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Condorcet winner configurations in the facility location problem

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Abstract. We consider the problem of locating a fixed number of facilities along a line to serve n players. We model this problem as a cooperative game and assume that any locational configuration can be eventually disrupted through a strict majority of players voting for an alternative configuration. A solution of such a voting location problem is called a Condorcet winner configuration. In this paper we state three necessary and one sufficient condition for a configuration to be a Condorcet winner. Consequently, we propose a fast algorithm, which enables us to verify whether a given configuration is a Condorcet winner, and can be efficiently used also for computing the (potentially empty) set of all Condorcet winner configurations.

Keywords: facility location, majority voting, algorithms

JEL classification codes: D7, D71, R53

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1 Introduction

Location problem in general relates to finding an optimal placement of a given number of public facilities to serve customers distributed at points of some space (a plane, a network, a rectilinear polygon). In particular, one can consider a problem of locating hospitals, schools, libraries, post offices or warehouses to serve citizens residing in a street system of a city.

In fact, there is an amount of different models treating this extensive problem (for a nice survey of this topic see Mirchandani and Francis (1990)). A great deal of the literature on location problems studies those situations where k public facilities have to be located on a network (an edge-weighted undirected graph without loops or multiple edges). Further, it is commonly assumed that the customers are located at the vertices of the network, while the facilities are to be located at any point of the network (i.e. at a vertex as well as at an interior point of an edge).

According to the type of the facilities to be located, a decision maker needs to choose an appropriate criterion for an optimal allocation. For example, when locating emergency facilities (such as hospitals, police stations or fire stations) it is reasonable to minimize the maximum travel distance to any customer from his nearest facility (Christofides and Viola,1971; Hakimi, 1964) while in the case of locating obnoxious facilities (e.g. nuclear reactors, chemical factories or garbage dumps) it is more eligible to maximize the minimum distance between the facilities and the citizens (Burkard et al., 1998; Tamir, 1991).

Another frequently considered objective is to minimize the total sum of distances from all customer to their nearest facility (Hakimi, 1964; Maranzana, 1964). This approach involves allocation of various service facilities, such as post offices, supply depots or switching centres in a telephone network. Solutions to this *minisum location problem* are called the k*-medians* of the network (in the case of $k = 1$, the term *median* is commonly used instead of 1-median).

However, there are many practical situations, where customers themselves have to cover their transportation costs to reach the desired public facility. Imagine people travelling from their homes to libraries, sport complexes, or municipal parks. In such circumstances, it seems to be more appropriate to determine the final allocation of the facilities as a result of a customers' collective decision. Given the assumption that each customer prefers to have the facility as close as possible to him, there exists no way in general to meet simultaneously the desires of all customers. In a democratic society, usually a suitable voting procedure is used to find a compromise among several candidate allocations. Perhaps the most popular voting mechanism is the *Condorcet voting rule*. Informally, *a Condorcet winner* of the facility location problem is such an allocation that no alternative allocation is preferred by a strict majority of customers.

As far as we know, there are only a few papers studying *the voting location problem*. In most of them it is supposed that customers vote in order to locate just a single public facility (Bandelt, 1985; Hansen and Thisse, 1981; Labbé, 1985). The main difficulty regarding the voting location models is that a Condorcet winner may fail to exist - the equilateral triangle network with the same number of customers residing at each vertex is a standard counterexample. The existence of a Condorcet winner clearly depends upon the structure of the network as well as upon the distribution of customers across the network.

Surprisingly, it turns out that in many networks, for which the existence of a Condorcet winner is guaranteed with respect to an arbitrary distribution of customers, the set of Condorcet winners equals the set of medians. More precisely, Hansen and Thisse (1981) proved that when the network is a tree, then medians and Condorcet winners coincide. Labbé (1985) extended this result to cactus networks (i.e. networks where no two cycles have more than one vertex in common). In particular, Labbé showed that in a cactus network the set of Condorcet winners is either empty or equals the set of medians of the network. Finally, Bandelt (1985) characterized all those networks, where Condorcet winners and medians coincide - he refers to such networks as to median networks of breadth at most two - examples of such networks are trees and rectangular networks.

In this paper we study the model introduced by Barberà and Beviá (2002), who examined a special voting location model, where k facilities are to be located along a real line. Realize that if $k \geq 2$, a customers' collective decision has to specify not only locations of the facilities, but also a partition of the customers into communities associated with each facility. A pair of k facility locations and corresponding k communities is called *a locational configuration*.

It is assumed that each customer has single peaked preferences over all possible locations of facilities, and when comparing two locational configurations he in fact compares only the locations of facilities he is assigned to in the considered configurations and does not care about the location of the rest of facilities.

Barberà and Beviá (2002, 2006) proved that Condorcet winner configurations satisfy a number of nice properties, whenever they exist. First, every Condorcet winner configuration is *Pareto efficient*, i.e. there exists no configuration unanimously preferred by all the customers (with at least one of them being strictly better off). In particular, this implies that every Condorcet winner configuration is *envy free*, i.e. no player could benefit from joining a different community than the one he was initially assigned to. Further, every Condorcet winner configuration is *Nash stable*, i.e. no player can assure a better outcome by becoming a member of another existing community and influencing the location of that facility in his favor. Finally, every Condorcet winner configuration is *internally consistent*, i.e. all the players inside each community agree, by majority, on the location of their own facility.

Further, Barberà and Beviá (2006) produced two counterexamples showing that 1) there exist instances, already for $k = 2$, with no Condorcet winner configurations and that 2) a k-median configuration need not be a Condorcet winner. Thus, the above mentioned result of Hansen and Thisse (1981) valid for trees when $k = 1$ cannot be extended to the case of $k \geq 2$ already for linear networks.

In our study we focus on the computational aspects of the considered multiple facility location problem. In particular, we aim to develop an efficient algorithm for computing the (possibly empty) set of Condorcet winner configurations. A polynomial algorithm for the problem of locating a single facility on a general network was proposed in Hansen and Labbé (1988).

The organization of the present paper is as follows. In Section 2 we introduce basic definitions and notations. In Section 3 we summarize the existing and prove some additional necessary conditions for a configuration to be a Condorcet winner. The proven propositions enable us to reduce, dramatically, the set of all feasible configurations to a small set of suitable candidates for Condorcet winner configurations.

In Section 4 we proceed to identify an easily checkable condition which is sufficient for a configuration to be a Condorcet winner. Consequently, in Section 5 we describe a fast algorithm for deciding whether a given configuration is a Condorcet winner of a given instance of the facility location problem. As the set of suitable candidates is really small, the proposed algorithm can be efficiently used also for computing the set of all Condorcet winner configurations.

2 Basic Definitions

A society S is composed of n customers, called players, labelled by natural numbers, i.e $S = \{1, 2, \ldots, n\}$. A natural number $k \in \mathbb{N}$ $(k < n)$ stands for the number of facilities, which have to be located along a real line.

