\infty} p(n) = 1/2$, and the Liapunov condition is fulfilled

$$* = \lim_{n\to\infty}\frac{\left(1-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}{\sqrt{n\frac{1}{2}\left(1-\frac{1}{2}\right)}} = 0\cdot$$

Now we can apply central limit theorem. Let us remind, that $\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j} = \overline{B}^{n,p}(a)$, what is the probability, that $j \ge a$. That is equivalent to $1-\overline{B}^{n,p}(a) = P(j \le a-1)$. This we can extend to

$$1-\overline{B}^{n,p}(a) = P\left(\frac{j-np}{\sqrt{np(1-p)}} \le \frac{a-1-np}{\sqrt{np(1-p)}}\right). \qquad (15)$$

When we express variable *j* from (11), we have $j = \left(\ln\frac{S^*}{S} - n\ln d\right)\Big/\ln\frac{u}{d}$. From (12), (13), (14) we get $E\left(\ln\frac{S^*}{S}\right) = E\left(\sum_{i=1}^{n}X_i\right) = n\hat{\mu}_p$, $D\left(\ln\frac{S^*}{S}\right) = D\left(\sum_{i=1}^{n}X_i\right) = n\hat{\sigma}_p^2$.

Substituting *j* into (15) and modifying we have

$$1-\overline{B}^{n,p}(a) = P\left(\frac{\ln\frac{S^*}{S} - \hat{\mu}_p n}{\hat{\sigma}_p\sqrt{n}} \le \frac{a-1-np}{\sqrt{np(1-p)}}\right).$$

We also know, that *a* is upper round part of the expression $\ln\frac{K}{d^n S}\Big/\ln\frac{u}{d}$, and so

$$a-1 = \frac{\ln\frac{K}{d^n S}}{\ln\frac{u}{d}} - \varepsilon = \frac{\ln\frac{K}{S} - n\ln d}{\ln\frac{u}{d}} - \varepsilon\text{, where } \varepsilon \in \langle 0,1\rangle.$$

Substituting into previous expression we have

$$1-\overline{B}^{n,p}(a) = P\left(\frac{\ln\frac{S^*}{S} - \hat{\mu}_p n}{\hat{\sigma}_p\sqrt{n}} \le \frac{\ln\frac{K}{S} - n\hat{\mu}_p - \varepsilon\ln\frac{u}{d}}{\hat{\sigma}_p\sqrt{n}}\right). \qquad (16)$$

Now we have to calculate limits of expressions $n\hat{\mu}_p$, $\hat{\sigma}_p\sqrt{n}$ and $\varepsilon\ln\frac{u}{d}$.

$$\lim_{n\to\infty}\ln\frac{u}{d} = \lim_{n\to\infty}2\sigma\sqrt{\frac{t}{n}} = 0 \Rightarrow \lim_{n\to\infty}\varepsilon\ln\frac{u}{d} = 0\,.$$

Because $p(n) \to 1/2$ for $n \to \infty$ we have:

$$\lim_{n\to\infty}\hat{\sigma}_p\sqrt{n}=\lim_{n\to\infty}\sqrt{np(1-p)}\ln\frac{u}{d}=\lim_{n\to\infty}\sqrt{np(1-p)}2\sigma\sqrt{\frac{t}{n}}=$$
$$=\lim_{n\to\infty}2\sigma\sqrt{tp(1-p)}=2\sigma\sqrt{t\frac{1}{2}\frac{1}{2}}=\sigma\sqrt{t}\,.$$

For $n\hat{\mu}_p$ we have

$$\lim_{n\to\infty}n\hat{\mu}_p=\lim_{n\to\infty}n\left(p\ln\frac{u}{d}+\ln d\right)=\lim_{n\to\infty}n\left(2p\sigma\sqrt{\frac{t}{n}}-\sigma\sqrt{\frac{t}{n}}\right)=\lim_{n\to\infty}2\sigma\sqrt{tn}\left(p-\frac{1}{2}\right)=\Delta\,,$$

and this gives us uncertain expression $\infty.0$ for $n\to\infty$. But using twice L`Hospital theorem for the function $f(x)=2\sigma\sqrt{tx}\left(p(x)-\frac{1}{2}\right)$ we get

