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**Abstract**

The aim of this paper is to study the qualitative behavior of solutions of nonlinear differential equations of the third order with quasiderivatives. In particular, we give the necessary and sufficient conditions for the existence of nonoscillatory solutions with given asymptotic behavior as $t \to \infty$. These conditions are presented as integral criteria involving only the coefficients of investigated differential equations. In order to prove some of the results, we use a topological approach based on the Schauder fixed point theorem.

**MSC (2000):** 34K11; 34K25
**Key words:** Third order nonlinear differential equation; Nonoscillatory solution; Asymptotic behavior; Quasiderivatives

## 1 Introduction

We consider the third order nonlinear differential equations with quasiderivatives of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(t)) = 0, \quad t \geq a. \tag{N}$$

Throughout the paper, we always assume that

$$r, p, q \in C([a,\infty),\mathbb{R}),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [a,\infty), \tag{H1}$$

$$f \in C(\mathbb{R},\mathbb{R}), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0. \tag{H2}$$

For the sake of brevity, we put

$$x^{[0]} = x,\ x^{[1]} = \frac{1}{r}x',\ x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)',\ x^{[3]} = \left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \left(x^{[2]}\right)'.$$

The functions $x^{[i]}$, i=0, 1, 2, 3, we call the *quasiderivatives* of $x$. In addition to (H1) and (H2), we will sometimes assume that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0. \tag{H3}$$

By a *solution* of an equation of the form $(N)$, we mean a function $w : [a, \infty) \to \mathbb{R}$ such that quasiderivatives $w^{[i]}(t)$, $0 \le i \le 3$, exist and are continuous on the interval $[a, \infty)$ and it satisfies the equation $(N)$ for all $t \ge a$. A solution $w$ of equation $(N)$ is said to be *proper* if it satisfies the condition

$$\sup \{|w(s)| : t \le s < \infty\} > 0 \quad \text{for any } t \ge a.$$

A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to $\infty$; otherwise it is said to be *nonoscillatory.* Furthermore, equation $(N)$ is called *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if all its nontrivial solutions are nonoscillatory.

Let $\mathcal{N}(N)$ denote the set of all proper nonoscillatory solutions of $(N)$. The set $\mathcal{N}(N)$ can be divided into the following four classes in the same way as in [2, 3, 5]:

$$\mathcal{N}_0 = \{x \in \mathcal{N}(N), \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \ge t_x\}$$

$$\mathcal{N}_1 = \{x \in \mathcal{N}(N), \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) < 0 \ \text{ for } t \ge t_x\}$$

$$\mathcal{N}_2 = \{x \in \mathcal{N}(N), \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \ge t_x\}$$

$$\mathcal{N}_3 = \{x \in \mathcal{N}(N), \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) < 0 \ \text{ for } t \ge t_x\}$$

Furthermore, with respect to asymptotic behavior of the solutions in the classes $\mathcal{N}_0 - \mathcal{N}_3$, we can divide the class $\mathcal{N}_0$ $[\mathcal{N}_3]$ into the following two disjoint subclasses

$$\mathcal{N}_0^B = \left\{x \in \mathcal{N}_0 : \lim_{t\to\infty} x(t) = l_x \neq 0\right\}, \quad \mathcal{N}_0^0 = \left\{x \in \mathcal{N}_0 : \lim_{t\to\infty} x(t) = 0\right\}$$

$$\left[\mathcal{N}_3^B = \left\{x \in \mathcal{N}_3 : \lim_{t\to\infty} x(t) = l_x \neq 0\right\}, \quad \mathcal{N}_3^0 = \left\{x \in \mathcal{N}_3 : \lim_{t\to\infty} x(t) = 0\right\}\right]$$

and also the class $\mathcal{N}_1$ $[\mathcal{N}_2]$ into the following two disjoint subclasses

$$\mathcal{N}_1^B = \left\{x \in \mathcal{N}_1 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\}, \ \mathcal{N}_1^\infty = \left\{x \in \mathcal{N}_1 : \lim_{t\to\infty} |x(t)| = \infty\right\}$$

$$\left[\mathcal{N}_2^B = \left\{x \in \mathcal{N}_2 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\}, \ \mathcal{N}_2^\infty = \left\{x \in \mathcal{N}_2 : \lim_{t\to\infty} |x(t)| = \infty\right\}\right].$$

If solution $x \in \mathcal{N}_0$, then its quasiderivatives satisfy the inequality $x^{[i]}(t)x^{[i+1]}(t) < 0$ for $i = 0, 1, 2$, for all sufficiently large $t$. Using the terminology as in [2, 3, 5, 14, 16], we call it a *Kneser solution.*

There are a lot of results (see, e.g., [2, 4, 5, 6, 17]) devoted to the oscillatory and asymptotic behavior of the linear case of equation $(N)$, namely of the linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0, \quad t \geq a.$$

The nonlinear case, i.e. equation $(N)$, has been largely studied in [2, 3, 5, 16]. In particular, many authors investigated the oscillatory and asymptotic properties of solutions of differential equations of the third order with deviating argument. Among the extensive literature on this field, we refer to [10, 11, 12, 14, 15, 18, 19, 20] and to the references contained therein.