A locational configuration, representing a collective decision of players, will be described by a k-tuple of pairs $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)]$, where $x_j \in \mathbb{R}$ for all $j \in \{1, 2, \ldots, k\}$, $x_1 < x_2 < \cdots < x_k$, and $(S_1, S_2, \ldots, S_k)$ is a partition of the society S. We interpret each x_j as the location of the j-th facility and each S_j as the community assigned to the j-th facility.

Let $\mathcal{L}(S,k)$ denote the set of all possible locational configurations feasible for the given society S and the given number of facilities k. Further, given a configuration $L = [(x_1,S_1),(x_2,S_2),\ldots,(x_k,S_k)] \in \mathcal{L}(S,k)$ and a player $i \in S$, the set S_j containing the player i will be denoted by $S(L,i)$ and the corresponding location x_j by $x(L,i)$.

Each player $i \in S$ is supposed to have a complete, reflexive and transitive preference relation $\succeq_i$ over $\mathbb{R}$, representing his preferences over all possible locations of the facilities. If $x \succeq_i y$ and $y \succeq_i x$, we say that player i is indifferent between the locations x and y and write $x \sim_i y$. If, on the other hand, $x \succeq_i y$ but not $y \succeq_i x$, we write $x \succ_i y$ and say that player i strictly prefers x to y.

In location problems it is natural to suppose that each preference relation $\succeq_i$ is **single-peaked**. That is, there exists a real number $p_i \in \mathbb{R}$, called player i's peak, which is the unique best element of $\mathbb{R}$ with respect to $\succeq_i$, and such that for any $x,y \in \mathbb{R}$: if $p_i \geq x > y$ then $x \succ_i y$, and similarly, if $y > x \geq p_i$ then $x \succ_i y$. In order to relate our model to the previously studied voting location models, we assume that all preference relations $\succeq_i$ are, in addition, **symmetric**, i.e $\forall x,y \in \mathbb{R} : x \succeq_i y \Leftrightarrow |x-p_i| \leq |y-p_i|$. In this case, $\succeq_i$ is uniquely determined by a specification of player i's peak p_i.

Further, we suppose that players do not care about community co-members or community sizes. Thus, each player i's preference relation $\sqsupseteq_i$ over the set $\mathcal{L}(S,k)$ is **singleton-based**, i.e. for any $L,L' \in \mathcal{L}(S,k) : L \sqsupseteq_i L' \Leftrightarrow x(L,i) \succeq_i x(L',i)$. Strict preference relation and indifference relation of player i over $\mathcal{L}(S,k)$ will be denoted by $\sqsupset_i$ and $\bowtie_i$, respectively.

Definition 1 An instance (called a game) of **the facility location problem** is given by a triple $G = (S,k,[p_1,p_2,\ldots,p_n])$.

Without loss of generality, we will always suppose that $p_1 \leq p_2 \leq \cdots \leq p_n$, i.e. we suppose that players in S are labelled by numbers $1,2,\ldots,n$ on the basis of the ordering of their peaks. We now define the concept of a Condorcet winner in the context of our special facility location problem.

Definition 2 Let us consider a game $G = (S, k, [p_1, p_2, \ldots, p_n])$. A configuration $L \in \mathcal{L}(S, k)$ is said to be a **Condorcet winner** (CW, for short) of the game G if $|\{i \in S; L \sqsupset_i L'\}| \geq |\{i \in S; L' \sqsupset_i L\}|$, for all $L' \in \mathcal{L}(S, k)$. The set of all Condorcet winner configurations of the game G will be denoted by $CW(G)$.

Example 1 Let us have the following game G_1 with

$$S = \{1, 2, 3, 4, 5, 6, 7\},\ k = 2,\ [p_1, p_2, \ldots, p_7] = [1, 3, 4, 7, 9, 16, 18].$$

We will prove that the configuration $L = [(4, \{1, 2, 3, 4, 5\}), (16, \{6, 7\})] \in \mathcal{L}(S, k)$ is not a Condorcet winner of the game G_1.

Let us consider e.g. the configuration $L' = [(3, \{1, 2, 3\}), (7, \{4, 5, 6, 7\})]$. Players 1 and 2 with peaks $p_1 = 1$ and $p_2 = 3$ prefer L' to L as they prefer $x'_1 = 3$ to $x_1 = 4$. On the other hand, player 3 with the peak $p_3 = 4$ prefers L to L' as he prefers $x_1 = 4$ to $x'_1 = 3$. Further, players 4 and 5 with peaks $p_4 = 7$ and $p_5 = 9$ prefer L' to L as they prefer $x'_2 = 7$ to $x_1 = 4$, while players 6 and 7 with peaks $p_6 = 16$ and $p_7 = 18$ prefer L to L' as they prefer $x_2 = 16$ to $x'_2 = 7$. To summarize, we have $\{i \in S;\ L \sqsupset_i L'\} = \{3, 6, 7\}$ and $\{i \in S;\ L' \sqsupset_i L\} = \{1, 2, 4, 5\}$. Hence, the configuration L' beats the configuration L in a majority voting in the ratio 4 : 3.

Let us now formally define several generally desired properties, which a reasonable collective decision should satisfy.

Definition 3 A configuration $L \in \mathcal{L}(S, k)$ is said to be **Pareto efficient** if there exists no configuration $L' \in \mathcal{L}(S, k)$ such that $L' \sqsupseteq_i L$ for every player $i \in S$ and $L' \sqsupset_l L$ for at least one player $l \in S$.

Definition 4 A configuration $L = [(x_1, S_1), (x_2, S_2), \dots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is said to be **envy-free** if $x(L, i) \succeq_i x_j$, for every player $i \in S$ and every facility $j \in \{1, 2, \dots, k\}$.

Realize, that a configuration which is not envy-free violates also the condition of Pareto efficiency. Hence, every Pareto efficient configuration is also envy-free.

To define the next two plausible conditions let us first introduce a useful notation. Given a set $T \subseteq S$, we denote by $[p_1, p_2, \dots, p_n]_T$ the reduced vector of players' peaks consisting of precisely those p_i for which $i \in T$.

Definition 5 A configuration $L = [(x_1, S_1), (x_2, S_2), \dots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is said to be **internally consistent** if $[(x_j, S_j)] \in CW(S_j, 1, [p_1, p_2, \dots, p_n]_{S_j})$, for every $j \in \{1, 2, \dots, k\}$ such that $S_j \neq \emptyset$.

Definition 6 A configuration $L = [(x_1, S_1), (x_2, S_2), \dots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is said to be **Nash stable** if for every player $i \in S$ and every $j \in \{1, 2, \dots, k\}$ there exists x'_j such that $[(x'_j, S_j \cup \{i\})] \in CW(S_j \cup \{i\}, 1, [p_1, p_2 \dots, p_n]_{S_j \cup \{i\}})$ but $x(L, i) \succeq_i x'_j$.

