On Coalition Formation Games *

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Abstract. In this paper we give an overview of various methods used to study cooperation within a set of players. Besides the classical games with transferable utility and games without transferable utility, recently new models have been proposed: the coalition formation games. In these, each player has his own preferences over coalitions to which he could belong and the quality of a coalition structure is evaluated according to its stability. We review various definitions of stability and restrictions of preferences ensuring the existence of a partition stable with respect to a particular stability definition. Further, we stress the importance of preferences over sets of players derived from preferences over individuals and review the known algorithmic results for special types of preferences derived from the best and/or the worst player of a coalition.

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1 Introduction

In many social, economic and political situations individuals prefer to carry out their activities in groups (coalitions) rather than on their own. Simple examples of coalitions from everyone's everyday life are families, households, social and sport clubs, firms, trade unions, research networks or political parties. Other important examples are various international agreements between countries, such as the European Union (EU), or the North Atlantic Treaty Organisation (NATO). Global environmental problems, such as the greenhouse effect, the ozone layer, or transboundary pollution urgently calling for a solution, lead to signing various international environmental agreements, which can be also considered as coalitions of countries.

These and many other situations can be modelled as coalition formation games. Already the founders of game theory, von Neumann and Morgenstern, recognised the central role and importance of coalition formation, and, indeed, a part of their monumental work Theory of Games and Economic Behaviour (1944) is devoted to a formal analysis of this topic. Their study of coalition formation was conducted mainly within the framework of games in characteristic function form. A game in characteristic function form is given by a set of players and a real-valued function, specifying for each coalition of players the total amount of utility that its members can jointly guarantee themselves and which can be transferred without loss between them. These games are usually called *n-person games with transferable utility* (TU games, for short).

Definition 1 *A* **TU game** *is given by a pair* (N, v)*, where* $N = \{1, 2, \ldots, n\}$ *denotes the set of players and* $v : 2^N \to R$ *is* **the characteristic function***, assigning to every coalition* $S \subseteq N$ *a value* $v(S)$*, representing the total payoff to this group of players when they cooperate. By convention* $v(\emptyset) = 0$*.*

There are many approaches used to study TU games. Early works on coalition formation supposed that the game is superadditive in the sense that any two disjoint coalitions, when acting together, can get at least as much as they can when acting separately. In such situations there are good reasons to expect the formation of the grand coalition N. Therefore these works focused on describing plausible ways of distributing the gain available to the grand coalition to individuals. Whatever subcoalitions can achieve independently has been viewed more as a bargaining power rather than as a possible outcome of the game itself. Among various solution notions, that have been defined for superadditive games, are the core, the kernel, the nucleolus, the Shapley value, von Neumann-Morgenstern solutions, various bargaining sets, and others.

Nevertheless, many situations are not superadditive. Aumann and Drèze (1974) warn that in some cases "acting together may be difficult, costly or illegal, or the players may for various personal reasons not wish to do so". A nice explanation of the formation of subgroups can be found in Demange (1994), who describes two opposing forces. First, the increasing power of coalitions which incites to cooperate; second, the heterogeneity of agents which leads to the formation of subcoalitions. Thus, for example, individuals form communities in order to share costs of the production of local public goods, political parties form coalitions in order to get more votes, countries sign international environmental agreements in order to regulate cooperatively cross-border pollution, etc. But on the other hand, a public good if used by many consumers may be far from some of them or it may be a subject of a congestion; or, in politics, forming a large coalition often means to agree on a neutral candidate who finally does not satisfy any voter. Thus, as a result, those who are close together either in their location, their revenues or in their tastes form subcoalitions. Therefore, given an n-person cooperative game, there are two fundamental questions that need to be answered: (1) Which coalitions can be expected to form? and (2) How will the players of coalitions that are actually formed apportion their joint profit?

Shenoy (1979) emphasises that these two aspects of coalitional behaviour are closely related. On one hand, the final allocation of payoffs to the players depends on the coalitions that finally form and, on the other hand, coalitions that finally form depend on the payoffs available to each player in each of these coalitions. Thus, the payoffs influence the coalition structure and vice versa.

However, most of the research in this field has been concerned with predicting players' payoffs while supposing that coalition structure is given exogenously. Exogenously given coalition structures were perhaps first studied in the context of the bargaining set (Aumann and Maschler(1964)), and subsequently in many contexts; a general treatment may be found in Aumann and Drèze (1974).