The aim of this paper is to continue in the study of equation $(N)$. Particularly, we investigate the asymptotic behavior of nonoscillatory solutions of equation $(N)$. To this aim, we divide all proper nonoscillatory solutions of $(N)$ into the above mentioned several classes with respect to their asymptotic behavior. Such a classification plays an important role in the study of the qualitative behavior of equation $(N)$. Further, we use a topological approach based on the following fixed point theorem:

**Theorem 1.1** *(Schauder theorem) Let $\Omega$ be a non-empty closed convex subset of a normed linear space $E$ and let $T : \Omega \to \Omega$ be a continuous mapping such that $T(\Omega)$ is relatively compact in $E$. Then $T$ has at least one fixed point in $\Omega$.*

After the summarization of some known definitions and notation, in Section 2 we present the necessary and sufficient conditions for the existence of nonoscillatory solutions of equation $(N)$ with a specified asymptotic behavior as $t \to \infty$. These results are interesting in themselves by virtue of their necessary and sufficient character. Furthermore, our results are presented as integral criteria that involve only the functions $p, r, q$. Several examples illustrating the main theorems are also provided.

We point out that our assumption on the nonlinearity $f$ is related with its behavior only in a neighbourhood of infinity. Moreover, not only monotonicity of the nonlinearity $f$ is unnecessary but also no assumptions on the behavior of $f$ in $\mathbb{R}$ are required. We also remark that the condition (H3) is needed only for the class $\mathcal{N}_2$.

We close the introduction with the following notation:

$$I(u_i) = \int_a^\infty u_i(t)\,dt, \qquad I(u_i, u_j) = \int_a^\infty u_i(t) \int_a^t u_j(s)\,ds\,dt, \quad i, j = 1, 2$$

$$I(u_i, u_j, u_k) = \int_a^\infty u_i(t) \int_a^t u_j(s) \int_a^s u_k(z)\,dz\,ds\,dt, \quad i, j, k = 1, 2, 3,$$

where $u_i$, $i = 1, 2, 3$, are continuous positive functions on the interval $[a, \infty)$. For simplicity, we will sometimes write $u(\infty)$ instead of $\lim_{t\to\infty} u(t)$.

# 2 Main results

We begin our consideration with several results concerning the asymptotic behavior of solutions of equation $(N)$ in the class $\mathcal{N}_1$. The following result provides sufficient conditions for the existence of solutions in the class $\mathcal{N}_1^B$.

**Theorem 2.1** *Let one of the following conditions be satisfied:*

*(a)* $I(p,q) < \infty$ *and* $I(r) < \infty$,

*(b)* $I(p,r) < \infty$ *and* $I(q) < \infty$.

*Then equation* $(N)$ *has a bounded solution* $x$ *in the class* $\mathcal{N}_1$*, i.e.* $\mathcal{N}_1^B \neq \emptyset$.

**Proof.** We prove the existence of a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_1$.

Suppose *a)*. Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(v)\, dv\, ds \leq \frac{d-c}{K} \quad \text{and} \quad \int_{t_0}^{\infty} r(s)\, ds \leq 1. \tag{1}$$

Let us define the set

$$\Delta = \{u \in C([t_0,\infty),\mathbb{R}) : c \leq u(t) \leq d\},$$

where $C([t_0,\infty),\mathbb{R})$ will denote the Banach space of all continuous and bounded functions defined on $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|, t \geq t_0\}$. Clearly, $\Delta$ is a non-empty closed, convex and bounded subset of $C([t_0,\infty),\mathbb{R})$. For every $u \in \Delta$ we consider a mapping $T : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (Tu)(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply to the mapping $T$ the Schauder fixed point theorem (Theorem 1.1), it is sufficient to prove that $T$ maps $\Delta$ into itself, $T$ is a continuous mapping in $\Delta$ and $T(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$.

*(i)* $T$ maps $\Delta$ into $\Delta$. In fact, $x_u(t) \geq c$ and in view of (1), we obtain

$$\begin{aligned}
x_u(t) &= c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau \\
&\leq c + K \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\
&\leq c + K \int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\
&\leq c + K \left( \int_{t_0}^{\infty} r(\tau)\, d\tau \right) \left( \int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds \right) \leq d.
\end{aligned}$$

*(ii)* $T$ is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of $\Delta$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Since $\Delta$ is closed, $u \in \Delta$. By the definition of $T$ and in view of (1), we see that

$$|(Tu_n)(t) - (Tu)(t)| \le \int_{t_0}^{\infty} G_n(s)\, ds, \quad t \ge t_0$$

where

$$G_n(s) = p(s) \int_{t_0}^{s} q(z) |f(u_n(z)) - f(u(z))|\, dz.$$

Thus

$$\|Tu_n - Tu\| \le \int_{t_0}^{\infty} G_n(s)\, ds. \tag{2}$$

It is easy to see that $\lim_{n\to\infty} G_n(s) = 0$, which is a consequence of the convergence $u_n \to u$ in $C([t_0, \infty), \mathbb{R})$ and that the following inequality holds

$$\int_{t_0}^{\infty} G_n(s)\, ds \le 2K \int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds.$$

Since $I(p, q) < \infty$, the Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{t_0}^{\infty} G_n(s)\, ds = 0.$$

Consequently, from (2), we have $\lim_{n\to\infty} \|Tu_n - Tu\| = 0$, i.e. $T$ is continuous.

*(iii)* $T(\Delta)$ is relatively compact. It suffices to show that the family of functions $T(\Delta)$ is uniformly bounded and equicontinuous on the interval $[t_0, \infty)$. The uniform boundedness of $T(\Delta)$ immediately follows from the facts that $T(\Delta) \subseteq \Delta$ and $\Delta$ is a bounded subset of $C([t_0, \infty), \mathbb{R})$. Now, we prove that $T(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$. This will be accomplished if we show that for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family $T(\Delta)$ have oscillations less than $\varepsilon$ (see, e.g. [13], p. 13).

