On arbitrarily vertex decomposable trees

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Abstract

A tree T is arbitrarily vertex decomposable if for any sequence τ of positive integers adding up to the order of T there is a sequence of vertex-disjoint subtrees of T whose orders are given by τ; from a result by Barth and Fournier it follows that $\Delta(T) \leq 4$. A necessary and a sufficient condition for being an arbitrarily vertex decomposable star-like tree have been exhibited. The conditions seem to be very close to each other.

1 Introduction

In this paper we deal with finite simple graphs only. Let G be a graph. For $V \subseteq V(G)$ we denote by $G\langle V\rangle$ the subgraph of G induced by V and by $G - V$ the graph $G\langle V(G) - V\rangle$. Further, for $E \subseteq E(G)$ we denote by $\langle E\rangle$ the subgraph of G induced by E, i.e., the union of all graphs K_2 corresponding to the edges of E (in fact, for the definition of $\langle E\rangle$ the structure of G is not important). A *graph property* is a set of (isomorphic types of) graphs. A graph property $\mathcal{P}$ is *hereditary* (*induced hereditary*) if $G \in \mathcal{P}$ implies $H \in \mathcal{P}$ for any subgraph (induced subgraph, respectively) H of G.

For $p, q \in \mathbb{Z}$ let $[p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : p \leq z\}$. If $m, n \in [0, \infty)$, $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$, we denote by AB the *concatenation* of the sequences A and B, i.e., the sequence $(a_1, \dots, a_m, b_1, \dots, b_n)$. Clearly, the concatenation of sequences is associative and this fact justifies the use of the notation $\prod_{i=1}^{k} A_i$ for the concatenation of sequences $A_1, \dots, A_k$ (in this order), $k \in [0, \infty)$. As usual, if $A_i = A$ for any $i \in [1, k]$, $\prod_{i=1}^{k} A_i$ is replaced by A^k; A^0 is the empty sequence ( ). If τ is a finite sequence of positive integers and $i \in [1, \infty)$, we use $f^i(\tau)$ to denote the number of terms of τ equal to i.

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Consider a graph G and a graph property $\mathcal{P}$. Let $\mathrm{Ei}(G,\mathcal{P})$ be the set of all positive integers e such that there is $E \subseteq E(G)$ with $|E| = e$ and $\langle E\rangle \in \mathcal{P}$. Let $\mathrm{Es}(G,\mathcal{P})$ be the set of all sequences whose terms belong to $\mathrm{Ei}(G,\mathcal{P})$ and add up to $|E(G)|$. A sequence $\varepsilon = (e_1,\dots,e_k) \in \mathrm{Es}(G,\mathcal{P})$ is $(G,\mathcal{P})$-*edge-realisable* if there is a $(G,\mathcal{P})$-*edge-realisation* of ε, i.e., a sequence $(G_1,\dots,G_k)$ of subgraphs of G such that $\{E(G_i) : i \in [1,k]\}$ is a decomposition of $E(G)$, $G_i \in \mathcal{P}$ and $|E(G_i)| = e_i$ for any $i \in [1,k]$. The graph G is *arbitrarily edge decomposable* with respect to $\mathcal{P}$ ($\mathcal{P}$-aed for short) if every sequence from $\mathrm{Es}(G,\mathcal{P})$ is $(G,\mathcal{P})$-edge-realisable. Note that if $\mathcal{P}$ is a hereditary property and $G \in \mathcal{P}$, then G is trivially $\mathcal{P}$-aed.

As an example consider the property $\mathcal{E}$ "to be Eulerian", i.e., to contain a closed Eulerian trail. If $n \in [3,\infty)$, $n \equiv 1 \,(\mathrm{mod}\, 2)$, it is easy to see that $\mathrm{Ei}(K_n,\mathcal{E}) = [3, \frac{n(n-1)}{2} - 3] \cup \{\frac{n(n-1)}{2}\}$. The well-known decomposition of K_5 into two C_5's shows that the sequence $(5,5) \in \mathrm{Es}(K_5,\mathcal{E})$ is $(K_5,\mathcal{E})$-edge- realisable.

There are some classes of graphs that are known to be $\mathcal{E}$-aed, namely complete graphs K_n with $n \equiv 1 \,(\mathrm{mod}\, 2)$, graphs $K_n - M_n$, where $n \equiv 0 \,(\mathrm{mod}\, 2)$ and M_n is a perfect matching in K_n (Balister [1]), complete bipartite graphs $K_{m,n}$ with $m,n \equiv 0 \,(\mathrm{mod}\, 2)$ (Horňák and Woźniak [9]), complete tripartite graphs $K_{n,n,n}$, where $n = 5 \cdot 2^l$ with $l \in [0,\infty)$ (Horňák and Kocková [7]). Moreover, in [7] it is shown that if $K_{p,q,r}$ with $p \le q \le r$ is $\mathcal{E}$-aed, then $(p,q,r) \in \{(1,1,3),(1,1,5)\}$ or $p = q = r$. Balister [2] proved that there are positive constants n and ε such that any *even* graph (having vertices of even degrees only) G, satisfying $|V(G)| \ge n$ and $\delta(G) \ge (1-\varepsilon)|V(G)|$, is $\mathcal{E}$-aed.

There is a natural analogy of the above notions in which edges are replaced by vertices. Thus, $\mathrm{Vi}(G,\mathcal{P})$ is the set of all positive integers v such that there is $V \subseteq V(G)$ with $|V| = v$ and $G\langle V\rangle \in \mathcal{P}$. Further, $\mathrm{Vs}(G,\mathcal{P})$ is the set of all sequences whose terms belong to $\mathrm{Vi}(G,\mathcal{P})$ and add up to $|V(G)|$. A sequence $\upsilon = (v_1,\dots,v_k) \in \mathrm{Vs}(G,\mathcal{P})$ is $(G,\mathcal{P})$-*vertex-realisable* if there is a $(G,\mathcal{P})$-*vertex-realisation* of υ, i.e., a sequence $(G_1,\dots,G_k)$ of induced subgraphs of G such that $\{V(G_i) : i \in [1,k]\}$ is a decomposition of $V(G)$, $G_i \in \mathcal{P}$ and $|V(G_i)| = v_i$ for any $i \in [1,k]$. The graph G is *arbitrarily vertex decomposable* with respect to $\mathcal{P}$ ($\mathcal{P}$-avd for short) if every sequence from $\mathrm{Vs}(G,\mathcal{P})$ is $(G,\mathcal{P})$-vertex-realisable. It should also be noted that if $\mathcal{P}$ is an induced hereditary property and $G \in \mathcal{P}$, then G is trivially $\mathcal{P}$-avd.

In the present paper we study trees that are $\mathcal{T}$-avd, where $\mathcal{T}$ is the property "to be a tree". Deleting a pendant vertex from a tree yields again a tree. Therefore, if T is a tree of order $t \ge 1$, then $\mathrm{Vi}(T,\mathcal{T}) = [1,t]$ and $\mathrm{Vs}(T,\mathcal{T}) = \bigcup_{k=1}^{t}\{(t_1,\dots,t_k) \in [1,t]^k : \sum_{i=1}^{k} t_i = t\}$. To simplify the notation we shall write avd, $\mathrm{Vs}(T)$, a T-realisable sequence and a T-realisation instead of $\mathcal{T}$-avd, $\mathrm{Vs}(T,\mathcal{T})$, a $(T,\mathcal{T})$-vertex-realisable sequence and a $(T,\mathcal{T})$-vertex-realisation, respectively.

A sequence $\tau = (t_1,\dots,t_k) \in \mathrm{Vs}(T)$ is *changeable* to a sequence $\tilde{\tau} = (\tilde{t}_1,\dots,$

$\tilde{t}_k) \in \mathrm{Vs}(T)$, in symbols $\tau \sim \tau'$, if there is a permutation π of the set $[1, k]$ such that $\tilde{t}_i = t_{\pi(i)}$ for any $i \in [1, k]$. In such a case, if $(T_1, \dots, T_k)$ is a T-realisation of the sequence τ, then $(T_{\pi(1)}, \dots, T_{\pi(k)})$ is a T-realisation of the sequence $\tilde{\tau}$. Therefore, we have the following evident statement:

Proposition 1 *If T is a tree, $\tau, \tilde{\tau} \in \mathrm{Vs}(T)$ and $\tau \sim \tilde{\tau}$, then τ is T-realisable if and only if $\tilde{\tau}$ is.* ■

Let T be a tree. A vertex $x \in V(T)$ is said to be *primary* if $\deg_T(x) \geq 3$, otherwise it is *secondary.* A subtree $\tilde{T}$ of T is an *end* of T if there is $n \in [1, \infty)$ such that $\tilde{T} \cong P_n$ (P_n denotes an n-vertex path) and, if y, z are endvertices of $\tilde{T}$, then $\min(\deg_T(y), \deg_T(z)) = 1$ and $\deg_T(w) = 2$ for any $w \in V(\tilde{T}) - (\{y\} \cup \{z\})$. In the partial ordering of subtrees of T determined by the binary relation "to be a subgraph", ends of T are grouped into disjoint chains; a maximal element of such a chain is called an *arm* of T. An end of T is *proper* if it is not an arm. If $T \cong P_n$, $n \in [1, \infty)$, T itself is the unique arm of T. Further, if $\Delta(T) \geq 3$, exactly one endvertex of an arm of T is primary in T.

It turned out that the class of star-like trees is crucial when analysing the property of a tree "to be avd". A *star-like tree* is a tree homeomorphic to a star $K_{1,q}$. If $q \geq 3$, such a tree has one primary vertex x and q arms A_i, $i = 1, \dots, q$, with endvertices x and y_i; let x_i be the neighbour of x in A_i and let a_i be the order of A_i (if $a_i = 2$, then $x_i = y_i$). The structure of a star-like tree is (up to isomorphism) determined by the non-decreasing sequence $(a_1, \dots, a_q)$ of orders of its arms. Let $\mathcal{A}$ be the set of all non-decreasing sequences with terms from $[2, \infty)$ that are finite and of length at least three. We denote the above defined star-like tree by $S(\alpha)$, where $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$. When speaking about a star-like tree $S(a_1, \dots, a_q)$, we use the presented notation without explicitly mentioning it and we denote by v the order of that tree, i.e., the number $1 + \sum_{i=1}^{q}(a_i - 1)$. The notation $S(a_1, \dots, a_q)$ can also be used for $q \in [1, 2]$; in such a case $S(a_1) \cong P_{a_1}$ and $S(a_1, a_2) \cong P_{a_1+a_2-1}$.

The maximum degree $\Delta(T)$ of an avd tree T cannot be arbitrarily large. Namely, we have proved in [10] that it is at most 6 and conjectured that that upper bound can even be lowered to 4. Rosenberg et al. in [12] have "halfway" succeeded by bounding $\Delta(T)$ from above by 5. The conjecture has been confirmed by Barth and Fournier in [4]:

Theorem 2 *If T is an avd tree, then $\Delta(T) \leq 4$. Moreover, if $\alpha = (a_1, a_2, a_3, a_4) \in \mathcal{A}$ and the star-like tree $S(\alpha)$ is avd, then $a_1 = 2$.* ■

There is also an on-line version of the problem of deciding whether a tree is avd, see Horňák et al. [8]. In that case it was (maybe a bit surprisingly) possible to solve the problem completely.

Let T be a tree and $\boldsymbol{T} = (T_1, \dots, T_k)$ a T-realisation of a sequence $\tau = (t_1, \dots, t_k) \in \mathrm{Vs}(T)$. If $w \in V(T)$, the *w-tree* of $\boldsymbol{T}$ is the unique tree of $\boldsymbol{T}$

containing w. Provided that T is a star-like tree, the x-tree of $\boldsymbol{T}$ is also called the *primary* tree of $\boldsymbol{T}$. A set $W \subseteq V(T)$ is said to be $\boldsymbol{T}$*-exact* if there is a subsequence of $\boldsymbol{T}$ that is a $T\langle W\rangle$-realisation of a subsequence of τ. In other words, W is $\boldsymbol{T}$-exact if there is $I \subseteq [1,k]$ such that $W = \bigcup_{i\in I} V(T_i)$.

A vertex of a path P_n, $n \in [5,\infty)$, is said to be *strongly internal* if it is neither an endvertex of P_n nor a neighbour of an endvertex of P_n. A subtree $\tilde{T}$ of a tree T is said to be *important* if there is an odd n such that $\tilde{T} \cong P_n$, endvertices of $\tilde{T}$ are pendant vertices of T and strongly internal vertices of $\tilde{T}$ are of degree 2 in T.