Among the first attempts to consider endogenous coalition structures was the model used by Hart and Kurz (1983). Their theory combines two kinds of game theoretic concepts: value and stability. They first evaluate the players' prospects in various coalition structures and then, based on these "values", they try to find which ones are stable. Thus, the coalition formation game in their model can be seen as a two-stage process. In the first stage, the players form coalitions and in the second stage they engage in a noncooperative game and their payoffs are determined according to the "value" that is defined with respect to the coalition structure that emerged in the first stage. Similar approach can be found also in Belleflamme (2000), Ray and Vohra (1999), Yi (1997) and others.

There are only few approaches that simultaneously provide answers to the question of payoff distribution as well as to the question of coalition formation. Among these is the concept of a bargaining aspiration outcome in Bennett and Zame (1988), the bargaining set defined in Zhou (1994), or C-solution from Gerber (2000).

In all the previously mentioned models it is supposed that the utility is freely transferable from one player to another. This is, in particular, possible in the presence of "ideal money", i.e. commodity whose utility is directly proportional to quantity, and independent of any other assets, which a player may have. In general, unfortunately, the situation is not so simple - players' utility for money may be not linear, it may depend on other assets of players, or, in some cases, side payments may even be forbidden. In such situations it is better to represent each coalition's possibilities not by a single number, but rather by a set of all payoff vectors, which the coalition can obtain for its members. We then speak about *n-person games with nontransferable utility* (NTU games, for short), or some authors use the concept of *games without side payments*.

Definition 2 *An* **NTU game** *is given by a pair* (N, V)*, where* $N = \{1, 2, \dots, n\}$ *is the set of players and* V *is the payoff map assigning to each coalition* $S \subseteq N$ *a subset* $V(S)$ *of* R^S *such that* $V(\emptyset) = \emptyset$ *and for all* $S \subseteq N, S \neq \emptyset$*:*

a) $V(S)$ *is a nonempty, closed and convex subset of* R^S

b) $V(S)$ *is comprehensive, i.e. if* $x \in V(S)$ *and* $y \leq x$ *then* $y \in V(S)$

c) $V(S) \cap R^S_+$ *is bounded.*

NTU games generalize TU games, in the sense that every TU game (N, v) can be reformulated as an NTU game by defining $V(S) = \{x \in R^S; \sum_{i \in S} x_i \leq v(S)\}$ for all $S \subseteq N, S \neq \emptyset$. NTU games are for example used in analysis of exchange or production economies or markets (Arrow and Debreu (1954)).

However, in a number of situations it is possible to specify neither a precise value for each coalition, nor a set of feasible payoffs for each player in each coalition he could belong to. Imagine people working in groups on some projects, individuals joining social or sport clubs, children competing in teams .... There are no measurable payoffs for players, rather the benefit of a player is his membership in a coalition itself. The game is then given by a finite set of players and their personal preferences for membership in specific coalitions. We speak about *coalition formation games* (CFG for short). A feasible allocation in such a game is a partition of players, i.e. it is supposed that each player belongs to one and only one coalition. Quality of a partition is evaluated on the basis of stability. Usually, desired are stable partitions, which are more likely to prevail.

Definition 3 *A* **coalition formation game** *is given by a pair* $(N, \mathcal{P})$*, where* $N = \{1, 2, \ldots, n\}$ *is the set of players, and* $\mathcal{P} = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ *denotes the preference profile, specifying for each player* $i \in N$ *his preference relation* $\succeq_i$*, i.e. a reflexive, complete and transitive binary relation on set* $\mathcal{N}_i = \{S \subseteq N : i \in S\}$*.*

Strict preference relation of a player i, and indifference relation of a player i are usually denoted by $\succ_i$ and $\sim_i$, respectively (i.e. $S \succ_i T$ if $[S \succeq_i T$ and not $T \succeq_i S]$ and $S \sim_i T$ if $[S \succeq_i T$ and $T \succeq_i S]$).

It is of course possible to think of situations where a player evaluates a coalition structure as a whole, but majority of works assume that player i only formulates his preferences over $\mathcal{N}_i$. Such coalition formation games are called *hedonic*. This terminology follows Drèze and Greenberg (1980), who introduced the hedonic aspect in players' preferences in a context concerning local public goods. Purely hedonic games were perhaps first studied in Bogomolnaia and Jackson (2002).