Let $u \in \Delta$ and $t_2 > t_1 \ge t_0$. Then, taking into account (1), we have

$$\begin{aligned} |(Tu)(t_2) - (Tu)(t_1)| &\le K \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\ &\le (d - c) \int_{t_1}^{\infty} r(\tau)\, d\tau \to 0 \qquad \text{as } t_1 \to \infty. \end{aligned} \tag{3}$$

We conclude from the above inequalities that for any given $\varepsilon > 0$ there exists $t^* > t_0$ such that for all $u \in \Delta$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon \qquad \text{if } t_2 > t_1 \ge t^*.$$

This shows that the oscillations of all functions of the family $T(\Delta)$ on $[t^*, \infty)$ are less than $\varepsilon$. Now, let $t_0 \le t_1 < t_2 \le t^*$. Then the inequalities (1) and (3) yield

$$|(Tu)(t_2) - (Tu)(t_1)| \le (d-c) \int_{t_1}^{t_2} r(\tau)\, d\tau \le (d-c) M_1 |t_2 - t_1|$$

where $M_1 = \max\{r(\tau) : \tau \in [t_1, t_2]\}$. Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon \quad \text{if} \quad |t_2 - t_1| < \delta.$$

Consequently, the interval $[t_0, \infty)$ can be divided into a finite number of subintervals on which every function of the family $T(\Delta)$ has oscillation less than $\varepsilon$. Therefore $T(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$. Hence $T(\Delta)$ is relatively compact.

Now, the Schauder theorem yields the existence of a fixed point $x \in \Delta$ for the mapping $T$ such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \ge t_0.$$

As

$$x^{[1]}(t) = \int_t^\infty p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds > 0, \quad x^{[2]}(t) = -\int_{t_0}^{t} q(z) f(x(z))\, dz < 0,$$

it is clear that $x$ is a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_1$, i.e. $x \in \mathcal{N}_1^B$.

Suppose *b)*. The proof is the same as in the case *a)* except for some minor changes. Therefore, we omit it. This completes the proof.

□

**Remark 1** We observe that the existence of a negative bounded solution of equation $(N)$ in the class $\mathcal{N}_1$ can be proved by using similar arguments. This fact about negative solution also holds for some next results.

The following example shows the meaning of Theorem 2.1.

**Example 1** We consider the differential equation

$$\Big( (t^2+1) \big( (t^2+1)\, x'(t) \big)' \Big)' + \frac{8t}{(2t^2+1)^2}\, x^2(t) \operatorname{sgn} x(t) = 0\,, \quad t \ge 2. \qquad (4)$$

This is the equation of the form $(N)$ where $r(t) = p(t) = \dfrac{1}{t^2+1}$, $q(t) = \dfrac{8t}{(2t^2+1)^2}$ and $f(u) = u^2 \operatorname{sgn} u$. It is easy to verify that the assumptions of Theorem 2.1 are fulfilled and so equation (4) has a solution in the class $\mathcal{N}_1^B$. One such solution is the function $x(t) = \dfrac{2t^2+1}{t^2+1}$.

For the class $\mathcal{N}_1^B$, we also have the following result.

**Theorem 2.2** *If $I(p,q)=\infty$, then $\mathcal{N}_1^B=\emptyset$.*

**Proof.** Assume that $x\in\mathcal{N}_1^B$. Without loss of generality, we suppose that there exists $T\ge a$ such that $x(t)>0$, $x^{[1]}(t)>0$, $x^{[2]}(t)<0$ for all $t\ge T$. Let $x(\infty)=M_x<\infty$. As $x$ is a positive increasing function and $f$ is a continuous function on the interval $[x(T),M_x]$, there exists a positive constant m such that

$$m=\min\left\{f(u):u\in[x(T),M_x]\right\}. \tag{5}$$

Integrating equation $(N)$ twice in $[T,t]$, we obtain

$$x^{[1]}(t)=x^{[1]}(T)+x^{[2]}(T)\int_T^t p(s)\,ds-\int_T^t p(s)\int_T^s q(k)f(x(k))\,dk\,ds$$

and therefore

$$x^{[1]}(t)<x^{[1]}(T)-\int_T^t p(s)\int_T^s q(k)f(x(k))\,dk\,ds.$$

Using this inequality with (5), we have

$$x^{[1]}(t)<x^{[1]}(T)-m\int_T^t p(s)\int_T^s q(k)\,dk\,ds,$$

which gives a contradiction as $t\to\infty$, because function $x^{[1]}(t)$ is a positive for all $t\ge T$. The case $x(t)<0$, $x^{[1]}(t)<0$, $x^{[2]}(t)>0$ for all $t\ge T^*$ (where $T^*\ge a$) can be treated similarly.

□

From Theorem 2.1 and Theorem 2.2, one gets immediately the following result.

**Corollary 2.1** *Let $I(r)<\infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_1^B$ is that $I(p,q)<\infty$.*

For the solutions in the class $\mathcal{N}_1^\infty$, the following theorem holds.

**Theorem 2.3** *If $I(r)<\infty$, then $\mathcal{N}_1^\infty=\emptyset$.*

**Proof.** Let $x\in\mathcal{N}_1^\infty$. Without loss of generality, we suppose that there exists $T\ge a$ such that $x(t)>0$, $x^{[1]}(t)>0$, $x^{[2]}(t)<0$ for all $t\ge T$. As $x^{[1]}$ is a positive decreasing function, we have that $x'(t)\le x^{[1]}(T)r(t)$ for all $t\ge T$. Integrating this inequality in the interval $[T,t]$, we obtain

$$x(t)\le x(T)+x^{[1]}(T)\int_T^t r(s)\,ds,$$

which gives a contradiction as $t\to\infty$ because $x$ is an unbounded solution. If $x(t)<0$, $x^{[1]}(t)<0$, $x^{[2]}(t)>0$ for all $t\ge T^*$ (where $T^*\ge a$), similar arguments hold.