2 Star-like trees

Proposition 3 *If $n \in [0,\infty)$, then P_n is avd.*

Proof. Suppose that $V(P_n) = [1,n]$ and $E(P_n) = \{\{i, i+1\} : i \in [1, n-1]\}$. For a sequence $\tau = (t_1,\dots,t_k) \in \mathrm{Vs}(P_n)$ and $j \in [0,k]$ define $\sigma_j := \sum_{i=1}^{j} t_i$. If, for $j \in [1,k]$, T_j is a subpath of P_n with $V(T_j) = [\sigma_{j-1}+1, \sigma_j]$, then evidently $(T_1,\dots,T_k)$ is a P_n-realisation of τ. ■

Lemma 4 *Let $q \in [3,\infty)$, $\alpha = (a_1,\dots,a_q) \in \mathcal{A}$, $T = S(\alpha)$ and let $\tau = (t_1,\dots,t_k) \in \mathrm{Vs}(T)$. If there are $s \in [q-1,q]$, $I \subseteq [1,k]$ and $p \in [1,k] - I$ such that $\sum_{i\in I} t_i \le a_s - 1$ and $\sum_{i\in I} t_i + t_p \ge 1 + \sum_{i=1}^{q-2}(a_i - 1) + (a_s - 1)$, then τ is T-realisable.*

Proof. Suppose that $I = \{i_j : j \in [1,m]\}$. Consider the subtree P of A_s of order $n := \sum_{i\in I} t_i$ satisfying $n \ge 1 \Rightarrow y_s \in V(P)$ (isomorphic to P_n), a P-realisation $(T_{i_1},\dots,T_{i_m})$ of the sequence $\tilde{\tau} := (t_{i_1},\dots,t_{i_m})$ (see Proposition 3) and the unique subtree T_p of T of order t_p containing all vertices of $(\bigcup_{i=1}^{q-2} V(A_i) \cup V(A_s)) - V(P)$ and $t_p - [1 + \sum_{i=1}^{q-2}(a_i - 1) + (a_s - 1) - \sum_{i\in I} t_i]$ vertices of the remaining arm of T. The rest of T is an end of T of order $v - \sum_{i\in I} t_i - t_p$, hence due to Proposition 3 we can easily find remaining trees of a T-realisation $(T_1,\dots,T_k)$ of the sequence τ. ■

Lemma 5 *Let P be a proper end of a tree T such that the tree $T - V(P)$ is avd. If $\tau = (t_1,\dots,t_k) \in \mathrm{Vs}(T)$ and there is $I \subseteq [1,k]$ such that $\sum_{i\in I} t_i = |V(P)|$, then τ is T-realisable.*

Proof. Suppose that $I = \{i_l : l \in [1,m]\}$ and pick a P-realisation $(T_{i_1}, \dots, T_{i_m})$ of $\tilde{\tau} := (t_{i_1},\dots,t_{i_m})$ (Lemma 3). Let $\hat{T} := T - V(P)$ and let $\hat{\tau} = (t_{j_1},\dots,t_{j_n}) \in \mathrm{Vs}(\hat{T})$ be the sequence created by deleting from τ all t_i's with $i \in I$. If $(T_{j_1},\dots,T_{j_n})$ is a $\hat{T}$-realisation of $\hat{\tau}$, then $(T_{i_1},\dots,T_{i_m},T_{j_1},\dots,T_{j_n})$ is a T-realisation of $\tilde{\tau}\hat{\tau} \sim \tau$, and so τ is T-realisable by Proposition 1. ■

For $k \in [1,\infty)$, $a_1 \in [3,\infty)$ and $a_2 \in [a_1,\infty)$ let the kth *obstacle* (for the pair (a_1,a_2)) be defined by $O_k(a_1,a_2) := [ka_2, k(a_1+a_2-2)]$, the kth *hole* by

$H_k(a_1,a_2) := [k(a_1+a_2-2)+1, (k+1)a_2-1]$ and the kth *parameter* by $p_k(a_1,a_2) := (k+1)a_2 - k(a_1+a_2-2) - 1 = a_2 - k(a_1-2) - 1$.

Let $\prec$ be the binary relation defined on the set of all nonempty subsets of $\mathbb{R}$ by $A \prec B \overset{\text{df.}}{\Leftrightarrow} (\forall a \in A\ \forall b \in B\ a < b)$. As an immediate consequence of the above definitions we obtain:

Proposition 6 *If $k, l \in [1,\infty)$, $a_1 \in [3,\infty)$ and $a_2 \in [a_1,\infty)$, then the following hold:*

1. *If $O_k(a_1,a_2) \prec O_{k+1}(a_1,a_2)$ and $H_k(a_1,a_2) \neq \emptyset$, then $O_k(a_1,a_2) \prec H_k(a_1,a_2) \prec O_{k+1}(a_1,a_2)$ and $\{O_k(a_1,a_2), H_k(a_1,a_2), O_{k+1}(a_1,a_2)\}$ is a decomposition of $[ka_2, (k+1)(a_1+a_2-2)]$.*
2. *$H_k(a_1,a_2) = \emptyset$ if and only if $p_k(a_1,a_2) \leq 0$.*
3. *If $H_k(a_1,a_2) \neq \emptyset$, then $|H_k(a_1,a_2)| = p_k(a_1,a_2)$.* ■

Lemma 7 *If $\alpha = (a_1,a_2,a_3) \in \mathcal{A}$, $a_1 \geq 3$ and $S(\alpha)$ is avd, then there is $k \in [2, \lfloor \frac{a_2-2}{a_1-2} \rfloor]$ such that $|V(S(\alpha))| \in H_k(a_1,a_2)$.*

Proof. Suppose there is $l \in [1,\infty)$ such that v belongs to $O_l(a_1,a_2)$. Then, clearly, there is a sequence $\tau = (t_1,\dots,t_l) \in [a_2, a_1+a_2-2]^l$ such that $\sum_{i=1}^{l} t_i = v$, and, consequently, there exists an $S(\alpha)$-realisation $\boldsymbol{T} = (T_1,\dots,T_l)$ of τ. Let T_j be the y_2-tree of $\boldsymbol{T}$. Since $|V(T_j)| = t_j \in [a_2, a_1+a_2-2]$, T_j is also the primary tree of $\boldsymbol{T}$; on the other hand, T_j contains at most a_1-2 secondary vertices of the arm A_1 (and certainly not y_1). Therefore, the y_1-tree of $\boldsymbol{T}$ is of order at most $a_1-1 \leq a_2-1$, a contradiction.

As $v = a_1+a_2+a_3-2 > 2a_2 \in O_2(a_1,a_2)$ and v belongs to no obstacle, we have $O_2(a_1,a_2) \prec \{v\}$. Let k be the maximum of the (finite) set $\{l \in [2,\infty) : O_l(a_1,a_2) \prec \{v\}\}$. Then $O_k(a_1,a_2) \prec \{v\} \prec O_{k+1}(a_1,a_2)$ and, by Proposition 6.1, 3, $v \in H_k(a_1,a_2)$ and $p_k(a_1,a_2) \geq 1$. Consider the decreasing sequence $\{a_2 - l(a_1-2) - 1\}_{l=1}^{\infty}$ of parameters and $m \in [2,\infty)$ with $p_m(a_1,a_2) \geq 1$ and $p_{m+1}(a_1,a_2) < 1$. The inequality $p_l(a_1,a_2) = a_2 - l(a_1-2) - 1 \geq 1$ is equivalent to $l \leq \frac{a_2-2}{a_1-2}$, and so $k \leq m = \lfloor \frac{a_2-2}{a_1-2} \rfloor$. ■

Theorem 8 *If $\alpha = (a_1,a_2,a_3) \in \mathcal{A}$ and $S(\alpha)$ is avd, then*

1. *$a_2 \geq 2a_1 - 2$;*
2. *$a_3 \geq a_1 + a_2 - 1$;*
3. *$a_1 + a_2 + a_3 - 2 = |V(S(\alpha))| \leq (\lfloor \frac{a_2-2}{a_1-2} \rfloor + 1)a_2 - 1$.*

Proof. Put $m := \lfloor \frac{a_2-2}{a_1-2} \rfloor$. By Lemma 7 there is $k \in [2,m]$ such that $v \in H_k(a_1,a_2)$. By Proposition 6.3 then $|H_k(a_1,a_2)| = a_2 - k(a_1-2) - 1 \geq 1$, $a_2 - 2(a_1-2) - 1 \geq a_2 - k(a_1-2) - 1 \geq 1$ and the first statement of our Theorem follows. Also, $v \in H_k(a_1,a_2)$ yields $2(a_1+a_2-2)+1 \leq k(a_1+a_2-1)+1 \leq v = a_1+a_2+a_3-2 \leq (k+1)a_2 - 1 \leq (m+1)a_2 - 1$, which, having in mind that $m \leq \lfloor \frac{a_2-2}{3-2} \rfloor = a_2 - 2$, implies the remaining two assertions. ■

Define $\mathcal{B}_i := \{(i)^{\lambda_0}(i+1)^{\lambda_1} : \lambda_0 \in [0,\infty), \lambda_1 \in [1,\infty)\}$ for $i \in [1,\infty)$ and $\bar{\mathcal{B}}_i := \{(m)(i)^{\lambda_0}(i+1)^{\lambda_1} : m \in [1, i-1], \lambda_0 \in [0,\infty), \lambda_1 \in [1,\infty)\}$ for $i \in [2,\infty)$. Further, with $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$, put $\mathcal{B}_i(\alpha) := \mathcal{B}_i \cap \mathrm{Vs}(S(\alpha))$ and $\bar{\mathcal{B}}_i(\alpha) := \bar{\mathcal{B}}_i \cap \mathrm{Vs}(S(\alpha))$. It turned out that deciding whether a star-like tree is avd only sequences belonging to $\mathcal{B}_i$ and $\bar{\mathcal{B}}_i$ are important.

Theorem 9 (see [3]) *If $\alpha = (a_1, a_2, a_3) \in \mathcal{A}$, then the following statements are equivalent:*

(1) *$S(\alpha)$ is avd.*

(2) *Any sequence belonging to $\mathcal{B}_i(\alpha)$ with $i \in [1, a_1 + a_2 - 2]$ or $\bar{\mathcal{B}}_i(\alpha)$ with $i \in [2, a_1 - 3]$ is $S(\alpha)$-realisable.* ∎

Theorem 10 (see [4]) *If $\alpha = (2, a_2, a_3, a_4) \in \mathcal{A}$, then the following statements are equivalent:*

(1) *$S(\alpha)$ is avd.*

(2) *$S(a_2, a_3, a_4)$ is avd and any sequence belonging to $\mathcal{B}_i(\alpha)$ with $i \in [1, a_2 + a_3 - 2]$ or $\bar{\mathcal{B}}_i(\alpha)$ with $i \in [2, a_2 - 3]$ is $S(\alpha)$-realisable.* ∎

Theorems 9 and 10 lead to algorithms able to decide whether a star-like tree with v vertices is avd in a polynomial time in v, in the case of star-like trees with three arms in a time at most $O(v^7)$. Let us mention also the following simple, but useful assertion of [3]:

Lemma 11 *If $q \in [3,\infty)$, $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$ and a sequence $\tau = (t_1, \dots, t_k) \in \mathrm{Vs}(S(\alpha))$ is $S(\alpha)$-realisable, there is an $S(\alpha)$-realisation $(T_1, \dots, T_k)$ of τ such that its primary tree is of order* $\max(t_i : i \in [1,k])$. ∎

For $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$, $i \in [1,\infty)$ and $j \in [1,q]$ let $r_j(i,\alpha) \in [0, i-1]$ be such that $a_j - 1 \equiv r_j(i,\alpha) \pmod{i}$. Further, let $r(i,\alpha) \in [1,i]$ be such that $v \equiv r(i,\alpha) \pmod{i}$. It is easy to see that $a_j - 1 = \rho_j(i,\alpha)i + r_j(i,\alpha)$, where $\rho_j(i,\alpha) := \lfloor \frac{a_j - 1}{i} \rfloor$ for $j \in [1,q]$, and $v = \rho(i,\alpha)i + r(i,\alpha)$, where $\rho(i,\alpha) := \lceil \frac{v}{i} \rceil - 1$. Clearly, $\{\rho_j(i,\alpha)\}_{i=1}^{\infty}$ is a non-increasing sequence for any $j \in [1,q]$.