The purpose of this paper is to survey results obtained for various models of hedonic CFG. The organization of the paper is as follows. Section 2 deals with various stability concepts used in the context of hedonic CFG. Section 3 surveys existing results regarding various restrictions posed on players' preferences in order to guarantee the existence of a stable partition. Section 4 stresses the importance of considering also the computational complexity of studied problems and explains the basic complexity concepts. Section 5 describes methods of extending players' preferences over individuals to their preferences over sets of players and Section 6 is devoted to algorithmic results concerning the existence of stable partitions with preferences over sets derived from the best and/or the worst player of a set.

2 Stability

2.1 Basic stability concepts

There are four main stability concepts studied in the hedonic games literature - *Nash stability, individual stability, contractual individual stability* and *core stability.* The first three notions are used in models where a partition is considered to be stable if it is immune to individual deviation. The last notion is used in models where immunity to coalition deviation is required.

There is a number of practical situations where it is useful to consider models in which only a single individual can change the existing partition by skipping from his current coalition to another existing one. For example, an unsatisfied professor considers moving to another university rather than establishing a new university, as well as a worker may decide to change his employer rather than found his own firm. Similarly, an individual may change a social club, which he belongs to, rather than start a new one, and a soccer player considers changing his team rather than creating a new sport club.

Thus allowing only individual deviations seems suitable whenever the individual is small relative to the size of existing coalitions or the cost of coordinating movements to form a new coalition is high, or there are some other economic restrictions for building new coalitions.

The simplest concept of stability was inspired by the classical notion of Nash equilibrium and introduced to purely hedonic games by Bogomolnaia and Jackson (2002):

Definition 4 *A partition is said to be* **Nash stable** *if no player can benefit from moving from his coalition S to another existing coalition T.*

The concept of individual stability (Bogomolnaia and Jackson (2002)) is based on the notion of "individually stable equilibrium" from a TU game model by Drèze and Greenberg (1980) but it is modified to apply to the purely hedonic setting where no allocation of goods needs to be considered.

Definition 5 *A partition is said to be* **individually stable** *if no player can benefit from moving from his coalition S to another existing coalition T (T may be empty) while not making the members of T worse off.*

Finally, the concept of contractual individual stability, introduced also by Bogomolnaia and Jackson (2002), is based on the notion of "individually stable contractual equilibrium" adapted also from Drèze and Greenberg (1980).

Definition 6 *A partition is said to be* **contractually individually stable** *if no player can benefit from moving from his coalition S to another existing coalition T (T may be empty) while making neither the members of S nor the members of T worse off.*

Obviously, each Nash stable partition is also individually stable and each individually stable partition is also contractually individually stable. Thus Nash stability is a stronger stability concept than individual stability since Nash stable partitions are immune even to those movements of individuals when a player who wants to change does not need permission to join an existing coalition.

On the other hand, contractual individual stability is a weaker stability concept than individual stability because it ensures only immunity to those movements of individuals where an unsatisfied player needs a permission of the new coalition to join as well as a permission of his original coalition to leave.

Each of these stability concepts can be linked to different institutional arrangements encountered in the real world. Thus, Nash stability analysis is relevant in situations where no permission is required to join or to leave a coalition. Examples, one can have in mind, include people moving from one city to another or individuals moving from one social club to another one. In contrast, individual stability is the most appropriate notion in any situation where a firm or other business or social entity hires an individual. Consider, for example, university departments, which may be viewed as coalitions of professors who are allowed to move when they receive an attractive offer (i.e. when the move is beneficial to the professor and to the department, which he joins, no matter whether the department, which he leaves, loses or gains). Finally, contractual individual stability can be used in the case where the unsatisfied individual (employee) has first to break his contract with his current coalition (employer). For instance, soccer teams typically "own" their players who are not allowed to move to another team (coalition) unless a proper compensation is paid.

The last and the most commonly used stability concept in wide coalition formation literature is the core stability, which extends individual stability to group stability. Allowing coalition deviations makes sense when players are able to coordinate their actions, i.e. if the set of players is not so large, and there is a possibility to communicate and to form new coalitions. Examples include people working in groups on some projects, children forming competition teams, groups of graduates planning to start their own firms, countries forming military coalitions, etc.