$$\Delta\sim\lim_{x\to\infty}4t\frac{\frac{1}{2}r^{\frac{t}{x}}\ln r+\frac{t}{x}(\ln r)^2r^{\frac{t}{x}}-\frac{\sigma^2}{4}r^{\frac{t}{x}}}{e^{\sigma\sqrt{\frac{t}{x}}}+e^{-\sigma\sqrt{\frac{t}{x}}}}=4t\frac{1}{2}\left(\frac{1}{2}\ln r-\frac{\sigma^2}{4}\right)=t\left(\ln r-\frac{\sigma^2}{2}\right).$$

Summarizing:

$$n\hat{\mu}_p\to t\left(\ln r-\frac{\sigma^2}{2}\right)\text{ for }n\to\infty,\ \hat{\sigma}_p\sqrt{n}\to\sigma\sqrt{t}\text{ for }n\to\infty,\ \varepsilon\ln\frac{u}{d}\to 0\text{ for }n\to\infty.$$

So we have

$$\frac{\ln\frac{K}{S}-n\hat{\mu}_p-\varepsilon\ln\frac{u}{d}}{\hat{\sigma}_p\sqrt{n}}\to\frac{\ln\frac{K}{S}-\left(\ln r-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}=\frac{\ln\frac{K}{Sr^t}}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}=Z\text{ for }n\to\infty.$$

Using the central limit theorem and from (16) we have:

$$\lim_{n\to\infty}\left[1-\bar{B}^{n,p}(a)\right]=1-\lim_{n\to\infty}\bar{B}^{n,p}(a)=\Phi(Z),$$

and that equals to $\lim_{n\to\infty}\bar{B}^{n,p}(a)=1-\Phi(Z)=\Phi(-Z)$. Substituting Z we have:

$$\lim_{n\to\infty}\bar{B}^{n,p}(a)=\Phi\left(-\frac{\ln\frac{K}{Sr^t}}{\sigma\sqrt{t}}-\frac{1}{2}\sigma\sqrt{t}\right)=\Phi\left(\frac{\ln\frac{Sr^t}{K}}{\sigma\sqrt{t}}-\frac{1}{2}\sigma\sqrt{t}\right)=\Phi\left(\frac{\ln\frac{Sr^t}{K}}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}-\sigma\sqrt{t}\right).$$

When we denote $x=\frac{\ln\frac{S}{Kr^{-t}}}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}$, then the previous expression can be rewritten to $\lim_{n\to\infty}\bar{B}^{n,p}(a)=\Phi\left(x-\sigma\sqrt{t}\right)$, what is (10). The prove of equation (9) is similar.

□

Calculating the limit for the binomial option pricing model in (7), we get

$$\lim_{n\to\infty}C(n)=\lim_{n\to\infty}\left(S\bar{B}^{n,p^*}(a)-K\hat{r}^{-n}\bar{B}^{n,p}(a)\right)=S\Phi(x)-Kr^{-t}\Phi\left(x-\sigma\sqrt{t}\right),$$

where $x=\frac{\ln\frac{S}{Kr^{-t}}}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}$. That is the continuous Black-Scholes valuation formula similar to formula (2), with the difference, that in (2) they compounded continuously.

Using real data from NYSE we can illustrate the speed of convergence of binomial model to Black-Scholes model. For this we use the stock of Wallmart. On the February 2nd 2002 the stock price was 62,1 USD. The price of european option on that stock with expiration in march and exercise price 55 USD was 7,1 USD. The price, Black-Scholes model gives us, is 7,14544095 USD and using binomial model with three periods to expiration we have 7,14253654 USD, with eight periods 7,14441930 USD and with thirty periods 7,14525244 USD what is now the same in three decimal positions as the price calculated using Black-Scholes model.

References

Black, F. and Scholes, M. (1973) The pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 3, 637-654.

Cox, J. C., Ross, S. A. and Rubinstein, M. (1979) Option pricing: A simplified approach, *Journal of Financial Economics*, 7, 229-268.

Neuts, M. (1972) *Probability*, Allyn and Bacon, Inc. Boston.