Theorem 12 *Suppose that $q \in [3,4]$, $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$, $S(\alpha)$ is avd and $i \in [1, a_{q-2} + a_{q-1} - 2]$. Then the following hold:*

1. *There exists a unique $\gamma(i,\alpha) \in [0,1]$ such that $\sum_{j=1}^{q} r_j(i,\alpha) = r(i,\alpha) - 1 + \gamma(i,\alpha)i$.*
2. *If $\gamma(i,\alpha) = 1$, there is $l \in [1,q]$ such that $r_l(i,\alpha) \geq r(i,\alpha)$.*
3. *If $\gamma(i,\alpha) = 0$ and $f^{i+1}(\tau) \leq i$ for some $\tau \in \mathcal{B}_i(\alpha)$, there is $l \in [1,q]$ such that $\rho_l(i+1,\alpha) \geq r_l(i,\alpha)$.*
4. *If $\gamma(i,\alpha) = 1$ and $f^{i+1}(\tau) \leq i$ for some $\tau \in \mathcal{B}_i(\alpha)$, then* $\sum_{j=1}^{q} \min(\rho_j(i+1,\alpha), r_j(i,\alpha)) \geq r(i,\alpha) - 1$.
5. *If $\gamma(i+1,\alpha) = 1$ and $f^{i}(\tau) \leq i$ for some $\tau \in \mathcal{B}_i(\alpha)$, there is $l \in [1,q]$ such that $r_l(i+1,\alpha) \geq r(i+1,\alpha)$ and $\rho_l(i,\alpha) + r_l(i+1,\alpha) \geq i+1$.*

Proof. 1, 2. We have $i \le 1 + \sum_{j=1}^{q}(a_j - 1) - a_q \le v - 2$, and so $s := \rho(i,\alpha) + 1 = \lceil \frac{v}{i} \rceil \ge \lceil \frac{v}{v-2} \rceil = 2$. By Lemma 11 there is an $S(\alpha)$-realisation $(T_1, \dots, T_s)$ of the sequence $(r(i,\alpha))(i)^{s-1} \in \mathrm{Vs}(S(\alpha))$ whose primary tree is of order i (we may suppose without loss of generality that it is T_s). Put $t_{s,j} := |V(T_s) \cap (V(A_j) - \{x\})|$ for $j \in [1,q]$. As $s \ge 2$, there is $l \in [1,q]$ such that $V(T_1) \subseteq V(A_l) - \{x\}$, hence $r(i,\alpha) + t_{s,l} \equiv r_l(i,\alpha) \pmod{i}$, $t_{s,j} \equiv r_j(i,\alpha) \pmod{i}$ and, consequently, $t_{s,j} = r_j(i,\alpha)$ for any $j \in [1,q] - \{l\}$.

If $r_l(i,\alpha) \ge r(i,\alpha)$, then from $r_l(i,\alpha) \le i - 1$ it follows that $t_{s,l} = r_l(i,\alpha) - r(i,\alpha)$, $i = t_s = 1 + r_l(i,\alpha) - r(i,\alpha) + \sum_{j \in [1,q]-\{l\}} r_j(i,\alpha)$, $\sum_{j=1}^{q} r_j(i,\alpha) = r(i,\alpha) - 1 + i$ and $\gamma(i,\alpha) = 1$.

On the other hand, $r_l(i,\alpha) < r(i,\alpha)$ implies $t_{s,l} + r(i,\alpha) = i + r_l(i,\alpha)$ (as $t_{s,l} + r(i,\alpha) \le 2i - 1$), $\sum_{j=1}^{q} r_j(i,\alpha) = r(i,\alpha) - 1$ and $\gamma(i,\alpha) = 0$. Since in this case $r_j(i,\alpha) \le r(i,\alpha) - 1$ for any $j \in [1,q]$, the assertions 1 and 2 of our Theorem follow.

For the cases 3–5 we use the fact that, again by Lemma 11, there is an $S(\alpha)$-realisation $\boldsymbol{T} = (T_1, \dots, T_k)$ of the sequence τ such that the primary tree of $\boldsymbol{T}$ is of order $i + 1$ (we may suppose without loss of generality that it is T_k). Put $t_{k,j} := |V(T_k) \cap (V(A_j) - \{x\})|$ and let f_j^s denote the number of trees of $\boldsymbol{T}$ of order s that are subtrees of A_j for $j \in [1,q]$ and $s \in [i, i+1]$.

If $f^{i+1}(\tau) \le i$ (the cases 3 and 4), from $v = f^i(\tau)i + f^{i+1}(\tau)(i+1) \equiv r(i,\alpha) \pmod{i}$ it follows that $f^{i+1}(\tau) \equiv r(i,\alpha) \pmod{i}$. As $f^{i+1}(\tau), r(i,\alpha) \in [1,i]$, we have $f^{i+1}(\tau) = r(i,\alpha)$. Because of the congruences $t_{k,j} + f_j^i i + f_j^{i+1}(i+1) = a_j - 1 \equiv r_j(i,\alpha) \pmod{i}$ and $t_{k,j} + f_j^{i+1} \equiv r_j(i,\alpha) \pmod{i}$ then (having in mind that $t_{k,j} + f_j^{i+1} \in [0, 2i-1]$: observe that $t_{k,j} = i$ implies $f_j^{i+1} \le i - 1$) there is $\lambda_j \in [0,1]$ satisfying $t_{k,j} + f_j^{i+1} = r_j(i,\alpha) + \lambda_j i$ for $j \in [1,q]$. Therefore, by Theorem 12.1, there is $\gamma(i,\alpha) \in [0,1]$ such that $r(i,\alpha) - 1 + \gamma(i,\alpha)i = \sum_{j=1}^{q} r_j(i,\alpha) = \sum_{j=1}^{q} t_{k,j} + \sum_{j=1}^{q} f_j^{i+1} - \sum_{j=1}^{q} \lambda_j i = i + f^{i+1}(\tau) - 1 - \sum_{j=1}^{q} \lambda_j i = r(i,\alpha) - 1 + (1 - \sum_{j=1}^{q} \lambda_j)i$ and $\gamma(i,\alpha) = 1 - \sum_{j=1}^{q} \lambda_j$.

3. If $\gamma(i,\alpha) = 0$, there is $l \in [1,q]$ such that $\lambda_l = 1$ and $\lambda_j = 0$ for any $j \in [1,q] - \{l\}$. Thus $t_{k,l} + f_l^{i+1} = r_l(i,\alpha) + i$, and so $t_{k,l} \le i$ implies $f_l^{i+1} \ge r_l(i,\alpha)$. Since $f_l^{i+1} \le \lfloor \frac{a_l - 1}{i+1} \rfloor = \rho_l(i+1,\alpha)$, the desired inequality follows.

4. If $\gamma(i,\alpha) = 1$, then $\lambda_j = 0$ and $f_j^{i+1} = r_j(i,\alpha) - t_{k,j} \le r_j(i,\alpha)$, so that from $f_j^{i+1} \le \rho_j(i+1,\alpha)$ we obtain $f_j^{i+1} \le \min(\rho_j(i+1,\alpha), r_j(i,\alpha))$ for any $j \in [1,q]$, and $r(i,\alpha) - 1 = f^{i+1}(\tau) - 1 = \sum_{j=1}^{q} f_j^{i+1} \le \sum_{j=1}^{q} \min(\rho_j(i+1,\alpha), r_j(i,\alpha))$.

5. In this case we deduce from $v = f^i(\tau)i + f^{i+1}(\tau)(i+1) \equiv r(i+1,\alpha) \pmod{i+1}$ that $f^i(\tau) + r(i+1,\alpha) \equiv 0 \pmod{i+1}$. As $f^i(\tau) \in [0,i]$ and $r(i+1,\alpha) \in [1, i+1]$, the last congruence implies $f^i(\tau) = i + 1 - r(i+1,\alpha)$. We have $t_{k,j} + f_j^i i + f_j^{i+1}(i+1) = a_j - 1 \equiv r_j(i+1,\alpha) \pmod{i+1}$, $t_{k,j} - f_j^i \equiv r_j(i+1,\alpha) \pmod{i+1}$, and so, as $t_{k,j}, r_j(i+1,\alpha), f_j^i \in [0,i]$, there is $\mu_j \in [0,1]$ such that $r_j(i+1,\alpha) = t_{k,j} - f_j^i + \mu_j(i+1)$ for any $j \in [1,q]$. Then, by Theorem 12.1, $r(i+1,\alpha) - 1 + i + 1 = \sum_{j=1}^{q} r_j(i+1,\alpha) = \sum_{j=1}^{q} t_{k,j} - \sum_{j=1}^{q} f_j^i + \sum_{j=1}^{q} \mu_j(i+1) = i - (i+1-r(i+1,\alpha)) + \sum_{j=1}^{q} \mu_j(i+1) = r(i+1,\alpha) - 1 + \sum_{j=1}^{q} \mu_j(i+1)$. Thus, there

is $l \in [1,q]$ such that $\mu_l = 1$ and $\mu_j = 0$ for any $j \in [1,q] - \{l\}$. Consequently, provided that $J := [1,q] - \{l\}$, $0 \le \sum_{j\in J} f_j^i = \sum_{j\in J} t_{k,j} - \sum_{j\in J} r_j(i+1,\alpha) \le i - \sum_{j=1}^{q} r_j(i+1,\alpha) + r_l(i+1,\alpha) = i - (r(i+1,\alpha) - 1 + i + 1) + r_l(i+1,\alpha) = r_l(i+1,\alpha) - r(i+1,\alpha)$, hence $r_l(i+1,\alpha) \ge r(i+1,\alpha)$. On the other hand, $f_l^i \le \rho_l(i,\alpha)$, and so $\sum_{j=1}^{q} f_j^i \le r_l(i+1,\alpha) - r(i+1,\alpha) + \rho_l(i+1,\alpha)$. Finally, $i+1-r(i+1,\alpha) = f^i(\tau) = \sum_{j=1}^{q} f_j^i \le \rho_l(i,\alpha) + r_l(i+1,\alpha) - r(i+1,\alpha)$, which immediately implies the desired inequality. ■

A sequence $\alpha = (a_1, \ldots, a_q) \in \mathcal{A}$ with $q \in [3,4]$ and $q = 4 \Rightarrow a_1 = 2$ is said to be *admissible* if for any $i \in [1, a_{q-2} + a_{q-1} - 2]$ all five assertions of Theorem 12 are true. Thus, if $S(\alpha)$ is avd, then α must be admissible.

Theorem 13 *The tree $S(\alpha)$ with $\alpha = (2, a_2, a_3) \in \mathcal{A}$ is avd if and only if* $\gcd(a_2, a_3) = 1$.

Proof. Put $T := S(\alpha)$ and $g := \gcd(a_2, a_3) \le a_2$. From $v = a_2 + a_3$ we obtain $g|v$. First assume that $g \ge 2$ and T is avd. Then $r_1(g,\alpha) = 1$, $r_2(g,\alpha) = r_3(g,\alpha) = g - 1$, $r(g,\alpha) = g$ and, by Theorem 12.1, $2g - 1 = g - 1 + \gamma(g,\alpha)g$, hence $\gamma(g,\alpha) = 1$. However, $r_j(g,\alpha) < r(g,\alpha)$, $j = 1,2,3$, which contradicts Theorem 12.2.

Now suppose that $g = 1$ and consider a non-decreasing sequence $\tau = (t_1, \ldots, t_k) \in \mathrm{Vs}(T)$. Let $m \in [1,k]$ be defined by the inequalities $\sum_{i=1}^{m-1} t_i \le a_2 - 1$ and $\sum_{i=1}^{m} t_i \ge a_2$. If $\sum_{i=1}^{m} t_i \ge a_2 + 1$, then τ is T-realisable by Lemma 4 with $q := 3$, $s := 2$, $I := [1, m-1]$ and $p := m$.

Otherwise we have $\sum_{i=1}^{m} t_i = a_2$. If $t_{m+1} > t_1$, then $\sum_{i=2}^{m} t_i = a_2 - t_1 \le a_2 - 1$ and $\sum_{i=2}^{m+1} t_i = \sum_{i=1}^{m} t_i + (t_{m+1} - t_1) \ge a_2 + 1$ and τ is T-realisable by Lemma 4 with $q := 3$, $s := 2$, $I := [2, m]$ and $p := m+1$. So, we may suppose that $t_{m+1} = t_1 = t_m$. If $t_k > t_m$, then $\sum_{i=1}^{m-1} t_i + t_k = \sum_{i=1}^{m} t_i + (t_k - t_m) \ge a_2 + 1$ and τ is T-realisable by Lemma 4 with $q := 3$, $s := 2$, $I := [1, m-1]$ and $p := k$. Finally, provided that $t_k = t_1 = t_i$ for any $i \in [1,k]$, $a_2 = mt_1$, $a_3 = (k-m)t_1$, $t_1|g$, $t_1 = 1$ and $\tau = (1)^v$ is trivially T-realisable. ■

An analogue of Theorem 13 with $a_1 = 3$ has been found by Cichacz et al. [6]. The corresponding necessary and sufficient condition is, however, much more complicated:

Theorem 14 *The tree $S(\alpha)$ with $\alpha = (3, a_2, a_3) \in \mathcal{A}$ is avd if and only if* $\gcd(a_2, a_3) \le 2$, $\gcd(a_2 + 1, a_3) \le 2$, $\gcd(a_2, a_3 + 1) \le 2$, $\gcd(a_2 + 1, a_3 + 1) \le 3$ *and there are no $\lambda_0, \lambda_1 \in [0, \infty)$ such that $|V(S(\alpha))| = \lambda_0 a_2 + \lambda_1(a_2 + 1)$.* ■

Consider a primary vertex x of a tree T that belongs to at least two arms A_1, A_2 of T. We adopt the notation used for star-like trees, i.e., we let x_i be the neighbour of x and y_i the pendant vertex in the arm A_i, $i = 1, 2$. By $T(A_1, A_2)$ we denote the tree with $V(T(A_1, A_2)) = V(T)$ and $E(T(A_1, A_2)) = E(T) - \{xx_2\} \cup \{y_1 y_2\}$ and by $A_{1,2}$ the arm of $T(A_1, A_2)$ with $V(A_{1,2}) = V(A_1) \cup V(A_2)$; we say that $T(A_1, A_2)$ is created from T by an *edge transportation*.