Let us remark that Nash stability is a little bit stronger requirement than the one asking for envy-free configurations, as it takes into account that player i, when joining a community S_j, may also change in his favor the location of the j-th facility through a majority voting involving players in $S_j \cup \{i\}$.

3 Necessary Conditions

In this section we examine properties of the Condorcet winner configurations in the defined facility location problem. Uppermost, let us briefly deal with the games, where $k = 1$.

Lemma 1 (Hansen and Thisse, 1981) Consider a problem of locating a single public facility on a tree network. Then every Condorcet winner is a median, and conversaly.

Corollary 1 Consider a game $G = (S, 1, [p_1, p_2, \ldots, p_n])$.

a) if $n = 2m + 1$ then there is a unique CW configuration $L = [(p_{m+1}, S)]$

b) if $n = 2m$ then $CW(G) = \{L = [(x, S)] : x \in \langle p_m, p_{m+1}\rangle\}$

In the rest of the paper we study games $G = (S, k, [p_1, p_2, \ldots, p_n])$ whith $k \geq 2$. In what follows we will use the following additional notations and concepts:

- Given a set of real numbers $A = \{a_1, a_2, \ldots, a_r\}$ with $a_1 \leq a_2 \leq \cdots \leq a_r$, we denote by $med(A)$ the median of the set A. More precisely, $med(A)$ will stand for the one-element set $\left\{a_{\frac{r+1}{2}}\right\}$ if r is odd, and the interval $\left\langle a_{\frac{r}{2}}, a_{\frac{r}{2}+1}\right\rangle$ if r is even.

- Given a game $G = (S, k, [p_1, p_2, \ldots, p_n])$ and a set of players $T \subseteq S$, we say that the community T is *connected* if for any two players $l, m \in T$ and any player $i \in S$, the ordering $p_l < p_i < p_m$ implies $i \in T$.

Lemma 2 (Barberà and Beviá, 2006) If a configuration $L \in \mathcal{L}(S, k)$ is a Condorcet winner of the game $G = (S, k, [p_1, p_2, \ldots, p_n])$, then L is Pareto efficient, internally consistent and Nash stable.

Theorem 1 If a configuration $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is a Condorcet winner of $G = (S, k, [p_1, p_2, \ldots, p_n])$, then $x_j \in med(\{p_i; i \in S_j\})$, for every $j \in \{1, 2, \ldots, k\}$.

Proof. Suppose $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is a Condorcet winner. Then, following Lemma 2, L is internally consistent, i.e. for all $j \in \{1, 2, \ldots, k\}$ we have $[(x_j, S_j)] \in CW(S_j, 1, [p_1, p_2, \ldots, p_n]_{S_j})$ (see Definition 5). Now, from Corollary 1 we directly obtain the desired assertion. ■

Theorem 2 If a configuration $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is a Condorcet winner of $G = (S, k, [p_1, p_2, \ldots, p_n])$, then every community S_j, $j \in \{1, 2, \ldots, k\}$, is connected.

Proof. Assume that for some $j \in \{1, 2, \ldots, k\}$ the community S_j is not connected. Then there exist players $l, m \in S_j$ and a player $i \notin S_j$ such that $p_l < p_i < p_m$. Let us denote $S_h = S(L, i)$. Since $i \notin S_j$ necessarily $x_h \neq x_j$.

Suppose first that $x_j < x_h$. If $x_h \leq p_m$ then $x_h \succ_m x_j = x(L, m)$, and so player m wants to move from S_j to S_h, what is a contradiction with L being envy-free. Thus, necessarily $x_h > p_m$. Similarly, if $p_i \leq x_j$ then $x_j \succ_i x_h = x(L, i)$ implying that player i can improve by switching from S_h to S_j, again a contradiction. Hence, we must have $p_i > x_j$. This, together with the previously derived inequality $x_h > p_m$ and the assumption $p_i < p_m$, leads to the following ordering $x_j < p_i < p_m < x_h$.

Further, if $|p_i - x_j| < |p_i - x_h|$ then $x_j \succ_i x_h = x(L, i)$, what again results in a contradiction. Therefore necessarily $|p_i - x_j| \geq |p_i - x_h|$. However, then we obtain $|p_m - x_h| < |p_i - x_h| \leq |p_i - x_j| < |p_m - x_j|$ implying that $x_h \succ_m x_j = x(L, m)$, giving a contradiction.

In a similar manner, we can prove that also in the case $x_j > x_h$, all possible positions of locations x_j and x_h with respect to positions of peaks p_l and p_i imply a contradiction. In particular, if $x_h \geq p_l$ then $x_h \succ_l x_j = x(L, l)$, if $p_i \geq x_j$ then $x_j \succ_i x_h = x(L, i)$, if $|p_i - x_j| < |p_i - x_h|$ then $x_j \succ_i x_h = x(L, i)$, and finally, if $x_h < p_l < p_i < x_j$ and $|p_i - x_j| \geq |p_i - x_h|$ then we obtain $|p_l - x_h| < |p_i - x_h| \leq |p_i - x_j| < |p_l - x_j|$ leading to a contradictory relation $x_h \succ_l x_j = x(L, l)$. ■

Theorem 3 Consider a game $G = (S, k, [p_1, p_2, \ldots, p_n])$ with $p_1 < p_2 < \cdots < p_n$. If $L = [(x_1, S_1), (x_2, S_2), \ldots (x_k, S_k)] \in \mathcal{L}(S, k)$ is a Condorcet winner configuration of G then $\big||S_g| - |S_h|\big| \leq 2$, for all $g, h \in \{1, 2, \ldots, k\}$.

Proof. Let $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)] \in \mathcal{L}(S, k)$ be a CW configuration with $|S_g| - |S_h| \geq 3$ for some $g, h \in \{1, 2, \ldots, k\}$. Let us denote $s = |S_h|$. Then $|S_g| \geq s + 3$.

Let us define configuration $L' = [(x'_1, S'_1), (x'_2, S'_2), \ldots, (x'_k, S'_k)]$ as follows:

- $x'_j = x_j$ and $S'_j = S_j$ for all $j \in \{1, 2, \ldots, k\}, j \neq g, j \neq h$
- $x'_g = x_g - \varepsilon$ and $x'_h = x_g + \varepsilon$, where $\varepsilon > 0$ (sufficiently small)
- $S'_g = \{i \in S_g \cup S_h; x'_g \succeq_i x'_h\}$ and $S'_h = \{i \in S_g \cup S_h; x'_g \prec_i x'_h\}$

First, realize that $L' \bowtie_i L$, for all $i \notin S_g \cup S_h$. Further, since players' peaks are mutually different, $x_g = p_l$ for at most one player $l \in S_g$, and so the rest of players from the set S_g prefer L' to L. Therefore $\{i \in S; L' \sqsupset_i L\} \supseteq S_g - \{l\}$. Consequently, $\{i \in S; L \sqsupset_i L'\} \subseteq S_h \cup \{l\}$. (Notice, that eventually some players in S_h can prefer L' to L.) This implies the following:

$$|\{i \in S; L' \sqsupset_i L\}| \geq |S_g - \{l\}| \geq s + 2 > s + 1 = |S_h \cup \{l\}| \geq |\{i \in S; L \sqsupset_i L'\}|$$

In other words, the defined configuration L' is preferred by a strict majority of players to the considered configuration L, what is a contradiction with the initial assumption that L is a CW configuration. ■

Example 2 Let us consider a game G_2 with $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $k = 3$ and $[p_1, p_2, \ldots, p_{10}] = [1, 2, 3, 11, 12, 13, 15, 16, 23, 24]$. Considering the distribution of players' peaks along the real line, it seems reasonable to partition the set S to $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 5, 6, 7, 8\}$ and $S_3 = \{9, 10\}$, and to locate the facilities to $x_1 = p_2 = 2$, $x_2 = p_6 = 13$ and $x_3 \in \langle p_9, p_{10}\rangle = \langle 23, 24\rangle$.