Given a partition of players $\mathcal{M}$ and a player $i \in N$, we will denote by $M(i)$ the coalition to which player i belongs in the partition $\mathcal{M}$. The classical definition of core stability (see Banerjee, Konishi and Sönmez (2001), Alcalde and Revilla (1999), Alcalde and Romero-Medina (2000), Cechlárová and Hajduková (1999, 2002, 2003, 2004)) is as follows:

Definition 7 *We say that a coalition $S \subseteq N$* **blocks** *a partition $\mathcal{M}$, if each player $i \in S$ strictly prefers the new coalition S to his current coalition $M(i)$ in the partition $\mathcal{M}$. A partition which admits no blocking coalition is said to be* **core stable***.*

This definition requires that each player from the blocking coalition S strictly improves his situation. However, in some cases it can be sufficient for S to be blocking even when at least one player from S improves and the others are not worse off than before. This is an idea of strong core stability (Roth and Postelwaite (1977)):

Definition 8 *We say that a coalition $S \subseteq N$* **weakly blocks** *a partition $\mathcal{M}$, if each player $i \in S$ either strictly prefers S to $M(i)$ or is indifferent between S and $M(i)$ and there exists at least one player $j \in S$ who strictly prefers S to his current coalition $M(j)$. A partition which admits no weakly blocking coalition is said to be* **strongly core stable***.*

Obviously, each strongly core stable partition is also core stable. Moreover, definitions 5 and 8 imply that each strongly core stable partition is also individually stable. The following example examines further relations between the defined concepts of stability:

Example 1 Consider the game $(N, \mathcal{P})$ with $N = \{1, 2, 3, 4\}$ and the following preference profile:

$$\{1,2\} \sim_1 \{1,2,3\} \succ_1 \{1,3\} \succ_1 \{1,4\} \succ_1 \{1,2,4\} \succ_1 \{1\} \succ_1 \ldots$$
$$\{1,2,4\} \succ_2 \{2,4\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2\} \succ_2 \ldots$$
$$\{1,2,3\} \succ_3 \{3,4\} \succ_3 \{1,3\} \succ_3 \{2,3,4\} \succ_3 \{3\} \succ_3 \ldots$$
$$\{3,4\} \succ_4 \{2,4\} \succ_4 \{1,2,4\} \succ_4 \{1,3,4\} \succ_4 \{4\} \succ_4 \ldots$$

First, let us consider partition $\{\{1,2\},\{3,4\}\}$. It is easy to check that this partition is core stable. To see this, realise that players 1 and 4 cannot be in any blocking coalition since they are in their favourite coalitions. Player 3 cannot improve his situation without player 1, and consequently the last player 2 has no possibility to form a blocking set.

However, this partition is not Nash stable, because player 3 has an incentive to leave coalition $\{3,4\}$ and join coalition $\{1,2\}$. Moreover, the arrival of player 3 hurts neither player 1 nor player 2 (player 1 is indifferent between being in $\{1,2\}$ and $\{1,2,3\}$, and player 2 prefers $\{1,2,3\}$ to $\{1,2\}$). Thus this partition is neither individually stable nor strongly core stable (as coalition $\{1,2,3\}$ weakly blocks the considered partition). Finally, since players 1 and 4 are in their favourite coalitions, they will not permit their partners to leave, and so this partition is contractually individually stable.

On the other hand, we can take partition $\{\{1,3\},\{2,4\}\}$, which is Nash stable (and hence also individually and contractually individually stable), because no player has incentive to leave his coalition and join another existing one, but it is not core stable (hence neither strongly core stable), since players 3 and 4 can do better by forming coalition $\{3,4\}$.

From *Example 1* one can see that neither core stability implies Nash stability nor Nash stability implies core stability. Also core stability does not imply individual stability. We can depict the existing relations between the defined stability concepts by a simple diagram as follows:

$$\begin{array}{ccccc} \textbf{Nash stability} & \Rightarrow & \textbf{individual stability} & \Rightarrow & \textbf{contractual individual stability} \\ & & \Uparrow & & \\ & & \textbf{strong core stability} & \Rightarrow & \textbf{core stability} \end{array}$$

As mentioned earlier, given a coalition formation game $(N, \mathcal{P})$, the main question is whether it is possible to find a partition of N that is stable. Unfortunately, in a general setting, without any restrictions on players' preferences, it is possible to guarantee the existence of neither core nor Nash nor individually stable partition. The following counterexample is taken from Bogomolnaia and Jackson (2002):

Example 2 Consider the game $(N, \mathcal{P})$ with $N = \{1, 2, 3\}$ and the following preference profile:

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3\}$$
$$\{2,3\} \succ_2 \{2,1\} \succ_2 \{2\} \succ_2 \{1,2,3\}$$
$$\{3,1\} \succ_3 \{3,2\} \succ_3 \{3\} \succ_3 \{1,2,3\}$$

These preferences contain a cycle: the first player prefers the second player to the third one, the second player prefers the third player to the first one, and the third player prefers the first one to the second one. Moreover, all players prefer to stay alone to being in the grand coalition.