Lemma 15 *Suppose that a tree T is avd and A_1, A_2 are arms of T that share a primary vertex of T. Then the tree $T(A_1, A_2)$ is avd, too.*

Proof. Consider a sequence $\tau = (t_1, \dots, t_k) \in \mathrm{Vs}(T(A_1, A_2)) = \mathrm{Vs}(T)$. There is a T-realisation $\boldsymbol{T} = (T_1, \dots, T_k)$ of τ. Let $I_j \subseteq [1, k]$, $j = 1, 2$, be defined by $i \in I_j \overset{\text{df.}}{\Leftrightarrow} V(T_i) \cap (V(A_j) - \{x\}) \neq \emptyset$ and let T_l be the primary tree of $\boldsymbol{T}$. Clearly, T_i is a path for any $i \in I_1 \cup I_2 - \{l\}$.

We define a $T(A_1, A_2)$-realisation $(\tilde{T}_1, \dots, \tilde{T}_k)$ of τ as follows: If $i \in [1, k] - (I_1 \cup I_2)$, then $\tilde{T}_i := T_i$. Put $B_2 := V(T_l) \cap (V(A_2) - \{x\})$, let B_1 be the set of $|B_2|$ vertices of $A_{1,2}$ that follow immediately after the vertices of T_l when passing from x to x_2 (which is the pendant vertex of $A_{1,2}$) and let $\tilde{T}_l$ be the subtree of $T(A_1, A_2)$ with $V(\tilde{T}_l) = V(T_l) - B_2 \cup B_1$. The remaining (not belonging to already defined $\tilde{T}_i$'s) vertices of $T(A_1, A_2)$ induce a subpath of $A_{1,2}$, hence to conclude the proof we use Proposition 3. ■

Note that Lemma 15 cannot be reversed in general. Indeed, if $(2, a_2, a_3) \in \mathcal{A}$ and $\gcd(a_2, a_3) \geq 2$, then $T = S(2, a_2, a_3)$ is not avd (Theorem 13), while $T(A_2, A_3) \cong P_{a_2 + a_3}$ is.

Proposition 16 *If $\alpha = (2, a_2, a_3, a_4) \in \mathcal{A}$, the tree $S(\alpha)$ is avd and $k, l \in [2, 4]$, $k \neq l$, then* $\gcd(a_k, a_l) = 1$.

Proof. Suppose that $g := \gcd(a_k, a_l) > 1$. Then $r_1(g, \alpha) = 1$, $r_m(g, \alpha) = g - 1$ for any $m \in \{k, l\}$ and $r(g, \alpha) \in [1, g]$. Therefore, by Theorem 12.1, $[0, 1] \ni \gamma(g, \alpha) = \frac{1}{g} \cdot \left(\sum_{j=1}^{4} r_j(g, \alpha) + 1 - r(g, \alpha)\right) \geq \frac{2g - r(g, \alpha)}{g}$, and so $\gamma(g, \alpha) = 1$ and $r(g, \alpha) = g$. Since $r_j(g, \alpha) \in [0, g - 1]$ for any $j \in [1, 4]$, we have obtained a contradiction with Theorem 12.2. ■

Theorem 17 *If $\alpha = (2, a_2, a_3, a_4) \in \mathcal{A}$ and $S(\alpha)$ is avd, then*

1. $a_3 \geq 2a_2$;
2. $a_4 \geq a_2 + a_3$;
3. $a_2 + a_3 + a_4 - 1 = |V(S(\alpha))| \leq (\lfloor \frac{a_3 - 2}{a_2 - 1} \rfloor + 1)a_3 - 1$.

Proof. From Proposition 16 it follows that $a_3 \geq a_2 + 1$. Therefore, by Lemma 15, the tree $S(a_2 + 1, a_3, a_4)$ is avd. So, our Theorem follows from Theorem 8.1, 2, 3. ■

Before proving our main theorem let us mention the following number-theoretical statement joined (in a more general setting, cf. Brauer [5]) with the name of Frobenius:

Proposition 18 *If $l \in [1, \infty)$, $m \in [l+1, \infty)$, $\gcd(l, m) = 1$ and $n \in [(l-1)(m-1), \infty)$, then there are $\lambda, \mu \in [0, \infty)$ such that $n = \lambda l + \mu m$.* ■

Theorem 19 *Let $q \in [3,4]$, let $\alpha = (a_1,\dots,a_q) \in \mathcal{A}$ be an admissible sequence with $a_{q-1} - 1 \geq (a_{q-2} - 3)(a_{q-2} - 2)$ and suppose that $q = 4$ implies the tree $S(a_2, a_3, a_4)$ is avd. Then the tree $S(\alpha)$ is avd.*

Proof. By Theorems 9 and 10 it is sufficient to show that any sequence $\tau = (t_1,\dots,t_k)$ with $\tau \in \mathcal{B}_i(\alpha)$, $i \in [1, a_{q-2} + a_{q-1} - 2]$, or $\tau \in \bar{\mathcal{B}}_i(\alpha)$, $i \in [1, a_{q-2} - 3]$, is realisable in the tree $T := S(\alpha)$. Recall that $f^{i+1}(\tau) \geq 1$.

(1) $\exists j \in [1,k]\ t_j = a_{q-2} - 1$

(11) If $q = 3$, then τ is T-realisable by Proposition 3 and Lemma 5 with $I := \{j\}$.

(12) If $q = 4$, Proposition 16 yields $\gcd(a_3, a_4) = 1$ so that τ is T-realisable by Theorem 13 and Lemma 5 with $I := \{j\}$.

(2) If $t_j \neq a_{q-2} - 1$ for any $j \in [1,k]$, then $i \neq a_{q-2} - 2$.

(21) If $i = a_{q-2} - 1$, then $\tau \in \mathcal{B}_i(\alpha)$ and $t_j = a_{q-2}$ for each $j \in [1,k]$, $v = k a_{q-2}$, $r(a_{q-2}, \alpha) = a_{q-2}$, $r_{q-2}(a_{q-2}, \alpha) = a_{q-2} - 1$ and, since α satisfies the assertions 1 and 2 of Theorem 12, we have necessarily $\gamma(a_{q-2}, \alpha) = 0$, $r_j(a_{q-2}, \alpha) = 0$ for any $j \in [1,q] - \{q-2\}$, hence $q = 3$ (if $q = 4$, then $r_1(a_{q-2}, \alpha) = 1$) and $a_j - 1 \equiv 0 \pmod{a_1}$, $j = 2, 3$. In such a case τ is T-realisable by Proposition 3 and Lemma 5 with $I := [1, \frac{a_2 - 1}{a_1}]$.

(22) If $i \in [1, a_{q-2} - 3] \cup [a_{q-2}, a_{q-2} + a_{q-1} - 2]$, then $\tau \sim (m)\tau'$, where $f^i(\tau') = f^i(\tau)$, $f^j(\tau') = 0$ for any $j \notin [i, i+1]$, $m \in [1, i-1] \cup \{i+1\}$ and $m = i+1$ if and only if $\tau \in \mathcal{B}_i(\alpha)$. Note also that $m + f^i(\tau')i + f^{i+1}(\tau')(i+1) = v = 1 + \sum_{j=1}^{q}(a_j - 1)$.

(221) $\min(f^i(\tau), f^{i+1}(\tau)) \geq i + 1$

(2211) If $a_{q-1} - 1 \geq i(i+1)$, by Proposition 18 there are $\lambda_0, \lambda_1 \in [0, \infty)$ such that $a_{q-1} - 1 = \lambda_0 i + \lambda_1 (i+1)$. The pair (λ_0, λ_1) is not necessarily unique, since $i(i+1) = (i+s)(i+1-s)$, $s = 0, 1$, and so with $\lambda_j \geq i+1-j$ for some $j \in [0,1]$ we have $a_{q-1} - 1 = \lambda_j(i+j) + \lambda_{1-j}(i+1-j) = (\lambda_j - i - 1 + j)(i+j) + (\lambda_{1-j} + i + j)(i+1-j)$, where $\lambda_j - i - 1 + j, \lambda_{1-j} + i + j \in [0, \infty)$. As $f^i(\tau') = f^i(\tau) \geq i+1$, we may suppose without loss of generality that $\lambda_0 \leq f^i(\tau')$, but $\lambda_0 + i + 1 > f^i(\tau')$, so that $\lambda_0 \geq f^i(\tau') - i$. Then $\lambda_1 \leq f^{i+1}(\tau')$. Indeed, the assumption $\lambda_1 \geq f^{i+1}(\tau') + 1$ would lead to $a_{q-1} - 1 = \lambda_0 i + \lambda_1(i+1) \geq (f^i(\tau') - i)i + (f^{i+1}(\tau') + 1)(i+1) = f^i(\tau')i + f^{i+1}(\tau')(i+1) + i + 1 - i^2 = 1 + \sum_{j=1}^{q}(a_j - 1) - m + i + 1 - i^2 > \sum_{j=1}^{q}(a_j - 1) - i^2 \geq a_{q-1} - 1 + \sum_{j=1}^{q-2}(a_j - 1) + i(i+1) - i^2 > a_{q-1} - 1$, a contradiction. Thus, there are $I_0, I_1 \subseteq [1,k]$ such that $|I_s| = \lambda_s$ and $t_j = i + s$ for any $j \in I_s$, $s = 0, 1$. Then $\sum_{j \in I_0 \cup I_1} t_j = a_{q-1} - 1$ and the sequence $(m)\tau' \sim \tau$ is T-realisable by Lemma 5 with $I := I_0 \cup I_1$ and either Proposition 3 ($q = 3$) or Proposition 16 and Theorem 13 ($q = 4$).

(2212) If $a_{q-1} - 1 < i(i+1)$, then $i \geq a_{q-2}$, since otherwise $i(i+1) \leq (a_{q-2} - 3)(a_{q-2} - 2) \leq a_{q-1} - 1$, a contradiction. Thus, $\tau \in \mathcal{B}_i(\alpha)$, $r_j(i, \alpha) = a_j - 1$ and $\rho_j(i+1, \alpha) = 0$ for $j \in [1, q-2]$, $\rho_{q-1}(i, \alpha) = \lfloor \frac{a_{q-1} - 1}{i} \rfloor < i + 1 \leq f^i(\tau)$ and $\rho_{q-1}(i+1, \alpha) = \lfloor \frac{a_{q-1} - 1}{i+1} \rfloor < i < f^{i+1}(\tau)$.

(22121) If $\gamma(i,\alpha)=0$, then $\sum_{j=1}^{q-2}(a_j-1)+\sum_{j=q-1}^{q} r_j(i,\alpha)=r(i,\alpha)-1$. Moreover, we have $\rho_{q-1}(i,\alpha)i=a_{q-1}-r_{q-1}(i,\alpha)\le a_{q-1}$, $\rho_{q-1}(i,\alpha)i+i+1=a_{q-1}-1-r_{q-1}(i,\alpha)+i+1=a_{q-1}-1-[r(i,\alpha)-1-\sum_{j=1}^{q-2}(a_j-1)-r_q(i,\alpha)]+i+1=\sum_{j=1}^{q-1}(a_j-1)+r_q(i,\alpha)-r(i,\alpha)+i+1\ge 1+\sum_{j=1}^{q-1}(a_j-1)$, and so τ is T-realisable by Lemma 4 with $s:=q-1$, $p\in[1,k]$ such that $t_p=i+1$ and $I\subseteq[1,k]-\{p\}$ such that $|I|=\rho_{q-1}(i,\alpha)$ and $t_j=i$ for any $j\in I$.

(22122) In the case $\gamma(i,\alpha)=1$ we have $\sum_{j=1}^{q-2}(a_j-1)+\sum_{j=q-1}^{q} r_j(i,\alpha)=r(i,\alpha)-1+i$ and, because α satisfies 4 of Theorem 12, $\sum_{j=q-1}^{q}\min(\rho_j(i+1,\alpha),r_j(i,\alpha))\ge r(i,\alpha)-1$. Since $i+1+2(a_q-1)\ge 1+\sum_{j=1}^{q}(a_j-1)=v=f^i(\tau)i+f^{i+1}(\tau)(i+1)\ge(i+1)(2i+1)$, we obtain $\frac{a_q-1}{i+1}\ge i>r_q(i,\alpha)$ and $\min(\rho_q(i+1,\alpha),r_q(i,\alpha))=r_q(i,\alpha)$.