However, Theorem 3 says that, no matter how players' peaks are distributed along the real line, in no CW configuration of G_2 the sizes of communities can differ by more than two. Indeed, one can easily verify that any configuration $L = [(2, \{1, 2, 3\}), (13, \{4, 5, 6, 7, 8\}), (x_3, \{9, 10\})]$, with $x_3 \in \langle 23, 24\rangle$, will be defeated by the configuration $L' = [(2, \{1, 2, 3\}), (12, \{4, 5, 6\}), (14, \{7, 8, 9, 10\})]$, because four players $4, 5, 7$ and 8 strictly prefer L' to L, while only three players $6, 9$ and 10 strictly prefer L to L'.

Corollary 2 Consider a game $G = (S, k, [p_1, p_2, \dots, p_n])$ with $p_1 < p_2 < \cdots < p_n$. If $L = [(x_1, S_1), (x_2, S_2), \dots, (x_k, S_k)] \in \mathcal{L}(S, k)$ is a Condorcet winner of G and $|S_g| > |S_h|$ for some $g, h \in \{1, 2, \dots, k\}$, then $x_g \in \{p_i; i \in S_g\}$.

Proof. Let $L = [(x_1, S_1), (x_2, S_2), \dots, (x_k, S_k)] \in \mathcal{L}(S, k)$ be a CW configuration with $|S_g| > |S_h|$ for some $g, h \in \{1, 2, \dots, k\}$ and $x_g \notin \{p_i; i \in S_g\}$. Then the configuration $L' = [(x'_1, S'_1), (x'_2, S'_2), \dots, (x'_k, S'_k)]$, defined equally as in the proof of Theorem 3, is strictly preferred to L by at least $|S_g|$ players and is strictly worse than L for at most $|S_h| < |S_g|$ players, a contradiction. ■

To summarize, we have proved that, given any instance of the facility location problem with mutually different players' peaks, every Condorcet winner configuration has to satisfy the following easily checkable conditions:

1. each community is connected
2. the difference between the sizes of any two communities is at most two
3. each facility is located in the median of the peaks of players assigned to this facility
4. if a community has even but not the minimal cardinality, then its facility must be located in some player's peak (i.e. it cannot be located inside the median peaks' interval)

Given a game $G = (S, k, [p_1, p_2, \dots, p_n])$ with $p_1 < p_2 < \cdots < p_n$, the set of those configurations, which satisfy all the above resumed necessary conditions imposed by Theorems 1, 2, 3, and Corollary 2 will be denoted by $\mathcal{L}_{\mathcal{C}}(S, k)$. Realize, that $\mathcal{L}_{\mathcal{C}}(S, k)$ contains only configurations, which still remain suitable candidates for a Condorcet winner, and constitutes really a very small subset of $\mathcal{L}(S, k)$.

Example 3 Let us consider a game G_3 with $S = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ and $[p_1, p_2, \dots, p_7] = [1, 3, 6, 7, 8, 12, 13]$. We will show that $CW(G_3) = \emptyset$.

Following Theorems 1, 2, 3, and Corollary 2, there are only four suitable candidates for CW configurations of the game G_3:

$$L_1 = [(3, \{1, 2, 3\}), (8, \{4, 5, 6, 7\})] \qquad L_3 = [(3, \{1, 2, 3, 4\}), (12, \{5, 6, 7\})]$$
$$L_2 = [(3, \{1, 2, 3\}), (12, \{4, 5, 6, 7\})] \qquad L_4 = [(6, \{1, 2, 3, 4\}), (12, \{5, 6, 7\})]$$

It is easy to see that L_1, L_2 and L_4 are not CW configurations as they are not envy-free. In particular, in L_1 player 3 with the peak $p_3 = 6$ prefers $x_2 = 8$ to $x(L_1, 3) = x_1 = 3$, in L_2 player 4 with the peak $p_4 = 7$ prefers $x_1 = 3$ to $x(L_2, 4) = x_2 = 12$, and finally, in L_4 player 5 with the peak $p_5 = 8$ prefers $x_1 = 6$ to $x(L_4, 5) = x_2 = 12$.

For what remains, configuration L_3 is not a CW configuration, since for configuration $L' = [(6, \{1, 2, 3, 4, 5\}), (13, \{6, 7\})]$ the set $\{i \in S; L' \sqsupset_i L_3\} = \{3, 4, 5, 7\}$ contains more players than the set $\{i \in S; L_3 \sqsupset_i L'\} = \{1, 2, 6\}$.

4 A Sufficient Condition

In this section we inspect those configurations, which are not Condorcet winners, although they belong to $\mathcal{L}_{\mathcal{C}}(S, k)$. As Example 3 illustrates, a configuration may fail to be a Condorcet winner because it is not envy-free. However, it is easy to see, that to verify envy-freenes of a configuration, it is sufficient to assure that the marginal players of the communities cannot improve by moving to the adjacent community, what can be checked in a linear time.

In what follows, we will prove that a configuration $L \in \mathcal{L}_{\mathcal{C}}(S, k)$, which is envy-free but is not a Condorcet winner, is always defeated in a majority voting by such a configuration L^*, that can be obtained from L by shifting of several successive facilities in the same direction.

To simplify many expressions, let us introduce the following useful notations.

Given two locational configurations $L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)]$ and $L' = [(x'_1, S'_1), (x'_2, S'_2), \ldots, (x'_k, S'_k)]$ from the set $\mathcal{L}(S, k)$, we will denote:

$D(L', L) = \{j \in \{1, 2, \ldots, k\} : x'_j \neq x_j\}$ ... the set of facilities with different locations under L' and L

$B(L', L) = \{i \in S : L' \sqsupset_i L\}$ ... the set of players who are strictly better off in L' than in L

$W(L', L) = \{i \in S : L \sqsupset_i L'\}$ ... the set of players who are strictly worse off in L' than in L

Realize that $D(L', L) = D(L, L')$, $B(L', L) = W(L, L')$ and $W(L', L) = B(L, L')$. Following this notation, a configuration L' beats a configuration L by a strict majority if and only if $\big|B(L', L)\big| > \big|W(L', L)\big|$.