There are five possible partitions of the three-element set $N = \{1, 2, 3\}$. Either every player is alone - partition $\{\{1\}, \{2\}, \{3\}\}$ - or two players are in a common set and the third one is alone - partitions $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$

and $\{\{2,3\},\{1\}\}$ - or all three players are together in the grand coalition - partition $\{\{1,2,3\}\}$. It is easy to check that any of these five partitions is neither core stable nor Nash stable nor individually stable. Nevertheless, there are three contractually individually stable partitions: $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$ and $\{\{2,3\},\{1\}\}$.

Bogomolnaia and Jackson (2002) showed that if each player's preferences over sets are strict then a contractually individually stable partition always exists. Moreover, they describe an algorithm, which identifies one such partition. But although we can guarantee the existence of a contractually individually stable partition, this positive result is rather unsatisfactory, since the concept of contractual individual stability is very restrictive as it puts great limits on mobility that players have.

Due to these negative results, several authors, including Alcalde and Revilla (1999), Alcalde and Romero-Medina (2000), Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002), Burani and Zwicker (2000), try to avoid the non-existence of a stable partition by imposing some plausible restrictions on players' preferences. Their results will be surveyed later in Section 3.

2.2 Farsighted stability

Another question, one can have in mind, is whether the proposed stability concepts really reflect the behaviour of rational players. Chwe (1994), as well as Diamantoudi and Xue (2000), argue that rational players will consider the possibility that once they will act (deviate), another coalition might react, a third coalition might in turn react and so on, without limit. In particular, suppose that a partition is unstable because there exists a set of players, which can induce a new partition beneficial to all of them. But this new partition is also unstable and a further deviation induces a third partition, which is for the originally deviating players even worse than it was in the very beginning. If players are able to predict this, they will rather not deviate. Alternatively, we can imagine a situation, where a group of players would deviate to a partition which does not necessarily immediately improve their situation, because they can foresee, that after this new partition becomes status quo, further deviations will occur and finally the originally deviating coalition will be better off than it was in the very beginning.

Thus, in some models, foresight (ability to look many steps ahead) can be an important aspect of players' rationality. This can be captured through the concept of indirect dominance defined in Diamantoudi and Xue (2000):

Definition 9 *A partition $\mathcal{M}$ is said to be* **indirectly dominated** *by another partition $\mathcal{M}'$ if there exists a sequence of partitions $\mathcal{M} = \mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k = \mathcal{M}'$ and a corresponding sequence of coalitions $S_1, S_2, \dots, S_{k-1}$ such that*

(1) every partition $\mathcal{M}_{j+1}$ is obtained from $\mathcal{M}_j$ by all the players from S_j leaving their coalitions in $\mathcal{M}_j$ and creating their own coalition S_j; while the other players remain in the rest of their original coalitions (i.e. $\mathcal{M}_{j+1} = \{S_j\} \cup \{T/S_j : T \in \mathcal{M}_j \text{ and } T/S_j \neq \emptyset\}$), and

(2) for all $j = 1, 2, \dots, k-1$, each player from S_j prefers the partition $\mathcal{M}'$ to the partition $\mathcal{M}_j$.

Example 3 To illustrate the outlined aspects of players' rationality let us consider the game $(N, \mathcal{P})$ with $N = \{1, 2, 3, 4\}$ and the following preference profile:

$$\{1,4\} \sim_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1,2\} \succ_1 \{1\} \succ_1 \dots$$

$$\{2,3,4\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2,4\} \succ_2 \{2\} \succ_2 \dots$$

$$\{1,3\} \succ_3 \{2,3,4\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3,4\} \succ_3 \{3\} \succ_3 \dots$$

$$\{2,3,4\} \succ_4 \{2,4\} \succ_4 \{1,4\} \succ_4 \{3,4\} \succ_4 \{4\} \succ_4 \dots$$