(221221) If $\rho_{q-1}(i+1,\alpha)\ge r_{q-1}(i,\alpha)$, consider the expression $a_{q-1}-1=\rho_{q-1}(i,\alpha)i+r_{q-1}(i,\alpha)=r_{q-1}(i,\alpha)(i+1)+(\rho_{q-1}(i,\alpha)-r_{q-1}(i,\alpha))i$. As $f^i(\tau)>i\ge\rho_{q-1}(i,\alpha)-r_{q-1}(i,\alpha)\ge\rho_{q-1}(i+1,\alpha)-r_{q-1}(i,\alpha)\ge 0$ and $f^{i+1}(\tau)>i>r_{q-1}(i,\alpha)$, there are $I_0,I_1\subseteq[1,k]$ such that $|I_0|=\rho_{q-1}(i,\alpha)-r_{q-1}(i,\alpha)$, $|I_1|=r_{q-1}(i,\alpha)$ and $t_j=i+s$ for any $j\in I_s$, $s=0,1$. Thus, τ is T-realisable similarly as in (2211).

(221222) If $\rho_{q-1}(i+1,\alpha)<r_{q-1}(i,\alpha)$, then $\rho_{q-1}(i+1,\alpha)+r_q(i,\alpha)\ge r(i,\alpha)-1$. We have $\frac{a_{q-1}-1}{i}-\frac{a_{q-1}-1}{i+1}=\frac{a_{q-1}-1}{i(i+1)}\in(0,1)$, and so $\rho_{q-1}(i+1,\alpha)\le\rho_{q-1}(i,\alpha)\le\rho_{q-1}(i+1,\alpha)+1$. Moreover, $a_{q-1}-1=\rho_{q-1}(i,\alpha)i+r_{q-1}(i,\alpha)=\rho_{q-1}(i,\alpha)(i+1)+r_{q-1}(i,\alpha)-\rho_{q-1}(i,\alpha)$, and also $a_{q-1}-1=\rho_{q-1}(i+1,\alpha)(i+1)+r_{q-1}(i+1,\alpha)$; having in mind that $i+1>r_{q-1}(i,\alpha)-\rho_{q-1}(i,\alpha)\ge r_{q-1}(i,\alpha)-\rho_{q-1}(i+1,\alpha)-1\ge 0$, we obtain $r_{q-1}(i+1,\alpha)=r_{q-1}(i,\alpha)-\rho_{q-1}(i+1,\alpha)$. Consider $I\subseteq[1,k]$ and $p\in[1,k]-I$ such that $|I|=\rho_{q-1}(i+1,\alpha)$ and $t_j=i+1$ for any $j\in I\cup\{p\}$ (notice that $\rho_{q-1}(i+1,\alpha)+1<f^{i+1}(\tau)$). Then $\sum_{j\in I}t_j=\rho_{q-1}(i+1,\alpha)(i+1)\le a_{q-1}-1$ and $\sum_{j\in I}t_j+t_p=a_{q-1}-1-r_{q-1}(i+1,\alpha)+i+1=a_{q-1}-1-r_{q-1}(i,\alpha)+\rho_{q-1}(i+1,\alpha)+i+1=1+\sum_{j=1}^{q-1}(a_j-1)+w$, where $w:=i-\sum_{j=1}^{q-1}r_j(i,\alpha)+\rho_{q-1}(i+1,\alpha)=\rho_{q-1}(i+1,\alpha)+r_q(i,\alpha)-(r(i,\alpha)-1)\ge 0$, so that the sequence τ is T-realisable by Lemma 4 with $s:=q-1$.

(222) $\min(f^i(\tau),f^{i+1}(\tau))\le i$

(2221) If $f^i(\tau)\le i$, then from $m+f^i(\tau)i+f^{i+1}(\tau')(i+1)=v\equiv r(i+1,\alpha)\pmod{i+1}$ it follows that $r(i+1,\alpha)\equiv m-f^i(\tau)\pmod{i+1}$.

(22211) If $m\ge r(i+1,\alpha)$, then $r(i+1,\alpha)=m-f^i(\tau)$.

(222111) If $\gamma(i+1,\alpha)=0$, then $\sum_{j=1}^{q}r_j(i,\alpha)=r(i,\alpha)-1$ and $\sum_{j=1}^{q}\rho_j(i+1,\alpha)=\sum_{j=1}^{q}\frac{a_j-1-r_j(i+1,\alpha)}{i+1}=\frac{1}{i+1}[\sum_{j=1}^{q}(a_j-1)-\sum_{j=1}^{q}r_j(i+1,\alpha)]=\frac{1}{i+1}(v-r(i+1,\alpha))=\frac{1}{i+1}[m+f^i(\tau)i+f^{i+1}(\tau')(i+1)-r(i+1,\alpha)]=f^i(\tau)+f^{i+1}(\tau')\ge f^i(\tau)$. From the obtained inequality it follows that for any $j\in[1,q]$ there is $b_j\in[0,\rho_j(i+1,\alpha)]$ such that $\sum_{j=1}^{q}b_j=f^i(\tau)$. Put $c_j:=\rho_j(i+1,\alpha)-b_j$; as $b_ji+c_j(i+1)=\rho_j(i+1,\alpha)(i+1)-b_j\le a_j-1$, there is a realisation $\boldsymbol{T}_j$ of the sequence $(i)^{b_j}(i+1)^{c_j}$ in the end $E_j\subseteq A_j$ (of the appropriate order) for $j\in[1,q]$. (Note that $\boldsymbol{T}_j$ may be an empty sequence: this is the case e.g. if $q=4$ and

$j=1$, since then $b_1=c_1=0$.) The remaining vertices of T induce the tree $\tilde{T}$ of order $v-\sum_{j=1}^q[b_ji+(\rho_j(i+1,\alpha)-b_j)(i+1)]=\sum_{j=1}^q b_j+v-\sum_{j=1}^q\rho_j(i+1,\alpha)(i+1)=f^i(\tau)+v-\sum_{j=1}^q(a_j-1-r_j(i+1,\alpha))=f^i(\tau)+r(i+1,\alpha)=m$. Therefore, $\sum_{j=1}^q c_j=f^{i+1}(\tau')$, $(\tilde{T})\prod_{j=1}^q\boldsymbol{T}_j$ is a T-realisation of the sequence $(m)\prod_{j=1}^q[(i)^{b_j}(i+1)^{c_j}]\sim\tau$ and τ is T-realisable by Proposition 1.

(222112) If $\gamma(i+1,\alpha)=1$, there is $l\in[1,q]$ such that $r_l(i+1,\alpha)\geq r(i+1,\alpha)$ (α satisfies 2 of Theorem 12).

(2221121) If $i\leq a_{q-2}-3$, then $a_{q-1}\geq i(i+1)$ and $f^{i+1}(\tau')=\frac{v-m-f^i(\tau)i}{i+1}\geq\frac{2i(i+1)-(i+1)-i^2}{i+1}>i-1$. Put $\lambda_0:=f^i(\tau)+i+1-m=i+1-r(i+1,\alpha)\in[1,i]$ and $\lambda_1:=f^{i+1}(\tau')-i+m\geq 1$. From $m+f^i(\tau)i+f^{i+1}(\tau')(i+1)=v=\lambda_0 i+\lambda_1(i+1)$ it follows that $(i)^{\lambda_0}(i+1)^{\lambda_1}\in\mathcal{B}_i(\alpha)$, and, as α satisfies 5 of Theorem 12, we may suppose without loss of generality that $\rho_l(i,\alpha)+r_l(i+1,\alpha)\geq i+1$. Pick $n\in[q-1,q]-\{l\}$; then the assumptions of our Theorem yield $a_n-1\geq(a_{q-2}-3)(a_{q-2}-2)$.

(22211211) If $m\geq r_l(i+1,\alpha)$, define $b_l:=m-r_l(i+1,\alpha)\in[0,f^i(\tau)]$ to obtain $a_l-1-b_li-m=a_l-1+r_l(i+1,\alpha)i-m(i+1)\equiv 0\,(\mathrm{mod}\ i+1)$, $\lfloor\frac{a_l-1}{i}\rfloor=\rho_l(i,\alpha)\geq i+1-r_l(i+1,\alpha)$ and $a_l-1\geq(i+1-r_l(i+1,\alpha))i$.

(222112111) If $a_l-1>(i+1-r_l(i+1,\alpha))i$ or $m\leq i-1$, then $a_l-1-b_li-m=a_l-1-(i+1-r_l(i+1,\alpha))i+(i+1-m)i-m>-i-1$, hence $c_l:=\frac{a_l-1-b_li-m}{i+1}\in[0,\infty)$. In such a case $a_l-1=m+b_li+c_l(i+1)$ and $c_l\leq f^{i+1}(\tau')$, since otherwise $f^{i+1}(\tau')-c_l\leq -1$ together with $f^i(\tau)-b_l\leq i$ would lead to $i(i+1)\leq(a_{q-2}-3)(a_{q-2}-2)\leq a_n-1<1+\sum_{j\in[1,q]-\{l\}}(a_j-1)=v-(a_l-1)=m+f^i(\tau)+f^{i+1}(\tau')(i+1)-[m+b_li+c_l(i+1)]\leq i^2-(i+1)$, a contradiction. Thus, there are pairwise disjoint sets $I_0,I_1,I_2\subseteq[1,k]$ such that $|I_0|=b_l$, $|I_1|=c_l$, $|I_2|=1$, $t_j=i+s$ for any $j\in I_s$, $s=0,1$, and $j\in I_2\Rightarrow t_j=m$; the sequence τ is T-realisable similarly as in (2211), but with $I:=I_0\cup I_1\cup I_2$.

(222112112) If $a_l-1=(i+1-r_l(i+1,\alpha))i$ and $m=i+1$, then $c_l:=\frac{a_l-1-b_li}{i+1}=0\leq f^{i+1}(\tau')$ and we can proceed as in (222112111), but with $I_2:=\emptyset$.

(22211212) If $m<r_l(i+1,\alpha)$, define $b_n:=m-r(i+1,\alpha)$, $b_j:=0$ for $j\in[1,q]-\{n\}$, $c_l:=\lfloor\frac{a_l-1-m}{i+1}\rfloor$, $\tau_l:=(m)(i+1)^{c_l}$, $c_j:=\lfloor\frac{a_j-1-b_ji}{i+1}\rfloor$ and $\tau_j:=(i)^{b_j}(i+1)^{c_j}$ for $j\in[1,q]-\{l\}$. Consider a realisation $\boldsymbol{T}_j$ of τ_j in the end $E_j\subseteq A_j$ for $j\in[1,q]$. Since $0\leq r_n(i+1,\alpha)+m-r(i+1,\alpha)<r_n(i+1,\alpha)+r_l(i+1,\alpha)-r(i+1,\alpha)\leq\sum_{j=1}^q r_j(i+1,\alpha)-r(i+1,\alpha)=i$, we have $c_n=\lfloor\frac{1}{i+1}[a_n-1-(m-r(i+1,\alpha))(i+1)+m-r(i+1,\alpha)]\rfloor=\lfloor\frac{1}{i+1}[\rho_n(i+1,\alpha)(i+1)+r_n(i+1,\alpha)+m-r(i+1,\alpha)]\rfloor-(m-r(i+1,\alpha))=\rho_n(i+1,\alpha))-(m-r(i+1,\alpha))$. Moreover, $c_j=\rho_j(i+1,\alpha)$ for any $j\in[1,q]-\{n\}$. Therefore, the rest of T is the tree $\tilde{T}$ of order $v-m-\sum_{j=1}^q[b_ji+c_j(i+1)]=v-m-(m-r(i+1,\alpha))i-\sum_{j=1}^q\rho_j(i+1,\alpha)(i+1)+(m-r(i+1,\alpha))(i+1)=v-r(i+1,\alpha)-\sum_{j=1}^q(a_j-1-r_j(i+1,\alpha))=i+1$. As $\sum_{j=1}^q b_j=f^i(\tau)$, $(\tilde{T})\prod_{j=1}^q\boldsymbol{T}_j$ is a T-realisation of the sequence $(i+1)\prod_{j=1}^q\tau_j\sim\tau$.

(2221122) If $i\geq a_{q-2}$, then $m=i+1$ and $r_j(i+1,\alpha)=a_j-1$ for $j\in[1,q-2]$. As α satisfies 5 of Theorem 12, we may suppose without loss of generality that

$\rho_l(i,\alpha)+r_l(i+1,\alpha) \geq i+1$, and hence $l \in [q-1,q]$ (note that $\rho_j(i,\alpha)+r_j(i+1,\alpha) = r_j(i+1,\alpha) \leq i$ for $j \in [1,q-2]$).