Given a configuration L, let us denote by $\mathcal{R}(L)$ the set of those configurations which beat L by a majority, i.e. $\mathcal{R}(L) = \{L' \in \mathcal{L}(S, k) : \big|B(L', L)\big| > \big|W(L', L)\big|\}$. Thus, a configuration L is a Condorcet winner if and only if $\mathcal{R}(L) = \emptyset$.

Definition 7 We say that a configuration $L^* = [(x^*_1, S^*_1), (x^*_2, S^*_2), \ldots, (x^*_k, S^*_k)]$ from the set $\mathcal{R}(L)$ is a **simple rival** of $L \in \mathcal{L}(S, k)$, if

1. $D(L^*, L) \neq \emptyset$ (i.e. at least one facility has changed its location)
2. $D(L^*, L)$ is connected, i.e. if $j_1 < j_2 < j_3$ and $j_1, j_3 \in D(L^*, L)$ then also $j_2 \in D(L^*, L)$, and
3. either $x^*_j > x_j$ for all $j \in D(L^*, L)$, or $x^*_j < x_j$ for all $j \in D(L^*, L)$ (i.e. all the facilities have been shifted in the same direction)

Theorem 4 Consider a game $G = (S, k, [p_1, p_2, \ldots, p_k])$ with $p_1 < p_2 < \cdots < p_n$. Suppose that L is an envy-free configuration from $\mathcal{L}_{\mathcal{C}}(S, k)$. Then $\mathcal{R}(L)$ is either empty or contains a simple rival L^* of L.

Proof. Let us have an envy-free configuration $L \in \mathcal{L}_{\mathcal{C}}(S,k)$. If L is a Condorcet winner of the given game G then $\mathcal{R}(L) = \emptyset$. Otherwise, $\mathcal{R}(L) \neq \emptyset$, and so there exists at least one configuration $L' \in \mathcal{R}(L)$ which beats L by a strict majority. Let us choose $L^* \in \mathcal{R}(L)$ such that $\left|D(L^*,L)\right| \leq \left|D(L',L)\right|$ for all $L' \in \mathcal{R}(L)$. Thus, L^* is a configuration which beats L and differs from L in minimum number of facility locations. Let us remark, that without loss of generality we can suppose that L^* is moreover envy-free, i.e. we can suppose that every player $i \in S$ is assigned in L^* to the facility located most closely to his peak p_i.

We will now prove, by a contradiction, that L^* is a simple rival of L. Suppose it is not. Then either $D(L^*,L) = \emptyset$, or $D(L^*,L)$ is not connected, or there exist indices $j_1, j_2 \in D(L^*,L)$ such that $x^*_{j_1} > x_{j_1}$ but $x^*_{j_2} < x_{j_2}$.

First, if $D(L^*,L) = \emptyset$, then $x^*_j = x_j$ for all $j \in \{1,2,\ldots,k\}$. Since $L^* \in \mathcal{R}(L)$, then $x_j = x^*_j \succ_i x(L,i)$ for some $i \in S$ and some $j \in \{1,2,\ldots,k\}$. However, this is a contradiction with L being envy-free.

Second, if $D(L^*,L)$ is not connected then there exist indices $j_1 < j_2 < j_3$ such that $j_1, j_3 \in D(L^*,L)$ but $j_2 \notin D(L^*,L)$. Thus $x^*_{j_2} = x_{j_2}$. Let us partition the set $B(L^*,L)$ into two disjoint subsets $B_1(L^*,L) = \{i \in B(L^*,L) : p_i < x^*_{j_2}\}$ and $B_2(L^*,L) = \{i \in B(L^*,L) : p_i > x^*_{j_2}\}$. Similarly, we can partition the set $W(L^*,L)$ into disjoint subsets $W_1(L^*,L) = \{i \in W(L^*,L) : p_i < x^*_{j_2}\}$ and $W_2(L^*,L) = \{i \in W(L^*,L) : p_i > x^*_{j_2}\}$. (Realize that if $p_i = x^*_{j_2} = x_{j_2}$ for some $i \in S$ then $L^* \bowtie_i L$, and so $i \notin B(L^*,L)$ and $i \notin W(L^*,L)$.)

Now, using the assumption that $L^* \in \mathcal{R}(L)$, we obtain the following inequality $\left|B_1(L^*,L)\right| + \left|B_2(L^*,L)\right| = \left|B(L^*,L)\right| > \left|W(L^*,L)\right| = \left|W_1(L^*,L)\right| + \left|W_2(L^*,L)\right|$. It is easy to see, that necessarily at least one of the following two inequalities holds: either $\left|B_1(L^*,L)\right| > \left|W_1(L^*,L)\right|$ or $\left|B_2(L^*,L)\right| > \left|W_2(L^*,L)\right|$.

If $\left|B_1(L^*,L)\right| > \left|W_1(L^*,L)\right|$ then we define a new configuration L' as follows: $x'_h = x^*_h$ for all $h \in \{1,2,\ldots,j_2\}$, $x'_h = x_h$ for all $h \in \{j_2+1, j_2+2, \ldots, k\}$, and $S'_h = \left\{i \in S : x'_h \succeq_i x'_j, \text{ for all } j \in \{1,2,\ldots,k\}\right\}$, i.e L' will be envy-free. However, then $\left|B(L',L)\right| = \left|B_1(L^*,L)\right| > \left|W_1(L^*,L)\right| = \left|W(L',L)\right|$, i.e

$L' \in \mathcal{R}(L)$. Moreover $\big|D(L', L)\big| < \big|D(L^*, L)\big|$, what is a contradiction with the assumption that L^* was the configuration from $\mathcal{R}(L)$ with the minimal possible cardinality of the set $D(L^*, L)$.

Similarly, if $\big|B_2(L^*, L)\big| > \big|W_2(L^*, L)\big|$, we can define a configuration L' with $x'_h = x_h$ for all $h \in \{1, 2, \ldots, j_2 - 1\}$, $x'_h = x^*_h$ for all $h \in \{j_2, j_2 + 1, \ldots, k\}$, and $S'_h = \big\{i \in S : x'_h \succeq_i x'_j, \text{ for all } j \in \{1, 2, \ldots, k\}\big\}$. Again, a contradiction arises as $\big|B(L', L)\big| = \big|B_2(L^*, L)\big| > \big|W_2(L^*, L)\big| = \big|W(L', L)\big|$, and moreover $\big|D(L', L)\big| < \big|D(L^*, L)\big|$.

Finally, let us assume that L^* is not a simple rival, because there exist indices $j_1, j_2 \in D(L^*, L)$ such that $x^*_{j_1} > x_{j_1}$ and $x^*_{j_2} < x_{j_2}$. Since we already know that $D(L^*, L)$ is connected, we can without loss of generality suppose that j_1 and j_2 are adjacent facilities. For a simplicity, let us denote $j = min\{j_1, j_2\}$.