Let $\mathcal{M}_0 = \{\{1,4\},\{2,3\}\}$ be the initial partition. There are various sequences of deviations that may take place. Let us look at some of them (the arrows are labelled by the deviating coalition):

1. $\mathcal{M}_0 = \{\{1,4\},\{2,3\}\} \xrightarrow{\{2,3,4\}} \mathcal{M}_1 = \{\{1\},\{2,3,4\}\} \xrightarrow{\{1,3\}} \mathcal{M}_2 = \{\{1,3\},\{2,4\}\}$

Since all three players 2, 3 and 4 prefer partition $\mathcal{M}_1$ to the initial partition $\mathcal{M}_0$, coalition $\{2,3,4\}$ blocks partition $\mathcal{M}_0$ and so $\mathcal{M}_0$ is not core stable. However, neither $\mathcal{M}_1$ is core stable as it is blocked by coalition $\{1,3\}$. So, if after the first action (deviation of players 2, 3 and 4) the second action (deviation of players 1 and 3) takes place, player 2 ends up in a worse situation since he prefers $\mathcal{M}_0$ to $\mathcal{M}_2$. So if player 2 is farsighted, he can refuse the deviation from $\mathcal{M}_0$ to $\mathcal{M}_1$ as he can be afraid of a possible outcome being $\mathcal{M}_2$. In the sense of the previous definition, $\mathcal{M}_2$ does not indirectly dominate $\mathcal{M}_0$.

2. $\mathcal{M}_0 = \{\{1,4\},\{2,3\}\} \xrightarrow{\{3,4\}} \mathcal{M}_3 = \{\{1\},\{2\},\{3,4\}\} \xrightarrow{\{2,4\}} \mathcal{M}_4 = \{\{1\},\{2,4\},\{3\}\}$ $\xrightarrow{\{1,3\}} \mathcal{M}_5 = \{\{1,3\},\{2,4\}\}$

Now, although $\mathcal{M}_3$ is for both players 3 and 4 worse than $\mathcal{M}_0$, they can agree to deviate as they can expect formation of $\mathcal{M}_5$, which they both prefer to $\mathcal{M}_0$. (Realise that not only players 3 and 4 prefer $\mathcal{M}_5$ to $\mathcal{M}_0$ but also players 2 and 4 prefer $\mathcal{M}_5$ to $\mathcal{M}_3$ and players 1 and 3 prefer $\mathcal{M}_5$ to $\mathcal{M}_4$.) In the terminology of Diamantoudi and Xue, partition $\mathcal{M}_5$ indirectly dominates partition $\mathcal{M}_0$.

3. $\mathcal{M}_0 = \{\{1,4\},\{2,3\}\} \xrightarrow{\{3,4\}} \mathcal{M}_3 = \{\{1\},\{2\},\{3,4\}\} \xrightarrow{\{1,2\}} \mathcal{M}_6 = \{\{1,2\},\{3,4\}\}$ $\xrightarrow{\{1,2,3\}} \mathcal{M}_7 = \{\{1,2,3\},\{4\}\}$

However, after the deviation of players 3 and 4 from partition $\mathcal{M}_0$ to partition $\mathcal{M}_3$, a different sequence of actions may also take place. Hence, players may also end up in partition $\mathcal{M}_7$, which is for initial deviators, players 3 and 4, less preferred than the initial partition $\mathcal{M}_0$. So partition $\mathcal{M}_0$ is not indirectly dominated by $\mathcal{M}_7$, however, it can be seen that partition $\mathcal{M}_7$ indirectly dominates partition $\mathcal{M}_3$.

In situations with several plausible sequences of deviations, players' decisions will depend on their attitude towards risk. If players are over-optimistic then a coalition of players will deviate if at least one of the ultimate outcomes makes all its members better off. However, this over-optimism is rather "dangerous". Diamantoudi and Xue assume that rational players are conservative (rather than over-optimistic) in the sense that a coalition will deviate only if each possible ultimate outcome makes its members better off.

This is the main idea of **the maximal conservative stable set** - the concept that represents a farsighted core stability. Farsighted individual stability is modelled similarly. It is based on the concept of indirect individual dominance, which is used to define **the maximal individually conservative stable set.**

Diamantoudi and Xue showed that for each preference profile there exists a maximal conservative stable set, as well as a maximal individually conservative stable set. Moreover, if all the preferences are strict then all core stable partitions belong to the maximal conservative stable set, i.e. in the case of strict preferences, core stable partitions are immune to deviations with foresight.