□

Combining Corollary 2.1 and Theorem 2.3, we get the following.

**Corollary 2.2** *Let $I(r) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_1$ is that $I(p,q) < \infty$.*

Now, we turn our attention to the class $\mathcal{N}_2$. For the existence of solutions of $(N)$ in the class $\mathcal{N}_2^B$, we state the following results.

**Theorem 2.4** *Let one of the following conditions be satisfied:*

*(a)* $I(q,p) < \infty$ *and* $I(r) < \infty$,

*(b)* $I(r,p) < \infty$ *and* $I(q) < \infty$.

*Then equation $(N)$ has a bounded solution $x$ in the class $\mathcal{N}_2$, i.e $\mathcal{N}_2^B \neq \emptyset$.*

**Proof.** We prove the existence of a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_2$.

Suppose *a)*. Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} q(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau \leq \frac{d-c}{K} \quad \text{and} \quad \int_{t_0}^{\infty} r(s)\, ds \leq 1. \tag{6}$$

Let us define the set $\Delta$ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_1 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_1 u)(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

Taking into account (6) and using similar arguments as in the proof of Theorem 2.1, it is easy to verify that $T_1$ maps $\Delta$ into itself, $T_1$ is a continuous mapping in $\Delta$ and $T_1(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$. Consequently, by the Schauder fixed point theorem, there exists a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = \int_{t_0}^{t} p(s) \int_{s}^{\infty} q(z) f(x(z))\, dz\, ds > 0, \quad x^{[2]}(t) = \int_{t}^{\infty} q(z) f(x(z))\, dz > 0,$$

it is clear that $x$ is a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_2$, i.e. $x \in \mathcal{N}_2^B$.

Suppose *b)*. Using similar arguments as in the case *a)*, we are led to the conclusion that $\mathcal{N}_2^B \neq \emptyset$. Therefore, we omit it. The proof is now complete.

$\square$

**Example 2** Let us consider the differential equation

$$\left(\left(t^2+1\right)\left(t^2x'(t)\right)'\right)' + \frac{2\left(t^2-1\right)}{\left(t^2+1\right)^2 \operatorname{arctg}^3 t}\, x^3(t) = 0, \quad t \geq 2. \tag{7}$$

This equation satisfies the conditions of Theorem 2.4. Hence, equation (7) has a solution in the class $\mathcal{N}_2^B$. Really, one such solution is the function $x(t) = \operatorname{arctg} t$.

**Theorem 2.5** *If $I(q) = \infty$, then $\mathcal{N}_2^B = \emptyset$.*

**Proof.** Assume that $x \in \mathcal{N}_2^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Let $x(\infty) = M_x < \infty$. As $x$ is a positive increasing function and $f$ is a continuous function on the interval $[x(T), M_x]$, there exists a positive constant m such that

$$m = \min\left\{f(u) : u \in [x(T), M_x]\right\}. \tag{8}$$

By integrating equation $(N)$ in the interval $[T, t]$, we get

$$x^{[2]}(t) = x^{[2]}(T) - \int_T^t q(s) f(x(s))\, ds.$$

This equality with (8) yields that

$$x^{[2]}(t) \leq x^{[2]}(T) - m \int_T^t q(s)\, ds,$$

which gives a contradiction as $t \to \infty$, because function $x^{[2]}(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way.

□

From Theorem 2.4 and Theorem 2.5, one gets immediately the following result.

**Corollary 2.3** *Let $I(r,p) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_2^B$ is that $I(q) < \infty$.*

The following result also holds.

**Theorem 2.6** *Let (H3) hold. If $I(q) = \infty$, then $\mathcal{N}_2^\infty = \emptyset$.*

**Proof.** Let $x \in \mathcal{N}_2^\infty$. Without loss of generality, we assume that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Because $\left(x^{[2]}(t)\right)' = -q(t)f(x(t)) < 0$ for all $t \geq T$, $x^{[2]}(t)$ is a positive decreasing function and thus $0 \leq x^{[2]}(\infty) < \infty$. As $x(\infty) = \infty$, the assumption (H3) implies that there exists a positive number K and $T_1 \geq T$ such that

$$\frac{f(x(t))}{x(t)} \geq K \qquad \text{for all } t \geq T_1. \tag{9}$$

Integrating equation $(N)$ in the interval $[T_1, t]$, we obtain

$$x^{[2]}(T_1) - x^{[2]}(t) = \int_{T_1}^{t} q(s) f(x(s))\, ds. \tag{10}$$

In view of (9) and the fact that $x$ is an increasing function, the equality (10) gives

$$x^{[2]}(T_1) - x^{[2]}(t) \geq K \int_{T_1}^{t} q(s) x(s)\, ds \geq K x(T_1) \int_{T_1}^{t} q(s)\, ds,$$

which is a contradiction as $t \to \infty$ because $I(q) = \infty$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated similarly. $\square$

Corollary 2.3 and Theorem 2.6 give the following result.

**Corollary 2.4** *Let (H3) hold and $I(r,p) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_2$ is that $I(q) < \infty$.*

In the sequel, we present several results regarding the asymptotic behavior of solutions of equation $(N)$ in the class $\mathcal{N}_3$. The following result gives sufficient condition for the existence of solutions in the class $\mathcal{N}_3^0$.