(22211221) If $\rho_l(i+1,\alpha) \geq i+1-r_l(i+1,\alpha) = f^i(\tau)$, put $b_l := i+1-r(i+1,\alpha)$, $b_j := 0$ for $j \in [1,q]-\{l\}$, and, with $c_j := \lfloor \frac{a_j-1-b_j i}{i+1} \rfloor$ consider a realisation $\boldsymbol{T}_j$ of the sequence $(i)^{b_j}(i+1)^{c_j}$ in the end $E_j \subseteq A_j$ for $j \in [1,q]$. Since $a_l - 1 - b_l i = a_l - 1 - (i+1-r(i+1,\alpha))i = a_l - 1 - r(i+1,\alpha) + (r(i+1,\alpha)-i)(i+1) = (\rho_l(i+1,\alpha) + r(i+1,\alpha)-i)(i+1) + r_l(i+1,\alpha) - r(i+1,\alpha)$ and $r_l(i+1,\alpha) - r(i+1,\alpha) \in [0,i]$, we have $c_l = \rho_l(i+1,\alpha) + r(i+1,\alpha) - i \geq 1$. Therefore, vertices that are not used yet induce the tree $\tilde{T}$ with $|V(\tilde{T})| = v - \sum_{j=1}^q [b_j i + c_j(i+1)] = v - (i+1-r(i+1,\alpha))i - \sum_{j=1}^q \rho_j(i+1,\alpha)(i+1) - (r(i+1,\alpha)-i)(i+1) = v - \sum_{j=1}^q (a_j - 1 - r_j(i+1,\alpha)) - r(i+1,\alpha) = i+1$. Thus, having in mind that $\sum_{j=1}^q b_j = f^i(\tau)$, $(\tilde{T}) \prod_{j=1}^q \boldsymbol{T}_j$ is a T-realisation of the sequence $(i+1) \prod_{j=1}^q [(i)^{b_j}(i+1)^{c_j}] \sim \tau$.

(22211222) If $\frac{a_l-1-r_l(i+1,\alpha)}{i+1} = \rho_l(i+1,\alpha) \leq i - r(i+1,\alpha)$, then $a_l - 1 \leq (i-r(i+1,\alpha))(i+1) + r_l(i+1,\alpha)$. From $\lfloor \frac{a_l-1}{i} \rfloor = \rho_l(i,\alpha) \geq i+1-r_l(i+1,\alpha)$ we obtain $a_l - 1 \geq (i+1-r_l(i+1,\alpha))i$, $0 \leq a_l - 1 - (i+1-r_l(i+1,\alpha))i = a_l - 1 - r_l(i+1,\alpha) + (r_l(i+1,\alpha)-i)(i+1) = (\rho_l(i+1,\alpha)+r_l(i+1,\alpha)-i)(i+1)$, hence $\kappa := \rho_l(i+1,\alpha) + r_l(i+1,\alpha) - i \geq 0$, $\kappa(i+1) = a_l - 1 - (i+1-r_l(i+1,\alpha))i \leq (i-r(i+1,\alpha))(i+1) + r_l(i+1,\alpha) - (i+1-r_l(i+1,\alpha))i = (r_l(i+1,\alpha) - r(i+1,\alpha))(i+1)$, and so $\kappa \in [0, r_l(i+1,\alpha) - r(i+1,\alpha)]$. With $b_l := i+1-r_l(i+1,\alpha)+\kappa = \rho_l(i+1,\alpha)+1 \leq i+1-r(i+1,\alpha) = f^i(\tau)$ we have $b_l i = a_l - 1 - \kappa \leq a_l - 1$ and $b_l i + i + 1 \geq a_l - 1 + r(i+1,\alpha) - r_l(i+1,\alpha) + i + 1 = a_l - 1 + 1 + \sum_{j \in [1,q]-\{l\}} r_j(i+1,\alpha) \geq 1 + \sum_{j=1}^{q-2}(a_j - 1) + a_l - 1$; as there are $I \subseteq [1,k]$ and $p \in [1,k] - I$ such that $|I| = b_l$ and $t_j = i$ for any $j \in I$, the sequence τ is T-realisable by Lemma 5 with $s := l$.

(22212) If $m < r(i+1,\alpha)$, then $r(i+1,\alpha) = m - f^i(\tau) + i + 1$, $\tau \in \bar{\mathcal{B}}_i(\alpha)$, $i \leq a_{q-2} - 3$, and so $a_{q-1} - 1 \geq i(i+1)$.

(222121) If $\gamma(i+1,\alpha) = 0$, put $b_j := 0$ for $j \in [1,q-2]$, $b_{q-1} := i+1-r(i+1,\alpha)$, $b_q := m$, $c_j := \rho_j(i+1,\alpha) - b_j$ for $j \in [1,q]$, $\tau_j := (i)^{b_j}(i+1)^{c_j}$ for $j \in [1,q-1]$, $\tau_q := (i)^{b_q}(i+1)^{c_q}(m)$ and consider a realisation $\boldsymbol{T}_j$ of the sequence τ_j in the end $E_j \subseteq A_j$ for $j \in [1,q]$; note that $b_j i + c_j(i+1) = \rho_j(i+1,\alpha)(i+1) - b_j \leq a_j - 1$ for any $j \in [1,q-1]$ and $b_q i + c_q(i+1) + m = \rho_q(i+1,\alpha)(i+1) \leq a_q - 1$. Let $\tilde{T}$ be the tree on the remaining vertices. Then $|V(\tilde{T})| = v - \sum_{j=1}^q [b_j i + c_j(i+1)] - m = v - \sum_{j=1}^q \rho_j(i+1,\alpha)(i+1) + \sum_{j=1}^{q-1} b_j = v - \sum_{j=1}^q (a_j - 1 - r_j(i+1,\alpha)) + i + 1 - r(i+1,\alpha) = 1 + \sum_{j=1}^q r_j(i+1,\alpha) - r(i+1,\alpha) + i + 1 = i+1$, and, as $\sum_{j=1}^q b_j = f^i(\tau)$, $(\tilde{T}) \prod_{j=1}^q \boldsymbol{T}_j$ is a T-realisation of the sequence $(i+1) \prod_{j=1}^q [(i)^{b_j}(i+1)^{c_j}](m) \sim \tau$.

(222122) If $\gamma(i+1,\alpha) = 1$, with $\lambda_0 := f^i(\tau) - m = i+1-r(i+1,\alpha) \in [0,i]$ and $\lambda_1 := f^{i+1}(\tau') + m$ we have $(i)^{\lambda_0}(i+1)^{\lambda_1} \in \mathcal{B}_i(\alpha)$; since α satisfies 5 of Theorem 12, there is $l \in [1,q]$ such that $r_l(i+1,\alpha) \geq r(i+1,\alpha)$ and $\rho_l(i,\alpha) + r_l(i+1,\alpha) \geq i+1$. Consequently, $a_l - 1 \geq (i+1-r_l(i+1,\alpha))i$ and $(i+1-r_l(i+1,\alpha))i \equiv r_l(i+1,\alpha) \equiv a_l - 1 \pmod{i+1}$, so that with $b_l := i+1-r_l(i+1,\alpha)$ and $c_l := \frac{a_l-1-b_l i}{i+1}$ we have $b_l, c_l \in [0,\infty)$ and $a_l - 1 = b_l i + c_l(i+1)$. Moreover, $c_l \leq f^{i+1}(\tau')$, since otherwise (having in mind that $a_j-1 \geq i(i+1)$ for $j \in [q-1,q]-\{l\}$) $i-1+i^2 \geq m+f^i(\tau)i = 1+\sum_{j=1}^q(a_j-1)-f^{i+1}(\tau')(i+1) > i(i+1)+a_l-1-f^{i+1}(\tau')(i+1) \geq i(i+1)+i+i+1$, a contradiction. Hence, there are $I_0, I_1 \subseteq [1,k]$ such that $|I_0| = b_l$, $|I_1| = c_l$ and $t_j = i+s$ for any $j \in I_s$, $s = 0,1$; the sequence τ is T-realisable as in (2211).

(2222) If $f^i(\tau) \geq i+1$ and $f^{i+1}(\tau) \leq i$, then from $m+f^i(\tau)i+f^{i+1}(\tau')(i+1) = v \equiv r(i,\alpha) \pmod{i}$ we obtain $r(i,\alpha) \equiv m + f^{i+1}(\tau') \pmod{i}$.

(22221) If $m > r(i,\alpha)$, then $r(i,\alpha) = m + f^{i+1}(\tau') - i$.

(222211) If $a_q - 1 \geq i(i+1)$, put $c_j := 0$ and $b_j := \rho_j(i,\alpha)$ for $j \in [1,q-1]$, $c_q := i+r(i,\alpha)-m = f^{i+1}(\tau')$, $b_q := \rho_q(i,\alpha)-c_q-1+\gamma(i,\alpha) \geq 1$, and consider a realisation $\boldsymbol{T}_j$ of the sequence $(i)^{b_j}(i+1)^{c_j}$ in the end $E_j \subseteq A_j$ for $j \in [1,q]$. The rest of T is the tree $\tilde{T}$ of order $v-\sum_{j=1}^q[(\rho_j(i,\alpha)-c_j)i+c_j(i+1)]+(1-\gamma(i,\alpha))i = v-\sum_{j=1}^q \rho_j(i,\alpha)i-(i+r(i,\alpha)-m)+i-\gamma(i,\alpha)i = v-\sum_{j=1}^q(a_j-1-r_j(i,\alpha)) - r(i,\alpha)-\gamma(i,\alpha)i+m = m$. Since $\sum_{j=1}^q c_j = f^{i+1}(\tau')$, $(\tilde{T})\prod_{j=1}^q \boldsymbol{T}_j$ is a T-realisation of the sequence $(m)\prod_{j=1}^q[(i)^{b_j}(i+1)^{c_j}] \sim (m)\tau' \sim \tau$.

(222212) If $a_q-1 < i(i+1)$, then also $a_{q-1}-1 < i(i+1)$, hence $i \leq a_{q-2}-3$ is impossible and we have $i \geq a_{q-2}$, $\rho_j(i,\alpha) = 0$ and $r_j(i,\alpha) = a_j-1$ for $j \in [1,q-2]$, $\tau \in \mathcal{B}_i(\alpha)$, $m = i+1$, $f^{i+1}(\tau') = r(i,\alpha)-1$ and $f^{i+1}(\tau) = r(i,\alpha)$.

(2222121) If $\gamma(i,\alpha) = 0$, then $\sum_{j=1}^q r_j(i,\alpha) = r(i,\alpha)-1$, and so $r(i,\alpha) > r_j(i,\alpha)$ for any $j \in [1,q]$. Further, as α satisfies the statement 3 of Theorem 12, there is $l \in [1,q]$ such that $\rho_l(i+1,\alpha) \geq r_l(i,\alpha)$. Put $c_l := r_l(i,\alpha)$ and $b_l := \rho_l(i,\alpha)-r_l(i,\alpha) \geq \rho_l(i+1,\alpha)-r_l(i,\alpha) \geq 0$. From $\rho_l(i+1,\alpha) < \frac{i(i+1)}{i+1} = i$ and $f^i(\tau) \geq i+1 > \frac{a_q-1}{i} \geq b_l$ it follows that there are $I_0, I_1 \subseteq [1,k]$ such that $|I_0| = b_l$, $|I_1| = c_l$ and $t_j = i+s$ for any $j \in I_s$, $s = 0,1$. Since $b_l i + c_l(i+1) = \rho_l(i,\alpha)i + r_l(i,\alpha) = a_l-1$, τ is T-realisable as in (2211).

(2222122) If $\gamma(i,\alpha) = 1$, then $\sum_{j=1}^q r_j(i,\alpha) = r(i,\alpha)-1+i$, and, as α satisfies 4 of Theorem 12, $\sum_{j=q-1}^q \min(\rho_j(i+1,\alpha), r_j(i,\alpha)) = \sum_{j=1}^q \min(\rho_j(i+1,\alpha), r_j(i,\alpha)) \geq r(i,\alpha)-1$. Therefore, there are $c_j \in [0, \min(\rho_j(i+1,\alpha), r_j(i,\alpha))]$, $j = q-1,q$, such that $c_{q-1}+c_q = r(i,\alpha)-1$. Put $b_j := \rho_j(i,\alpha)-c_j \geq 0$ and consider a realisation $\boldsymbol{T}_j$ of the sequence $(i)^{b_j}(i+1)^{c_j}$ in the end $E_j \subseteq A_j$, $j = q-1,q$. What remains from T is the tree $\tilde{T}$ of order $v-\sum_{j=q-1}^q[b_j i+c_j(i+1)] = v-\sum_{j=q-1}^q \rho_j(i,\alpha)i-(c_{q-1}+c_q) = 1+\sum_{j=1}^q(a_j-1)-\sum_{j=q-1}^q(a_j-1-r_j(i,\alpha)) - r(i,\alpha)+1 = 1+\sum_{j=1}^{q-2}(a_j-1)+\sum_{j=q-1}^q r_j(i,\alpha)-r(i,\alpha)+1 = i+1$. Thus, $(\tilde{T})\boldsymbol{T}_{q-1}\boldsymbol{T}_q$ is a T-realisation of the sequence $(i+1)\prod_{j=q-1}^q[(i)^{b_j}(i+1)^{c_j}] \sim \tau$.

(22222) If $m \leq r(i,\alpha)$, then $r(i,\alpha) = m+f^{i+1}(\tau')$, $m \leq i-1$, $\tau \in \bar{\mathcal{B}}_i(\alpha)$, $f^{i+1}(\tau') \geq 1$, $i \leq a_{q-2}-3$, and so $a_{q-1}-1 \geq i(i+1)$. With $\lambda_0 := f^i(\tau)-m \geq 2$ and $\lambda_1 = f^{i+1}(\tau')+m = r(i,\alpha) \in [2,i]$ we have $(i)^{\lambda_0}(i+1)^{\lambda_1} \in \mathcal{B}_i(\alpha)$.