Let us now partition the sets $B(L^*, L)$ and $W(L^*, L)$ in the following manner:

$$
\begin{aligned}
B_1(L^*, L) &= \big\{i \in B(L^*, L) : p_i \leq \tfrac{x^*_j + x^*_{j+1}}{2}\big\}, \\
B_2(L^*, L) &= \big\{i \in B(L^*, L) : p_i > \tfrac{x^*_j + x^*_{j+1}}{2}\big\}, \\
W_1(L^*, L) &= \big\{i \in W(L^*, L) : p_i \leq \tfrac{x^*_j + x^*_{j+1}}{2}\big\}, \\
W_2(L^*, L) &= \big\{i \in W(L^*, L) : p_i > \tfrac{x^*_j + x^*_{j+1}}{2}\big\}
\end{aligned}
$$

Again, the fact that the cofiguration L^* belongs to the set $\mathcal{R}(L)$ implies $\big|B_1(L^*, L)\big| + \big|B_2(L^*, L)\big| = \big|B(L^*, L)\big| > \big|W(L^*, L)\big| = \big|W_1(L^*, L)\big| + \big|W_2(L^*, L)\big|$, and thus either $\big|B_1(L^*, L)\big| > \big|W_1(L^*, L)\big|$ or $\big|B_2(L^*, L)\big| > \big|W_2(L^*, L)\big|$.

If $\big|B_1(L^*, L)\big| > \big|W_1(L^*, L)\big|$, then we define the envy-free configuration L' with $x'_h = x^*_h$ for $h \in \{1, 2, \ldots, j\}$, and $x'_h = x_h$ for $h \in \{j + 1, j + 2, \ldots, k\}$. Since $\big|B(L', L)\big| = \big|B_1(L^*, L)\big| > \big|W_1(L^*, L)\big| = \big|W(L', L)\big|$, and $\big|D(L', L)\big| < \big|D(L^*, L)\big|$, we obtain a contradiction.

Similarly, if $\big|B_2(L^*, L)\big| > \big|W_2(L^*, L)\big|$, then we define the envy-free configuration L' with $x'_h = x_h$ for $h \in \{1, 2, \ldots, j\}$, and $x'_h = x^*_h$ for $h \in \{j + 1, j + 2, \ldots, k\}$. This again leads to a contradiction, because $\big|B(L', L)\big| = \big|B_2(L^*, L)\big| > \big|W_2(L^*, L)\big| = \big|W(L', L)\big|$, and $\big|D(L', L)\big| < \big|D(L^*, L)\big|$. ■

As a consequence of Theorem 4, we obtain the following sufficient condition for a Condorcet winner configuration:

Corollary 3 Consider a game $G = (S, k, [p_1, p_2, \ldots, p_n])$ with $p_1 < p_2 < \cdots < p_n$. If L is an envy-free configuration from $\mathcal{L}_{\mathcal{C}}(S, k)$ and there exists no simple rival L^* of L, then the configuration L is a Condorcet winner of the game G.

Example 4 Let us consider a game G_4 with $S = \{1, 2, \ldots, 17\}$, $k = 4$ and

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}
1	3	6	8	9	11	12	14	17	18	20	22	23	25	26	28	30

Let us examine the following envy-free configuration

$$L = [(3, \{1, 2, 3\}), (11, \{4, 5, 6, 7, 8\}), (20, \{9, 10, 11, 12, 13\}), (27, \{14, 15, 16, 17\})].$$

We will illustrate how to check whether there exists a simple rival L^* of L, e.g. with the set $D(L^*, L) = \{2, 3\}$ and with $x_2^* > x_2$, $x_3^* > x_3$.

We aim to find such locations x_2^* and x_3^*, for which the number of players who improve is maximal possible while the number of players who become worse is minimal possible. Clearly, we can assure that $(S_1 \cup S_4) \cap W(L^*, L) = \emptyset$, i.e. players from S_1 and S_4 need not be worse off in L^* than in L. Further, $S_1 \cap B(L^*, L) = \emptyset$ and $S_4 \cap B(L^*, L) \subseteq \{14, 15\}$, because movements of the second and the third facility to the right can help neither players from S_1 nor those players from S_4 with the peaks greater than x_4.

It is easy to see, that already a very small shift of x_2 to the right surely harms players 4, 5 and 6 and a shift of x_3 to the right harms at least player 11, i.e. $W(L^*, L) \supseteq \{4, 5, 6, 11\}$. However these shifts can improve to at least four players from the rest of players, i.e $B(L^*, L) \subseteq \{7, 8, 9, 10, 12, 13, 14, 15\}$ and $|B(L^*, L)| \geq 4$. So the question is: What is the maximum number of players with peaks between x_2 and x_3 (and between x_3 and x_4, respectively), who can simultaneously improve?

Realize, that if players $i \in S_2$ and $m \in S_3$ prefer x_2^* to the original locations of their facilities, then necessarily $x_2^* \in \big(x_2, p_i + (p_i - x_2)\big) \cap \big(p_m - (x_3 - p_m), x_3\big)$. Obviously, the intersection of these two intervals is nonempty if and only if $p_m - (x_3 - p_m) < p_i + (p_i - x_2)$ what is equivalent to $p_m - p_i < \frac{x_3 - x_2}{2} = \frac{9}{2}$. Hence, we cannot simultaneously improve to all three players 7, 8 and 9 because $p_9 - p_7 = 17 - 12 = 5 > \frac{9}{2}$, however we can simultaneously improve to players 8, 9 and 10 as $p_{10} - p_8 = 18 - 14 = 4 < \frac{9}{2}$. This can be achieved by locating the second facility into the interval $(p_{10} - (x_3 - p_{10}), p_8 + (p_8 - x_2))$. Thus $x_2^* \in (16, 17)$.

Similarly, if players $i \in S_3$ and $m \in S_4$ have to prefer x_3^* to the locations of their facilities in L, then necessarily $p_m - p_i < \frac{x_4 - x_3}{2} = \frac{7}{2}$. Thus, we can assure the improvement for players 12, 13 and 14 because $p_{14} - p_{12} = 25 - 22 = 3 < \frac{7}{2}$. In this case, it is sufficient to locate the third facility at $x_3^* \in (23, 24)$.

To summarize, by shifting x_2 to $x_2^* \in (16, 17)$ and x_3 to $x_3^* \in (23, 24)$, players $8, 9, 10, 12, 13$ and 14 improve while players $4, 5, 6, 7$ and 11 become worse. Thus $L^* = [(3, \{1, 2, 3\}), (\frac{33}{2}, \{4, 5, 6, 7, 8, 9, 10\}), (\frac{47}{2}, \{11, 12, 13, 14\}), (27, \{15, 16, 17\})]$ is a simple rival of L.