Another approach can be found in Barberà and Gerber (2001), who use in their model of hedonic coalition formation games a combination of foresight and extreme risk aversion. They assume that rational players will not disrupt a partition if they can foresee that they may fall in a situation that makes them worse off. In other words, a coalition (or coalitions) will only deviate if no matter which partition is reached later, no deviating player will ever be worse off (i.e. also in intermediate partitions) than in his present partition. In this sense a partition is considered to be potentially unstable, what the authors call **transient,** if it can be disrupted by some coalition S (or more coalitions) without risk of a future "loss" (after some steps) for any deviating player of S. A partition, which is not transient, is said to be **durable.**

Barberà and Gerber proved that from every transient partition one can obtain in a finite number of deviations a partition that is durable. So for any hedonic game the set of durable partitions is nonempty. Moreover each core stable partition is also durable; therefore the core is always a subset of the set of durable partitions.

3 Restrictions of preferences

This section reviews the results regarding various restrictions imposed on players' preferences over sets in order to guarantee the existence of stable partitions.

Perhaps the majority of results have been obtained for coalition formation games with separable preferences (Banerjee, Konishi and Sönmez (2001), Burani and Zwicker (2000), Dimitrov et al. (2004)). Players' preferences are considered separable if each player i can divide remaining players into three disjoint sets of good, bad and neutral players in such a way that adding a good player (a friend) to a coalition always makes player i better off, adding a bad player (an enemy) always makes him worse off, whereas adding a neutral player never changes his situation. More precisely:

Definition 10 *A game* $(N, \mathcal{P})$ *is* **separable** *if for any player* $i \in N$*, for any coalition* $S \in \mathcal{N}_i$ *and for any player* $j \notin S$*:*

(1) $S \cup \{j\} \succ_i S \Leftrightarrow \{i, j\} \succ_i \{i\}$,

(2) $S \cup \{j\} \prec_i S \Leftrightarrow \{i, j\} \prec_i \{i\}$, *and*

(3) $S \cup \{j\} \sim_i S \Leftrightarrow \{i, j\} \sim_i \{i\}$.

A stronger notion of separability is additive separability (Bogomolnaia and Jackson (2002), Banerjee, Konishi and Sönmez (2001), Burani and Zwicker (2000)). Here, each player attaches a precise value to each individual in the society, and then, his valuation of a given coalition is simply the sum of the values that he assigns to the members of this coalition. Since the value that a player assigns to himself has no effect on his ranking, it is commonly assumed to be zero.

Definition 11 *A game* $(N, \mathcal{P})$ *is* **additively separable** *if for each player* $i \in N$*, there exists a utility function* $v_i : N \to R$ *such that* $v_i(i) = 0$ *and for any two coalitions* $S, T \in \mathcal{N}_i : S \succeq_i T \Leftrightarrow \sum_{j \in S} v_i(j) \geq \sum_{k \in T} v_i(k)$.

A profile of additively separable preferences, represented by utility functions $(v_1, v_2, \ldots, v_n)$, satisfies **mutuality**, if $v_i(j) \geq 0 \Leftrightarrow v_j(i) \geq 0$ for any two players $i, j \in N$, and it satisfies **symmetry**, if $v_i(j) = v_j(i)$ for any two players $i, j \in N$. Thus symmetry implies mutuality.

Bogomolnaia and Jackson showed that additively separable and symmetric preferences guarantee the existence of a Nash stable (and hence also the existence of an individually stable and a contractually individually stable) partition. But if the requirement of symmetry is weakened to mutuality then individually stable partitions may fail to exist even if an additional strong requirement of *single-peakedness on a tree* is imposed (for a definition see Bogomolnaia and Jackson (2002)).

Banerjee et al. investigated core stability under additively separable preferences and they found that additive separability and symmetry are not sufficient to guarantee the existence of a core stable partition even if additional strong conditions such as *single-peakedness on a tree* and *the tree intermediate preference property* are imposed (for definitions see Banerjee, Konishi and Sönmez (2001)).

Interesting results concerning core stability and separable preferences can be found in Burani and Zwicker (2000). They introduced a decomposition of the utility functions representing symmetric additively separable preferences into two components, namely *the cardinal component* and *the alternating component.* They showed that, when the alternating component is the only one present, then the core might be empty. However, if players' preferences are restricted to be purely cardinal (i.e. symmetric, additively separable and with zero alternating components in the decomposed utility functions), then there always exists a core stable partition.