(222221) If $\gamma(i,\alpha) = 0$, then $\sum_{j=1}^q r_j(i,\alpha) = r(i,\alpha)-1$ and $r(i,\alpha) \geq r_j(i,\alpha)+1$ for $j \in [1,q]$. Since 3 of Theorem 12 holds for α, there is $l \in [1,q]$ such that $\rho_l(i+1,\alpha) \geq r_l(i,\alpha)$.

(2222211) If $r_l(i,\alpha) < r(i,\alpha)-m = f^{i+1}(\tau')$, put $c_l := r_l(i,\alpha)$ and $b_l := \rho_l(i,\alpha)-r_l(i,\alpha) \geq \rho_l(i+1,\alpha)-r_l(i,\alpha) \geq 0$. Then $b_l i + c_l(i+1) = \rho_l(i,\alpha)i + r_l(i,\alpha) = a_l-1$ and $b_l \leq f^i(\tau)$, since otherwise $i-1+i(i+1) \geq m+f^{i+1}(\tau')(i+1) =$

$1+\sum_{j=1}^{q}(a_j-1)-f^i(\tau)i > i(i+1)+a_l-1-f^i(\tau)i \ge i(i+1)+i$, a contradiction. As a consequence, there are $I_0, I_1 \subseteq [1,k]$ such that $|I_0| = b_l$, $|I_1| = c_l$ and $t_j = i+s$ for any $j \in I_s$, $s = 0,1$, and we are done as in (2211).

(2222212) If $r_l(i,\alpha) \ge r(i,\alpha) - m$, put $c_l := r(i,\alpha) - m \ge 0$, $c_j := 0$ for $j \in [1,q]-\{l\}$ and $b_j := \rho_j(i,\alpha)-c_j$ for $j \in [1,q]$. As $b_l \ge \rho_l(i+1,\alpha)-r_l(i,\alpha) \ge 0$, we have $b_j \ge 0$ for any $j \in [1,q]$. Consider a realisation $\boldsymbol{T}_j$ of the sequence $(i)^{b_j}(i+1)^{c_j}$ in the end $E_j \subseteq A_j$ for $j \in [1,q]$. The rest of T forms the tree $\tilde{T}$ of order $v-\sum_{j=1}^{q}[b_j i+c_j(i+1)] = 1+\sum_{j=1}^{q}(a_j-1)-\sum_{j=1}^{q}\rho_j(i,\alpha)i-r(i,\alpha)+m = 1+\sum_{j=1}^{q}r_j(i,\alpha)-r(i,\alpha)+m = m$ so that $(\tilde{T})\prod_{j=1}^{q}\boldsymbol{T}_j$ is a T-realisation of the sequence $(m)\prod_{j=1}^{j}[(i)^{b_j}(i+1)^{c_j}] \sim (m)\tau' \sim \tau$.

(222222) If $\gamma(i,\alpha)=1$, then $\sum_{j=1}^{q}r_j(i,\alpha) = r(i,\alpha)-1+i$. As α satisfies 2 and 4 of Theorem 12, there is $l \in [1,q]$ such that $r_l(i,\alpha) \ge r(i,\alpha)$ and $\sum_{j=1}^{q}\min(\rho_j(i+1,\alpha), r_j(i,\alpha)) \ge r(i,\alpha)-1 \ge r(i,\alpha)-m$. With $\mu_j := \min(\rho_j(i+1,\alpha), r_j(i,\alpha))$ for $j \in [1,q]-\{l\}$ and $\mu_l := \min(\rho_l(i+1,\alpha), r_l(i,\alpha)-m) \ge 0$ we have $\mu_l \ge \min(\rho_l(i+1,\alpha), r_l(i,\alpha))-m$, and so $\sum_{j=1}^{q}\mu_j \ge r(i,\alpha)-1-m = f^{i+1}(\tau')-1 \ge 0$. Thus, for any $j \in [1,q]$ there is $c_j \in [0,\mu_j]$ such that $\sum_{j=1}^{q}c_j = r(i,\alpha)-1-m$. Let us show that c_l can be chosen so that $c_l \le \rho_l(i+1,\alpha)-1$. Since $c_l \le r(i,\alpha)-1-m \le i-2$ and $\rho_j(i+1,\alpha) \ge i$, $j = q-1,q$, the choice is possible if $l \ge q-1$. Notice that otherwise $l = q-2$: if $q=4$, then $r_1(i,\alpha) = 1 < r(i,\alpha)$. In such a case from $\mu_j = r_j(i,\alpha)$, $j=q-1,q$, and $\mu_{q-1}+\mu_q-(r(i,\alpha)-1-m) = \sum_{j=q-1}^{q}r_j(i,\alpha)+m-(\sum_{j=1}^{q}r_j(i,\alpha)-i) = m+i-\sum_{j=1}^{q-2}r_j(i,\alpha) \ge m+i-1-r_{q-2}(i,\alpha) \ge m$ it follows that we can choose c_{q-1} and c_q in such a way that $c_{q-1}+c_q = r(i,\alpha)-1-m$; therefore, with $c_j := 0$ for $j \in [1,q-2]$ we have $c_l = c_{q-2} = 0 \le \lfloor\frac{i+2}{i+1}\rfloor \le \lfloor\frac{a_{q-2}-1}{i+1}\rfloor-1 = \rho_l(i+1,\alpha)-1$. Now put $b_j := \lfloor\frac{a_j-1-c_j(i+1)}{i}\rfloor$ for $j \in [1,q]-\{l\}$ and $b_l := \lfloor\frac{a_l-1-c_l(i+1)-m}{i}\rfloor$. From $a_j-1-c_j(i+1) \ge a_j-1-\rho_j(i+1,\alpha)(i+1) \ge 0$ and $0 \le c_j \le r_j(i,\alpha)$ it is easily seen that $b_j = \rho_j(i,\alpha)-c_j \ge 0$ for $j \in [1,q]-\{l\}$; on the other hand, $a_l-1-c_l(i+1)-m \ge a_l-1-(\rho_l(i+1,\alpha)-1)(i+1)-m = r_l(i+1,\alpha)+i+1-m \ge 2$ together with $0 \le c_l \le r_l(i,\alpha)-m$ yields $b_l = \rho_l(i,\alpha)-c_l$. Define $\tau_j := (i)^{b_j}(i+1)^{c_j}$ for $j \in [1,q]-\{l\}$, $\tau_l := (i)^{b_l}(i+1)^{c_l}(m)$ and consider a realisation $\boldsymbol{T}_j$ of the sequence τ_j in the end $E_j \subseteq A_j$ for $j \in [1,q]$. The remaining vertices of T induce the tree $\tilde{T}$ with $|V(\tilde{T})| = v-\sum_{j=1}^{q}[b_j i+c_j(i+1)]-m = v-\sum_{j=1}^{q}[(\rho_j(i,\alpha)-c_j)i+c_j(i+1)]-m = v-\sum_{j=1}^{q}\rho_j(i,\alpha)i-\sum_{j=1}^{q}c_j-m = v-\sum_{j=1}^{q}\rho_j(i,\alpha)i-(r(i,\alpha)-1) = v-\sum_{j=1}^{q}\rho_j(i,\alpha)i-(\sum_{j=1}^{q}r_j(i,\alpha)-i) = v-\sum_{j=1}^{q}(\rho_j(i,\alpha)i+r_j(i,\alpha))+i = i+1$. As $\sum_{j=1}^{q}c_j = f^{i+1}(\tau')-1$, $(\tilde{T})\prod_{j=1}^{q}\boldsymbol{T}_j$ is a T-realisation of a sequence changeable to $(m)\tau' \sim \tau$. ■

Proposition 20 *If* $q \in [3,4]$, $\alpha = (a_1,\ldots,a_q) \in \mathcal{A}$, $i \in [1,\infty)$ *and* $\tau \in \mathcal{B}_i(\alpha)$, *then the following hold:*

1. $f^i(\tau) \le i$ *if and only if* $1+\sum_{j=1}^{q}(a_j-1) \ge (i+1)^2-r(i+1,\alpha)$.
2. $f^{i+1}(\tau) \le i$ *if and only if* $1+\sum_{j=1}^{q}(a_j-1) \ge r(i,\alpha)(i+1)$.

Proof. Put $v := 1+\sum_{j=1}^{q}(a_j-1)$.

1. If $f^i(\tau) \leq i$, then from $f^i(\tau)i + f^{i+1}(\tau)(i+1) = v \equiv r(i+1,\alpha) \pmod{i+1}$ and $r(i+1,\alpha) \in [1, i+1]$ it follows that $f^i(\tau) = i+1-r(i+1,\alpha)$. As $f^{i+1}(\tau) \geq 1$, we have $\frac{v-(i+1)}{i} \geq \lfloor \frac{v-(i+1)}{i} \rfloor \geq f^i(\tau) = i+1-r(i+1,\alpha)$, and so $v \geq (i+1)^2 - r(i+1,\alpha)i$.

If $v \geq (i+1)^2 - r(i+1,\alpha)i$, put $\lambda_0 := i+1-r(i+1,\alpha) \in [0,i]$ and $\lambda_1 := \frac{v-\lambda_0 i}{i+1} = \frac{v+r(i+1,\alpha)i}{i+1} - i \geq 1$; from $v \equiv r(i+1,\alpha) \pmod{i+1}$ we have $\frac{v+r(i+1,\alpha)i}{i+1} \in \mathbb{Z}$ so that $\lambda_1 \in [1,\infty)$ and $\tau := (i)^{\lambda_0}(i+1)^{\lambda_1} \in \mathcal{B}_i(\alpha)$.

2. If $f^{i+1}(\tau) \leq i$, then from $f^i(\tau)i + f^{i+1}(\tau)(i+1) = v \equiv r(i,\alpha) \pmod{i}$ and $r(i,\alpha) \in [1,i]$ we obtain $f^{i+1}(\tau) = r(i,\alpha)$, hence $\frac{v}{i+1} \geq \lfloor \frac{v}{i+1} \rfloor \geq f^{i+1}(\tau) = r(i,\alpha)$ and $v \geq r(i,\alpha)(i+1)$.

If $v \geq r(i,\alpha)(i+1)$, put $\lambda_1 := r(i,\alpha) \in [1,i]$ and $\lambda_0 := \frac{v-r(i+1,\alpha)}{i} = \frac{v-r(i,\alpha)}{i} - r(i,\alpha) \geq 0$; then $v \equiv r(i,\alpha) \pmod{i}$ yields $\frac{v-r(i,\alpha)}{i} \in \mathbb{Z}$, $\lambda_0 \in [0,\infty)$ and $\tau := (i)^{\lambda_0}(i+1)^{\lambda_1} \in \mathcal{B}_i(\alpha)$. ■

Because of Theorem 12 and Proposition 20, for a star-like tree on v vertices that is not avd it is possible to check this fact in a time $O(v)$. We have written a computer programme to (try to) recognise the admissibility of a sequence $\alpha = (a_1, \dots, a_q) \in \mathcal{A}$ with $q \in [3,4]$. Almost all admissible sequences $\alpha = (a_1,a_2,a_3) \in \mathcal{A}$ the computer has found satisfy the inequality $a_2 - 1 \geq (a_1-3)(a_1-2)$; in such a case, by Theorem 19, the tree $S(\alpha)$ is avd. The only exception is the admissible sequence $(6,10,15)$. Reanalysing the proof of Theorem 19 we see that to verify that the tree $S(6,10,15)$ is avd it is sufficient to show that the sequences $(1)(3)^8(4)$, $(3)^7(4)^2$, $(2)(3)^5(4)^2$, $(1)(3)^4(4)^4$, $(3)^3(4)^5$, $(2,3)(4)^6$, $(1)(4)^7$ are $S(6,10,15)$-realisable. Since any such sequence $(t_1,\dots,t_k)$ admits a set $I \subseteq [1,k]$ with $\sum_{i\in I} t_i = 9$, we are done by using Lemma 5 and Proposition 3.

Moreover, all admissible sequences $\alpha = (2,a_2,a_3,a_4) \in \mathcal{A}$ found so far for which $S(a_2,a_3,a_4)$ is avd, satisfy the inequality $a_3 - 1 \geq (a_2-3)(a_2-2)$, and are therefore avd by Theorem 19.