**Theorem 2.7** *If $I(r,p) < \infty$ and $I(q) < \infty$, then equation (N) has a solution $x$ in the class $\mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) = 0$, i.e. $\mathcal{N}_3^0 \neq \emptyset$.*

**Proof.** We prove the existence of a positive solution of equation $(N)$ in the class $\mathcal{N}_3$ which approaches to zero as $t \to \infty$.

Let $t_0 \geq a$ be such that

$$K \int_{t_0}^{\infty} q(t)\, dt \leq 1, \tag{11}$$

where

$$K = \max\left\{ f(u) : u \in \left[0;\, 2\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau\right]\right\}.$$

For convenience, we make use of the following notation:

$$H(t) = \int_{t}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau, \quad t \geq t_0.$$

Let us define the set

$$\Delta = \{u \in C([t_0,\infty),\mathbb{R}) \,:\, H(t) \leq u(t) \leq 2H(t)\},$$

where $C([t_0,\infty),\mathbb{R})$ denotes the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|,\, t \geq t_0\}$.

Clearly, $\Delta$ is a non-empty closed, convex and bounded subset of $C([t_0,\infty),\mathbb{R})$. For every $u \in \Delta$ we consider a mapping $T_2 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_2u)(t) = H(t) + \int_t^\infty r(\tau)\int_{t_0}^\tau p(s)\int_{t_0}^s q(z)f(u(z))\,dz\,ds\,d\tau, \quad t \ge t_0.$$

In order to apply to the mapping $T_2$ the Schauder fixed point theorem (Theorem 1.1), it is sufficient to prove that $T_2$ maps $\Delta$ into itself, $T_2$ is a continuous mapping in $\Delta$ and $T_2(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$.

*(i)* $T_2$ maps $\Delta$ into $\Delta$. In fact, $x_u(t) \ge H(t)$ and in view of (11), we have

$$\begin{aligned} x_u(t) &= H(t) + \int_t^\infty r(\tau)\int_{t_0}^\tau p(s)\int_{t_0}^s q(z)f(u(z))\,dz\,ds\,d\tau \\ &\le H(t) + K\int_t^\infty r(\tau)\int_{t_0}^\tau p(s)\int_{t_0}^s q(z)\,dz\,ds\,d\tau \\ &\le H(t) + K\left(\int_{t_0}^\infty q(z)\,dz\right)\left(\int_t^\infty r(\tau)\int_{t_0}^\tau p(s)\,ds\,d\tau\right) \le 2H(t). \end{aligned}$$

*(ii)* $T_2$ is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of $\Delta$ such that $\lim_{n\to\infty}\|u_n - u\| = 0$. Since $\Delta$ is closed, $u \in \Delta$. From the definition of $T_2$, we obtain

$$|(T_2u_n)(t) - (T_2u)(t)| \le \int_{t_0}^\infty G_n(\tau)\,d\tau, \quad t \ge t_0$$

where

$$G_n(\tau) = r(\tau)\int_{t_0}^\tau p(s)\int_{t_0}^s q(z)|f(u_n(z)) - f(u(z))|\,dz\,ds.$$

Thus

$$\|T_2u_n - T_2u\| \le \int_{t_0}^\infty G_n(\tau)\,d\tau. \tag{12}$$

It is easy to see that $\lim_{n\to\infty} G_n(\tau) = 0$, which is a consequence of the convergence $u_n \to u$ in $C([t_0,\infty),\mathbb{R})$ and that the following inequality holds

$$\int_{t_0}^\infty G_n(\tau)\,d\tau \le 2K\int_{t_0}^\infty r(\tau)\int_{t_0}^\tau p(s)\int_{t_0}^s q(z)\,dz\,ds\,d\tau.$$

Since $I(r,p,q) < \infty$, the Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty}\int_{t_0}^\infty G_n(\tau)\,d\tau = 0.$$

Consequently, from (12), we have $\lim_{n\to\infty}\|T_2u_n - T_2u\| = 0$, i.e. $T_2$ is continuous.

*(iii)* $T_2(\Delta)$ is relatively compact. It suffices to show that the family of functions

$T_2(\Delta)$ is uniformly bounded and equicontinuous on the interval $[t_0, \infty)$. The uniform boundedness of $T_2(\Delta)$ immediately follows from the facts that $T_2(\Delta) \subseteq \Delta$ and $\Delta$ is a bounded subset of $C([t_0, \infty), \mathbb{R})$. Now, we prove that $T_2(\Delta)$ is an equicontinuous family of functions on the interval $[t_0, \infty)$.

Let $u \in \Delta$ and $t_2 > t_1 \geq t_0$. From the definition of $T_2$, we have

$$\begin{aligned}(T_2u)(t_2) - (T_2u)(t_1) = & - \int_{t_1}^{t_2} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau \\ & - \int_{t_1}^{t_2} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau \end{aligned} \tag{13}$$

and so, taking into account (11), we obtain

$$\begin{aligned}|(T_2u)(t_2) - (T_2u)(t_1)| &\leq H(t_1) + \int_{t_1}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau \\ &\leq H(t_1) + K \left( \int_{t_0}^{\infty} q(t)\, dt \right) \left( \int_{t_1}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau \right) \\ &\leq 2H(t_1).\end{aligned}$$

Since $H(t_1) \to 0$ as $t_1 \to \infty$, for any given $\varepsilon > 0$ there exists $T > t_0$ such that for all $u \in \Delta$, we have

$$|(T_2u)(t_2) - (T_2u)(t_1)| < \varepsilon \qquad \text{if } \ t_2 > t_1 \geq T.$$