In what follows, we prove that in order to assure that there exists no simple rival of a configuration L, it is sufficient to examine only those configurations L^*, in which at most one player not belonging to the communities, whose facilities were shifted, strictly prefers L^* to L, i.e.

$$\big|B(L^*, L)\big| + \big|W(L^*, L)\big| \ \leq\ 1 + \sum_{j \in D(L^*, L)} \big|S_j\big|$$

Given a simple rival L^* of a configuration L, let us define an index $next(L^*)$ as follows:

1. if $x_j^* > x_j$ for all $j \in D(L^*, L)$ and $k \notin D(L^*, L)$,
   then $next(L^*) = min\big\{h \in \{1, 2, \dots, k\} : h \notin D(L^*, L)\big\}$
2. if $x_j^* < x_j$ for all $j \in D(L^*, L)$ and $1 \notin D(L^*, L)$,
   then $next(L^*) = max\big\{h \in \{1, 2, \dots, k\} : h \notin D(L^*, L)\big\}$
3. in all other cases, $next(L^*)$ is simply not defined

Lemma 3 Consider a game $G = (S, k, [p_1, p_2, \ldots, p_k])$ with $p_1 < p_2 < \cdots < p_n$ and an envy-free configuration $L \in \mathcal{L}_{\mathcal{C}}(S, k)$. Suppose L^* is a simple rival of L with the defined index $next(L^*)$. If $\left|S_{next(L^*)} \cap B(L^*, L)\right| \geq 2$ then there exists a simple rival L^{**} with $D(L^{**}, L) = D(L^*, L) \cup \{next(L^*)\}$.

Proof. For an abreviation of the indexing, let us denote $g = next(L^*)$. Since $g \notin D(L^*, L)$, necessarily $x_g^* = x_g$. Further, let us distinguish two cases:

1) If $x_j^* > x_j$ for all $j \in D(L^*, L)$ and $k \notin D(L^*, L)$, then we define L^{**} as follows: $x_h^{**} = x_h^*$ for all $h \in \{1, 2, \ldots, k\}, h \neq g$, and $x_g^{**} = x_g^* + \varepsilon$, where ε is a sufficiently small positive real number. Obviously, $D(L^{**}, L) = D(L^*, L) \cup \{g\}$, $D(L^{**}, L)$ is nonempty and connected, and $x_j^{**} > x_j$ for all $j \in D(L^{**}, L)$. Hence, it remains to show that $L^{**} \in \mathcal{R}(L)$.

Realize that $B(L^{**}, L) = B(L^*, L) \cup \{i \in S_g : p_i > x_g^* = x_g\}$ and
$W(L^{**}, L) \subseteq W(L^*, L) \cup \{i \in S_g : p_i \leq x_g^* = x_g\}$.

Since L is internally consistent, i.e. $x_g \in med(\{p_i; i \in S_g\})$, we obtain the following inequalities: $\left|\{i \in S_g : p_i > x_g\}\right| \geq \frac{|S_g|}{2} - 1$ and $\left|\{i \in S_g : p_i \leq x_g\}\right| \leq \frac{|S_g|}{2} + 1$. From the assumption $\left|S_g \cap B(L^*, L)\right| \geq 2$ it follows that at least two players from S_g with $p_i \leq x_g$ does belong to $B(L^*, L) \subseteq B(L^{**}, L)$, and thus they cannot belong to $W(L^{**}, L)$. Therefore $\left|W(L^{**}, L)\right| \leq \left|W(L^*, L)\right| + \frac{|S_g|}{2} + 1 - 2$. Now, using the fact that $L^* \in \mathcal{R}(L)$, we finally obtain that

$$\left|B(L^{**}, L)\right| \geq \left|B(L^*, L)\right| + \frac{|S_g|}{2} - 1 > \left|W(L^*, L)\right| + \frac{|S_g|}{2} - 1 \geq \left|W(L^{**}, L)\right|$$

and so $L^{**} \in \mathcal{R}(L)$.

2) If, on the other hand, $x_j^* < x_j$ for all $j \in D(L^*, L)$ and $1 \notin D(L^*, L)$, we will consider the following configuration L^{**}: $x_h^{**} = x_h^*$ for all $h \in \{1, 2, \ldots, k\}$, $h \neq g$, and $x_g^{**} = x_g^* - \varepsilon$, where ε is again a sufficiently small positive real number. Using similar arguments than in the first case, one can easily prove that the defined configuration L^{**} is the simple rival of L with the desired properties.

■

5 Algorithm CWTEST

Input: $n,\ k,\ p_1 < p_2 < \cdots < p_n,\ L = [(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)]$

Output: answer whether L is a Condorcet winner configuration of the game $G = (S, k, [p_1, p_2, \ldots, p_n])$ or not

Step 1: *(connectivity)*

For each facility $j \in \{1, 2, \ldots, k\}$ compute the bottom player $b_j = min\{i :\ i \in S_j\}$ and the top player $t_j = max\{i :\ i \in S_j\}$ of the community S_j.

If $\exists\, i \in S$ such that $b_j \leq i \leq t_j$ and $i \notin S_j$, then *STOP: "L is not CW".*

Step 2: *(sizes of the communities)*

For each $j \in \{1, 2, \ldots, k\}$ compute the size $s_j = t_j - b_j + 1$ of the community S_j.

Compute $s_{min} = min\big\{s_j : j \in \{1, 2, \ldots, k\}\big\}$, $s_{max} = max\big\{s_j : j \in \{1, 2, \ldots, k\}\big\}$.

If $s_{max} - s_{min} > 2$, then *STOP: "L is not CW".*

Step 3: *(internal consistency)*

For each $j \in \{1, 2, \ldots, k\}$ check whether $x_j \in med(\{p_i; i \in S_j\})$. In particular,

if s_j is odd and $x_j \neq p_{\frac{b_j+t_j}{2}}$ then *STOP: "L is not CW"*

if s_j is even then

if $s_j = s_{min}$ and $x_j \notin \left\langle p_{\frac{b_j+t_j-1}{2}}, p_{\frac{b_j+t_j+1}{2}} \right\rangle$ then *STOP: "L is not CW"*

if $s_j > s_{min}$ and $x_j \notin \left\{ p_{\frac{b_j+t_j-1}{2}}, p_{\frac{b_j+t_j+1}{2}} \right\}$ then *STOP: "L is not CW"*

Step 4: *(envy-freeness)*

For each facility $j \in \{1, 2, \ldots, k-1\}$ check whether its top player t_j wants to move to the facility $j + 1$. In particular,

if $(p_{t_j} - x_j) > (x_{j+1} - p_{t_j})$ then *STOP: "L is not CW"*

For each facility $j \in \{2, \ldots, k\}$ check whether its bottom player b_j wants to move to the facility $j - 1$. In particular,

if $(x_j - p_{b_j}) > (p_{b_j} - x_{j-1})$ then *STOP: "L is not CW"*

Step 5: *(simple rival)*

For each facility $j \in \{1, 2, \ldots, k\}$ compute $l_j = \big|\{i \in S_j : p_i < x_j\}\big|$ and $r_j = \big|\{i \in S_j : p_i > x_j\}\big|$. Further, for each $j \in \{1, 2, \ldots, k-1\}$ compute $f_j = max\,\big\{m - i + 1 : \ (x_j < p_i < p_m < x_{j+1}) \ \text{ and } \ \big(p_m - p_i < \frac{x_{j+1} - x_j}{2}\big)\big\}$.