For $a_1 \in [2,\infty)$ and $a_2 \in [a_1,\infty)$ define $A_3(a_1,a_2) := \{a_3 \in [a_2,\infty) : S(a_1,a_2,a_3) \text{ is avd}\}$ and $A_2(a_1) := \{a_2 \in [a_1,\infty) : A_3(a_1,a_2) \neq \emptyset\}$. From Theorem 8 we know that $A_3(a_1,a_2)$ can be nonempty only if $a_2 \geq 2a_1 - 2$ and that $a_1 \geq 3$ implies $A_3(a_1,a_2) \subseteq [a_1+a_2-1, \lfloor \frac{a_2-2}{a_1-2} \rfloor a_2 + 1 - a_1]$. The set $A_3(a_1,a_2)$ may contain both extremal values a_1+a_2-1 and $\lfloor \frac{a_2-2}{a_1-2} \rfloor a_2 + 1 - a_1$, for example $A_3(3,5) = \{7,8,13\}$. For $a_1 = 3$ and $a_2 = 2k+1$ we have $A_3(3,2k+1) \subseteq [2k+3, 4k^2-3]$; using Theorem 14 it is easy to check that $4k^2-3 \in A_3(3,2k+1)$ for any $k \in [2,\infty)$. It is unclear whether $A_2(a_1) \neq \emptyset$ for every $a_1 \in [2,\infty)$ or at least for infinitely many a_1's. Nevertheless, $A_2 \neq \emptyset$ for any $a_1 \in [2,28]$. Given $a_1 \in [2,28]$ we have computed the lexicographical minimum of the set $\{(a_2,a_3) : (a_1,a_2,a_3) \in \mathcal{A}, S(a_1,a_2,a_3) \text{ is avd}\}$. The results are presented in Table 1.

Further, for $a_2 \in [2,\infty)$ and $a_3 \in [a_2,\infty)$ define $A_4(a_2,a_3) := \{a_4 \in [a_3,\infty) : S(2,a_2,a_3,a_4) \text{ is avd}\}$ and $A_3(a_2) := \{a_3 \in [a_2,\infty) : A_4(a_2,a_3) \neq \emptyset\}$. Because of Theorem 17, the set $A_4(a_2,a_3) \subseteq [a_2+a_3, \lfloor \frac{a_3-2}{a_2-1} \rfloor a_3 - a_2]$ can be nonempty only

a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3
2	2	3	9	92	100	16	705	6326	23	7777	20306
3	4	6	10	110	211	17	991	10882	24	8401	150977
4	6	9	11	145	155	18	1981	25708	25	18851	18875
5	8	12	12	211	222	19	2081	12674	26	23410	1452961
6	10	15	13	577	2942	20	4621	18701	27	25201	722305
7	49	92	14	706	1871	21	5377	7570	28	36863	1916641
8	73	80	15	706	1871	22	5153	41042			

Table 1: Star-like tree $S(a_1, a_2, a_3)$ is avd.

a_2	a_3	a_4	a_2	a_3	a_4	a_2	a_3	a_4	a_2	a_3	a_4
2	5	7	8	145	211	14	1201	13161	20	6579	57541
3	13	16	9	110	211	15	1777	9181	21	12559	138601
4	25	31	10	529	3251	16	2081	6121	22	21253	266137
5	31	57	11	379	1105	17	1981	25708	23	8401	150977
6	73	211	12	1201	4915	18	3601	21737			
7	73	80	13	785	3241	19	4621	18701			

Table 2: Star-like tree $S(2, a_2, a_3, a_4)$ is avd.

if $a_3 \geq 2a_2$. Also here both extremal values can be present in $A_4(a_2, a_3)$, e.g. $A_4(2, 7) = \{9, 17, 25, 33\}$. Analogously as in the case of star-like trees with three arms, given $a_2 \in [2, 23]$ we have computed the lexicographical minimum of the set $\{(a_3, a_4) : (a_2, a_3, a_4) \in \mathcal{A}, S(2, a_2, a_3, a_4) \text{ is avd}\}$ with output in Table 2.

3 General trees

Theorem 21 *If a tree T is avd, it contains at most one important subtree.*

Proof. If there is $n \in [1, \infty)$ such that $T \cong P_n$, then the only important subtree of T can be T itself (if n is odd). Suppose therefore that $\delta(T) \geq 3$ and T has an important subtree. Put $v := |V(T)|$, let $r \in [1, 2]$ be such that $v \equiv r \pmod 2$ and let $k := \frac{v-r}{2}$. Consider a realisation $\boldsymbol{T} = (T_1, \ldots, T_{k+1})$ of the sequence $(r)(2)^k \in \mathrm{Vs}(T)$.

Claim *If $\tilde{T}$ is an important subtree of T, then the set $V(\tilde{T})$ is $\boldsymbol{T}$-exact.*

Proof. Let $\tilde{y}_1, \tilde{y}_2$ be the two endvertices of $\tilde{T}$ and let $\tilde{z}_i$ be the neighbour of $\tilde{y}_i$, $i = 1, 2$. Since $\Delta(T) \geq 3$, we have $\max(\deg_T(\tilde{z}_1), \deg_T(\tilde{z}_2)) \geq 3$ and we may assume without loss of generality that $\deg_T(\tilde{z}_1) \geq 3$. Let T_l be the $\tilde{y}_1$-tree of $\boldsymbol{T}$ and T_m the $\tilde{y}_2$-tree of $\boldsymbol{T}$.

If $t_l = 1$, then $t_m = 2$, the set $V(\tilde{T}) - \{\tilde{y}_1, \tilde{y}_2, \tilde{z}_2\}$ is $\boldsymbol{T}$-exact (its vertices except maybe for $\tilde{z}_1$ are of degree 2 in T), and, consequently, the same is true for $V(\tilde{T})$.

If $(t_l, t_m) = (2, 1)$ and T_n is the $\tilde{z}_2$-tree of $\boldsymbol{T}$, then $V(T_n) \subseteq V(\tilde{T})$ (if $\tilde{z}_2 \neq \tilde{z}_1$, the set $V(\tilde{T}) - \{\tilde{y}_1, \tilde{z}_1, \tilde{y}_2, \tilde{z}_2\}$ is of odd cardinality, so that it cannot be $\boldsymbol{T}$-exact), and hence both $V(\tilde{T}) - (\{\tilde{y}_1, \tilde{z}_1, \tilde{y}_2\} \cup V(T_n))$ and $V(\tilde{T})$ are $\boldsymbol{T}$-exact.

Finally, if $(t_l, t_m) = (2, 2)$, then both $V(\tilde{T}) - \{\tilde{y}_1, \tilde{z}_1, \tilde{y}_2, \tilde{z}_2\}$ and $V(\tilde{T})$ are $\boldsymbol{T}$-exact. ■

Since T has an important subtree, from Claim it follows that $r = 1$ and the unique vertex of T_1 belongs to any important subtree of T. Therefore, T cannot have two vertex-disjoint important subtrees.

Suppose that T has two distinct (but having a common vertex) important subtrees $\tilde{T}$ and $\hat{T}$. Let $\tilde{y}_1, \tilde{y}_2$ be the two endvertices of $\tilde{T}$, $\hat{y}_1, \hat{y}_2$ the two endvertices of $\hat{T}$. Let $\tilde{z}_i$ be the neighbour of $\tilde{y}_i$ and $\hat{z}_i$ the neighbour of $\hat{y}_i$, $i = 1, 2$. Further, let T_m be the $\tilde{y}_1$-tree and T_n the $\hat{y}_1$-tree of $\boldsymbol{T}$ (so that $m \neq n$).

If $\tilde{T}$ and $\hat{T}$ have a common edge that is not pendant, then the sets of non-pendant edges of $\tilde{T}$ and $\hat{T}$ are equal (each non-pendant edge is incident with at least one strongly internal vertex that is of degree 2 in T). Therefore, $|V(\tilde{T})| = |V(\hat{T})|$ and we may assume without loss of generality that $\tilde{z}_1 = \hat{z}_1$ and $\tilde{y}_1 \neq \hat{y}_1$. Since $\{\tilde{y}_1, \hat{y}_1\} \cap (V(\tilde{T}) \cap V(\hat{T})) = \emptyset$, we obtain $t_m = t_n = 2$ and $V(T_m) \cap V(T_n) = \{\tilde{z}_1\} \neq \emptyset$, a contradiction.

If $\tilde{T}$ and $\hat{T}$ have a common pendant edge (but they differ in non-pendant edges), we may suppose without loss of generality that $\tilde{y}_1 = \hat{y}_1$, $\tilde{z}_1 = \hat{z}_1$ and $(V(\tilde{T}) - \{\tilde{y}_1, \tilde{z}_1\}) \cap (V(\hat{T}) - \{\hat{y}_1, \hat{z}_1\}) = \emptyset$ (note that T is a tree). As $V(T_1) \subseteq \{\tilde{y}_1, \tilde{z}_1\}$, we have necessarily $t_m (= t_n) = 1$. Let T_p be the $\tilde{z}_1$-tree of $\boldsymbol{T}$. Then $t_p = 2$ and, using Claim, $V(T_p) \subseteq \{\tilde{z}_1\}$, a contradiction.

If $\tilde{T}$ and $\hat{T}$ have a common vertex, but they are edge-disjoint, that common vertex can only be $\tilde{z}_1$ or $\tilde{z}_2$, so that we may assume without loss of generality that $\tilde{z}_1 = \hat{z}_1$, $\tilde{y}_1 \neq \hat{y}_1$ and $\tilde{y}_2 \neq \hat{y}_2$. Then $V(T_1) = \{\tilde{z}_1\}$, $t_m = 2$, $m \neq 1$ and $V(T_m) \cap V(T_1) = \{\tilde{z}_1\} \neq \emptyset$, a contradiction. ■

Corollary 22 *If a tree T is avd and y is a primary vertex of T, then T has at most two arms of order 2 with primary vertex y.*

Proof. If yy_1, yy_2 and yy_3 are three distinct pendant edges of T, then $T\langle\{y_1, y, y_i\}\rangle$, $i = 2, 3$, are distinct important subtrees of T in contradiction with Theorem 21. ■

A *caterpillar* is a tree in which there is a longest path P (a *spine* of T) such that any vertex either belongs to P or is a neighbour of a vertex of P.

Corollary 23 *If a caterpillar T is avd, then T has at most one vertex of degree 4.*

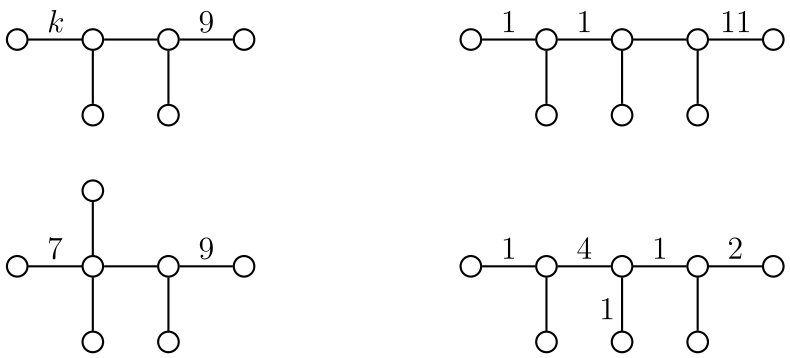

Figure 1: Some avd trees ($k = 2$ or $k = 3$).

Proof. If y, z are distinct vertices of degree 4 in T and yy_i, zz_i, $i = 1, 2$, are four distinct pendant edges in T, then $T\langle\{y_1, y, y_2\}\rangle$ and $T\langle\{z_1, z, z_2\}\rangle$ are distinct important subtrees of T which contradicts Theorem 21. ∎

Let $\tilde{T}$ be an important subtree of a caterpillar T that is avd and is not a path. Let $\tilde{y}_1, \tilde{y}_2$ be the two endvertices of $\tilde{T}$ and let $\tilde{z}_i$ be the neighbour of $\tilde{y}_i$, $i = 1, 2$, $\deg_T(\tilde{z}_1) \geq \deg_T(\tilde{z}_2)$. Then $\tilde{T}$ can be of one of the following three possible types: (i) $\tilde{z}_1 = \tilde{z}_2$ and $\deg_T(\tilde{z}_1) = 4$; (ii) $\deg_T(\tilde{z}_1) = \deg_T(\tilde{z}_2) = 3$; (iii) $\deg_T(\tilde{z}_1) = 3$ and $\deg_T(\tilde{z}_2) = 2$. All three types really do exist. This is illustrated in Fig. 1 where an edge labelled with l is to be subdivided by l vertices of degree 2 and the label k (in the left upper tree) is either 2 or 3. All trees of Fig. 1 are easily seen to be avd. If $k = 3$, the left upper tree of Fig. 1 is an avd caterpillar with no important subtree. We have been informed by Marczyk (see [11]) that there are also trees that are avd, but are neither star-like, nor caterpillars. His example contains two vertices of degree 4.

4 Concluding remarks

Performed computations suggest the following two conjectures:

Conjecture 1 *If a sequence sequence $\alpha = (a_1, a_2, a_3) \in \mathcal{A}$ is admissible, then the tree $S(\alpha)$ is avd.*

Conjecture 2 *If sequences $\alpha = (2, a_2, a_3, a_4) \in \mathcal{A}$ and (a_2, a_3, a_4) are admissible, then the tree $S(\alpha)$ is avd.*

The following problems arise naturally from our analysis:

Problem 1 *Do there exist infinitely many $a_1 \in [2, \infty)$ such that $A_2(a_1) \neq \emptyset$?*

Problem 2 *Do there exist infinitely many $a_2 \in [2, \infty)$ such that $A_3(a_2) \neq \emptyset$?*

Problem 3 *Does there exist a constant c such that any avd tree has at most c vertices of degree four?*

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