This shows that the oscillations of all functions of the family $T_2(\Delta)$ on $[T, \infty)$ are less than $\varepsilon$. Now, let $t_0 \leq t_1 < t_2 \leq T$. Then the equality (13) yields

$$|(T_2u)(t_2) - (T_2u)(t_1)| \leq M_1 |t_2 - t_1| + K M_2 |t_2 - t_1|$$

where

$$\begin{aligned}M_1 &= \max \left\{ r(\tau) \int_{t_0}^{\tau} p(s)\, ds : \tau \in [t_1,\, t_2] \right\}, \\ M_2 &= \max \left\{ r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\, dz\, ds : \tau \in [t_1,\, t_2] \right\}.\end{aligned}$$

Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(T_2u)(t_2) - (T_2u)(t_1)| < \varepsilon \quad \text{if } \ |t_2 - t_1| < \delta.$$

Consequently, we can divide the interval $[t_0, \infty)$ into a finite number of subintervals on which every function of the family $T_2(\Delta)$ has oscillation less than $\varepsilon$. Therefore $T_2(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$ (see, e.g. [13], p. 13). Hence $T_2(\Delta)$ is relatively compact.

Now, according to the Schauder fixed point theorem there exists $x \in \Delta$ such that

$$x(t) = \int_t^\infty r(\tau) \int_{t_0}^\tau p(s)\, ds\, d\tau + \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) f(x(z))\, dz\, ds\, d\tau, \ t \geq t_0.$$

It is clear that $x$ is a positive solution of the equation $(N)$ in the class $\mathcal{N}_3$ which approaches to zero as $t \to \infty$, i.e. $x \in \mathcal{N}_3^0$. This completes the proof.

□

We have the following result for solutions of equation $(N)$ in the class $\mathcal{N}_3$.

**Theorem 2.8** *If $I(r,p) = \infty$, then $\mathcal{N}_3 = \emptyset$.*

**Proof.** The proof is the same as the one of the case IV of Theorem 3.1 in [16] and hence it is omitted.

□

As a consequence of Theorems 2.7 and 2.8, we get the following result.

**Corollary 2.5** *Assume that $I(q) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_3^0$ is that $I(r,p) < \infty$.*

The next results deal with the existence of solutions of $(N)$ in the class $\mathcal{N}_3^B$.

**Theorem 2.9** *If $I(r,p,q) = \infty$, then $\mathcal{N}_3^B = \emptyset$.*

**Proof.** Let $x \in \mathcal{N}_3^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. Let $x(\infty) = l_x > 0$. Integrating equation $(N)$ three times in the interval $[T,t]$, we get

$$\begin{aligned} x(t) = x(T) + x^{[1]}(T) \int_T^t r(s)\, ds + x^{[2]}(T) \int_T^t r(s) \int_T^s p(z)\, dz\, ds \\ - \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau) f(x(\tau))\, d\tau\, dz\, ds. \end{aligned}$$

Thus

$$x(t) < x(T) - \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau) f(x(\tau))\, d\tau\, dz\, ds \qquad \text{for all } t \geq T. \tag{14}$$

The continuity of the function $f(u)$ on the interval $[l_x, x(T)]$ ensures the existence of a positive constant $K$ such that

$$K = \min\left\{ f(u) : u \in [l_x, x(T)] \right\}. \tag{15}$$

The inequality (14) with (15) yields

$$x(t) < x(T) - K \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau)\, d\tau\, dz\, ds \qquad \text{for all } t \geq T.$$

When $t \to \infty$, we get a contradiction because the function $x(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way.

□

**Theorem 2.10** *If $I(r,p,q) < \infty$, then equation (N) has a solution $x$ in the class $\mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) \neq 0$, i.e. $\mathcal{N}_3^B \neq \emptyset$.*

**Proof.** We prove the existence of a positive solution of equation $(N)$ in the class $\mathcal{N}_3$ which approaches to positive constant as $t \to \infty$.

Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \leq \frac{d-c}{K}. \tag{16}$$

Let us define the set $\Delta$ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_3 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_3u)(t) = c + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

Taking into account (16) and using similar arguments as in the proof of Theorem 2.1, it is easy to verify that $T_3$ maps $\Delta$ into itself, $T_3$ is a continuous mapping in $\Delta$ and $T_3(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$. Consequently, the Schauder fixed point theorem ensures the existence of a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = -\int_{t_0}^{t} p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds < 0, \;\; x^{[2]}(t) = -\int_{t_0}^{t} q(z) f(x(z))\, dz < 0,$$

it is clear that $x$ is a positive solution of the equation $(N)$ in the class $\mathcal{N}_3$ which approaches to positive constant as $t \to \infty$, i.e. $x \in \mathcal{N}_3^B$. This completes the proof.

□

Theorem 2.10 is illustrated by the following example.
**Example 3** Let us consider the differential equation

$$\left(\frac{1}{t}\left(t^6 x'(t)\right)'\right)' + \frac{8t}{\operatorname{arctg}\frac{t+1}{t}} \operatorname{arctg} x(t) = 0, \quad t \geq 1. \tag{17}$$

This equation has the form $(N)$ where $f(u) = \operatorname{arctg} u$, $r(t) = 1/t^6$, $p(t) = t$ and $q(t) = \dfrac{8t}{\operatorname{arctg} \frac{t+1}{t}}$. As $I(r,p,q) < \infty$, Theorem 2.10 secures that equation (17) has a solution in the class $\mathcal{N}_3^B$. One such solution is the function $x(t) = \dfrac{t+1}{t}$.