For each $j \in \{1, 2, \ldots, k-1\}$ and for each $h \in \{j, j+1, \ldots, k\}$ verify whether there exists a simple rival L^* with $D(L^*, L) = \{j, j+1, \ldots, h\}$ and such that the set $B(L^*, L)$ contains at most one player from $S_{next(L^*)}$. In particular,

if $\big(f_j + f_{j+1} + \cdots + f_{h-1} + r_h\big) > \frac{s_j + s_{j+1} + \cdots + s_h}{2}$ then *STOP: "L is not CW"*

if $\big(f_j + f_{j+1} + \cdots + f_{h-1} + r_h\big) = \frac{s_j + s_{j+1} + \cdots + s_h}{2}$ and $(h \neq k)$ and
$\big(p_{b_{h+1}} - p_{t_h - r_h + 1}\big) < \frac{x_{h+1} - x_h}{2}$ then *STOP: "L is not CW"*

if $\big(l_j + f_{j+1} + f_{j+2} + \cdots + f_h\big) > \frac{s_j + s_{j+1} + \cdots + s_h}{2}$ then *STOP: "L is not CW"*

if $\big(l_j + f_{j+1} + f_{j+2} + \cdots + f_h\big) = \frac{s_j + s_{j+1} + \cdots + s_h}{2}$ and $(j \neq 1)$ and
$\big(p_{b_j + l_j - 1} - p_{t_{j-1}}\big) < \frac{x_j - x_{j-1}}{2}$ then *STOP: "L is not CW"*

Step 6: *(positive answer)*

If L passes successfully over all five steps of the algorithm then *"L is CW"*.

Theorem 5 The algorithm *CWTEST* runs in $\mathcal{O}\big(max\{n, k^2\}\big)$ time.

Proof. Let us examine the algorithm *CWTEST* step by step. First, we need $\mathcal{O}\big(|S_j|\big)$ instructions to compute b_j and t_j, and to verify that every player i between b_j and t_j belongs to S_j. This together, summed over all communities, leads to the complexity $\mathcal{O}\big(\sum_{j=1}^{k} |S_j|\big) = \mathcal{O}\big(n\big)$ for Step 1. Further, it is easy to see, that complexity of each of the Steps 2, 3 and 4 is $\mathcal{O}\big(k\big)$.

Finally, let us consider Step 5. Realize, that to compute l_j and r_j we do not need to use a cycle, because these numbers are determined already by x_j and s_j. In particular: $l_j = \frac{s_j - 1}{2}$ if s_j is odd; $l_j = \frac{s_j}{2} - 1$ if s_j is even and $x_j = p_{\frac{b_j + t_j - 1}{2}}$; and $l_j = \frac{s_j}{2}$ if s_j is even and $x_j \neq p_{\frac{b_j + t_j - 1}{2}}$. Similarly, $r_j = \frac{s_j - 1}{2}$ if s_j is odd; $r_j = \frac{s_j}{2} - 1$

if s_j is even and $x_j = p_{\frac{b_j+t_j+1}{2}}$; and $r_j = \frac{s_j}{2}$ if s_j is even and $x_j \neq p_{\frac{b_j+t_j+1}{2}}$. Hence, the complexity of computing all l_j and r_j, for $j \in \{1, 2, \dots, k\}$, is $\mathcal{O}(k)$.

In order to compute f_j we can use the following simple procedure

$i := t_j - r_j + 1; \quad f_j := r_j;$

for $m := b_{j+1}$ to $b_{j+1} + l_{j+1} - 1$ **do**

  if $p_m - p_i < \frac{x_{j+1}-x_j}{2}$ **then** **if** $m-i+1 > f_j$ **then** $f_j := m-i+1$

  else $inc(i)$;

with the complexity $\mathcal{O}(|S_{j+1}|)$. Hence, computing all f_j, $j \in \{1, 2, \dots, k-1\}$, takes $\mathcal{O}(n)$ instructions.

Finally, notice that to verify, whether there exists a special simple rival with $D(L^*, L) = \{j, j+1, \dots, h\}$, only a constant time is necessary. Indeed, all the sums $f_j + f_{j+1} + \cdots + f_h$ as well as $s_j + s_{j+1} + \cdots + s_h$ can be computed sequentially, within the cycles running through j and h (i.e. we do not need additional cycles). The number of simple rival tests equals the number of possible nonempty and connected sets $D(L^*, L)$, i.e. we need $k + k - 1 + \cdots + 2 + 1 = \frac{k.(k+1)}{2}$ tests. Thus, looking for a simple rival takes $\mathcal{O}(k^2)$ instructions.

To summarize, we have showed that the complexity of the algorithm *CWTEST* is $\mathcal{O}(n) + \mathcal{O}(k) + \mathcal{O}(k^2) = \mathcal{O}(max\{n, k^2\})$. ∎

6 Conclusion

In this paper, we have for the first time considered computational questions connected with the multiple facility voting location problem. In particular, we have studied the problem of locating $k \geq 2$ facilities into a linear environment to serve n players.

First, we have explored properties of Condorcet winner configurations. Based on the proven results (Theorems 1, 2, 3, and Corollary 2), the set of suitable candidates for a CW configuration has been significantly reduced . The following table indicates the number of suitable partitions of the society S to k connected communities, for $k = 2, 3, 4, 5$:

k	$p =$ the number of suitable partitions
2	$p = 3$ if $n \equiv 0 \pmod 2$, $p = 2$ if $n \equiv 1 \pmod 2$
3	$p = 7$ if $n \equiv 0 \pmod 3$, $p = 6$ if $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$
4	$p = 19$ if $n \equiv 0 \pmod 4$, $p = 16$ if $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$, $p = 18$ if $n \equiv 2 \pmod 4$
5	$p = 51$ if $n \equiv 0 \pmod 5$, $p = 45$ if $n \equiv 1 \pmod 5$ or $n \equiv 4 \pmod 5$, $p = 35$ if $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$

Hence, the number of candidates expands exponentially with the increasing k, however it practically does not change with the increasing n. Therefore, for many practical situations, where only a few facilities are to be located, one can compute the set of all Condorcet winner configurations simply by examining a small number of suitable candidates.

The second part of the paper has been devoted to identifying an easily verifiable sufficient condition for a configuration to be a Condorcet winner. Consequently, we have proposed the efficient algorithm $CWTEST$ for checking whether a given configuration is a Condorcet winner of the given instance of the considered facility location problem.

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