Theorems 2.9 and 2.10 give the following result.

**Corollary 2.6** *A necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_3^B$ is that $I(r,p,q) < \infty$.*

The following also holds.

**Corollary 2.7** *Assume that $I(q) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_3$ is that $I(r,p) < \infty$.*

Finally, we deal with solutions of equation $(N)$ in the class $\mathcal{N}_0$. We state the following results for the existence of solutions of $(N)$ in the class $\mathcal{N}_0^B$.

**Theorem 2.11** *If $I(q,p,r) < \infty$, then equation (N) has a solution $x$ in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) \neq 0$, i.e. $\mathcal{N}_0^B \neq \emptyset$.*

**Proof.** We prove the existence of a positive solution of equation $(N)$ in the class $\mathcal{N}_0$ which approaches to positive constant as $t \to \infty$.

Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} q(z) \int_{t_0}^{z} p(s) \int_{t_0}^{s} r(\tau)\, d\tau\, ds\, dz \leq \frac{d-c}{K}. \tag{18}$$

Let us define the set $\Delta$ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_4 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_4 u)(t) = c + \int_t^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_s^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

Taking into account (18) and using similar arguments as in the proof of Theorem 2.1, it is easy to verify that $T_4$ maps $\Delta$ into itself, $T_4$ is a continuous mapping in $\Delta$ and $T_4(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$. Now, the Schauder fixed point theorem (Theorem 1.1) can be applied to the mapping $T_4$. Hence, there exists a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_t^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_s^{\infty} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

It is clear that $x$ is a positive solution of the equation $(N)$ in the class $\mathcal{N}_0$ which approaches to positive constant as $t \to \infty$, i.e. $x \in \mathcal{N}_0^B$. This completes the proof.

□

**Theorem 2.12** *If $I(q,p,r) = \infty$, then $\mathcal{N}_0^B = \emptyset$.*

**Proof.** Let $x \in \mathcal{N}_0^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Let $x(\infty) = l_x > 0$. From equation $(N)$, it follows that $\left(x^{[2]}(t)\right)' < 0$ for all $t \geq T$. Hence, $x^{[2]}(t)$ is a positive decreasing function. Integrating equation $(N)$ three times in $[t, \infty)$ and taking into account the facts that $0 < x(\infty) < \infty$, $0 \leq x^{[2]}(\infty) < \infty$ and $-\infty < x^{[1]}(\infty) \leq 0$, we obtain

$$x(t) \geq \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(x(z))\, dz\, ds\, d\tau. \tag{19}$$

The continuity of the function $f(u)$ on the interval $[l_x, x(T)]$ ensures the existence of a positive constant $K$ such that

$$K = \min\left\{ f(u) : u \in [l_x, x(T)] \right\}. \tag{20}$$

In view of (19) and (20), we have

$$x(t) \geq K \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z)\, dz\, ds\, d\tau \qquad \text{for all } t \geq T.$$

Hence, by interchanging the order of integration, we get that $I(q,p,r) < \infty$. For the case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$), similar arguments hold.

□

Theorems 2.11 and 2.12 give the following result.

**Corollary 2.8** *A necessary and sufficient condition for equation (N) to have a solution $x$ in the class $\mathcal{N}_0^B$ is that $I(q,p,r) < \infty$.*

**Example 4** The differential equation

$$\left( t^2 \left( \frac{1}{t} x'(t) \right)' \right)' + \frac{6}{(2t+1)^3}\, x^3(t) = 0, \quad t \geq 1 \tag{21}$$

satisfies the condition of Theorem 2.11. Therefore, the equation (21) has a solution in the class $\mathcal{N}_0^B$. In fact, one such solution is the function $x(t) = \dfrac{2t+1}{t}$.

Now, we prove sufficient condition for the existence of solutions of $(N)$ in the class $\mathcal{N}_0^0$.

**Theorem 2.13** *If $I(p,r) < \infty$ and $I(q) < \infty$, then equation (N) has a solution $x$ in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = 0$, i.e. $\mathcal{N}_0^0 \neq \emptyset$.*

**Proof.** We prove the existence of a positive solution of equation $(N)$ in the class $\mathcal{N}_0$ which approaches to zero as $t \to \infty$.

Let $t_0 \geq a$ be such that

$$K \int_{t_0}^{\infty} q(t)\, dt \leq 1, \tag{22}$$

where

$$K = \max\left\{ f(u) : u \in \left[0;\, 2\int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau\right]\right\}.$$

We note that $\displaystyle\int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau = \int_{t_0}^{\infty} p(s) \int_{t_0}^{s} r(\tau)\, d\tau\, ds.$
For convenience, we make use of the following notation:

$$H(t) = \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau, \qquad t \geq t_0.$$

Let us define the set

$$\Delta = \{u \in C([t_0, \infty), \mathbb{R}) \,:\, H(t) \leq u(t) \leq 2H(t)\},$$

where $C([t_0, \infty), \mathbb{R})$ denotes the Banach space of all continuous and bounded functions defined on the interval $[t_0, \infty)$ with the sup norm $\|u\| = \sup\{|u(t)|,\, t \geq t_0\}$. Clearly, $\Delta$ is a non-empty closed, convex and bounded subset of $C([t_0, \infty), \mathbb{R})$. For every $u \in \Delta$ we consider a mapping $T_5 : \Delta \to C([t_0, \infty), \mathbb{R})$ given by

$$x_u(t) = (T_5 u)(t) = H(t) + \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply the Schauder fixed point theorem to the mapping $T_5$, it is sufficient to prove that $T_5$ maps $\Delta$ into itself, $T_5$ is a continuous mapping in $\Delta$ and $T_5(\Delta)$ is a relatively compact set in $C([t_0, \infty), \mathbb{R})$.

*(i)* $T_5$ maps $\Delta$ into $\Delta$. In fact, $x_u(t) \geq H(t)$ and in view of (22), we have

$$\begin{aligned}
x_u(t) &= H(t) + \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau \\
&\leq H(t) + K \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z)\, dz\, ds\, d\tau \\
&\leq H(t) + K \left(\int_{t}^{\infty} q(z)\, dz\right) \left(\int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau\right) \\
&\leq H(t) + K \left(\int_{t_0}^{\infty} q(z)\, dz\right) \left(\int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau\right) \leq 2H(t).
\end{aligned}$$

*(ii)* $T_5$ is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of $\Delta$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Since $\Delta$ is closed, $u \in \Delta$. The definition of $T_5$ yields that

$$|(T_5u_n)(t) - (T_5u)(t)| \le \int_{t_0}^{\infty} G_n(z)\,dz, \quad t \ge t_0$$

where

$$G_n(z) = q(z)|f(u_n(z)) - f(u(z))| \int_{t_0}^{z} p(s) \int_{t_0}^{s} r(\tau)\,d\tau\,ds.$$

Thus, we have the following

$$\|T_5u_n - T_5u\| \le \int_{t_0}^{\infty} G_n(z)\,dz. \tag{23}$$

It is obvious that $\lim_{n\to\infty} G_n(z) = 0$ and

$$\int_{t_0}^{\infty} G_n(z)\,dz \le 2K \int_{t_0}^{\infty} q(z) \int_{t_0}^{z} p(s) \int_{t_0}^{s} r(\tau)\,d\tau\,ds\,dz.$$

Since $I(q,p,r) < \infty$, applying the Lebesgue's dominated convergence theorem, we obtain from (23) that $\lim_{n\to\infty} \|T_5u_n - T_5u\| = 0$ which means that $T_5$ is continuous.

*(iii)* $T_5(\Delta)$ is relatively compact. It suffices to show that the family of functions $T_5(\Delta)$ is uniformly bounded and equicontinuous on $[t_0,\infty)$. The uniform boundedness of $T_5(\Delta)$ follows from the facts that $T_5(\Delta) \subseteq \Delta$ and $\Delta$ is a bounded subset of $C([t_0,\infty),\mathbb{R})$. Now, we prove that $T_5(\Delta)$ is an equicontinuous family of functions on $[t_0,\infty)$.

Let $u \in \Delta$ and $t_2 > t_1 \ge t_0$. From the definition of $T_5$, we have

$$\begin{aligned}(T_5u)(t_2) - (T_5u)(t_1) = &- \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s)\,ds\,d\tau \\ &- \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\,dz\,ds\,d\tau \end{aligned} \tag{24}$$

and so, taking into account (22), we obtain

$$\begin{aligned}|(T_5u)(t_2) - (T_5u)(t_1)| &\le H(t_1) + \int_{t_1}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\,dz\,ds\,d\tau \\ &\le H(t_1) + K \left(\int_{t_1}^{\infty} q(z)\,dz\right) \left(\int_{t_1}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\,ds\,d\tau\right) \\ &\le 2H(t_1).\end{aligned}$$

Since $H(t_1) \to 0$ as $t_1 \to \infty$, for any given $\varepsilon > 0$ there exists $T > t_0$ such that for all $u \in \Delta$, we have

$$|(T_5u)(t_2) - (T_5u)(t_1)| < \varepsilon \qquad \text{if } t_2 > t_1 \ge T. \tag{25}$$

Now, let $t_0 \le t_1 < t_2 \le T$. The equality (24) and the facts that $I(p) < \infty$ and $I(q,p) < \infty$ (it follows from $I(q) < \infty$ and $I(p,r) < \infty$) yield

$$|(T_5 u)(t_2) - (T_5 u)(t_1)| \le M_1 |t_2 - t_1| + K M_2 |t_2 - t_1|$$

where

$$M_1 = \max\left\{ r(\tau) \int_\tau^\infty p(s)\, ds : \tau \in [t_1,\, t_2] \right\},$$
$$M_2 = \max\left\{ r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z)\, dz\, ds : \tau \in [t_1,\, t_2] \right\}.$$

Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(T_5 u)(t_2) - (T_5 u)(t_1)| < \varepsilon \quad \text{if} \quad |t_2 - t_1| < \delta. \tag{26}$$

In view of (25) and (26), we are able to decompose the interval $[t_0, \infty)$ into a finite number of subintervals on which every function of the family $T_5(\Delta)$ has oscillation less than $\varepsilon$. It follows that $T_5(\Delta)$ is relatively compact.

Now, according to the Schauder fixed point theorem there exists $x \in \Delta$ such that

$$x(t) = H(t) + \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \ge t_0.$$

It is clear that $x$ is a positive solution of the equation $(N)$ in the class $\mathcal{N}_0$ which approaches to zero as $t \to \infty$, i.e. $x \in \mathcal{N}_0^0$. The proof is now complete.

□

**Remark 2** Similar investigation of the asymptotic behavior of solutions of the second order differential equations

$$(r(t)x'(t))' + q(t)x(t) = 0 \quad \text{and} \quad (r(t)x'(t))' + q(t)f(x(t)) = 0, \quad t \ge a$$

where $r, q, f$ satisfy (H1), (H2), has been given in [7, 9] and [8, 9], respectively. We also refer the reader to [1, 13] for other results on this topic.

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