Burani and Zwicker also proved that the existence of a core stable and a Nash stable partition is guaranteed even under a weaker set of requirements, called *descending separable preferences* (for a definition see Burani and Zwicker (2000)). They also describe a very simple algorithm that, given a set of players and descending separable preferences, provides a partition that is both core stable and Nash stable.

Dimitrov et al. (2004) studied two special classes of additively separable preferences. The first class is based on *appreciation of friends* while the second class is based on *aversion to enemies.* Under the first preference domain, when comparing two coalitions, a player prefers a coalition containing more friends, and if the two coalitions have the same number of friends a player prefers a coalition with less enemies. Under the second preference domain the player declares as better the coalition that contains less enemies, and if the two coalitions have the same number of enemies, the coalition with more friends wins the comparison. The authors proved that friend appreciation is a sufficient condition for the existence of a strongly core stable partition and that enemies aversion guarantees the existence of a core stable partition.

Bogomolnaia and Jackson (2002) and Banerjee, Konishi and Sönmez (2001) examined also another restriction of preferences, called anonymity, which requires players to care only about the size of a coalition, but not about the identities of the members of the coalition.

Definition 12 *A game* $(N, \mathcal{P})$ *satisfies* **anonymity** *if for any player* $i \in N$*, and for any two coalitions* $S, T \in \mathcal{N}_i$ *: if* $|S| = |T|$ *then* $S \sim_i T$*.*

Although for the number of players not greater than 7, anonymity alone suffices for the existence of an individually stable partition, this is not true for arbitrary number of players (Bogomolnaia and Jackson (2002)). So additional conditions are necessary to be imposed.

Bogomolnaia and Jackson (2002) showed that anonymity and *single peakedness* are not sufficient for the existence of a Nash stable partition. Nevertheless, these requirements guarantee the existence of an individually stable partition. Banerjee et al. (2001) provided a counterexample showing that a core stable partition may fail to exist even if players' preferences satisfy anonymity, *single peakedness on population* and *the population's intermediate preference property* (for definitions see Banerjee et al. (2001)).

Since neither additive separability nor anonymity with additional strong conditions do not guarantee the existence of a core stable partition, Banerjee et al. looked for other possible restrictions. They were inspired by a positive result of Farrell and Scotchmer (1988) who showed that *the common ranking property* (requiring the existence of a linear ordering over all the coalitions which coincides with any player's preferences) guarantees the existence of a core stable partition. Banerjee et al. defined two relaxed versions of the common ranking property, that are called the top coalition property and the weak top coalition property. The first condition requires that for any non-empty group of players there is a subgroup that is unanimously the best one for all its members. The second condition is a weaker version of the first one.

Definition 13 *Given a non-empty set of players $S \subseteq N$, a non-empty subset $T \subseteq S$ is said to be a* **top-coalition** *of S if any player from T prefers T to any other subset $U \subseteq S$.*
A game $(N, \mathcal{P})$ satisfies **the top coalition property** *if for any non-empty set of players $S \subseteq N$, there exists a top-coalition of S.*

Definition 14 *Given a non-empty set of players $S \subseteq N$, a non-empty subset $T \subseteq S$ is said to be a* **weak top-coalition** *of S if the set T can be partitioned into sets $T_1, T_2, \ldots, T_m$ in such a way that:*

(1) any player from T_1 prefers T to any other subset $U \subseteq S$, and

(2) any player from T_k, for any $k \leq m$, needs cooperation of at least one player from $\cup_{j<k} T_j$ to form $U \subseteq S$ that is better than T.

A game $(N, \mathcal{P})$ satisfies **the weak top coalition property** *if for any non-empty set of players $S \subseteq N$, there exists a weak top-coalition of S.*

Banerjee et al. proved that for each coalition formation game satisfying the weak top coalition property there exists a core stable partition. Moreover, if the game satisfies the top coalition property and the preferences are strict, then the game has a unique core stable partition. They also showed that if players' preferences are anonymous and separable then the top coalition property is satisfied. Hence anonymity and separability together guarantee the existence of a core stable partition. However, it is not very surprising, because these two restrictions are very strong. In particular, Banerjee et al. show that if players' preferences over sets are simultaneously anonymous and separable, then the set of players can be divided into two disjoint sets N^+ and N^- in such a way that every player in N^+ has anonymous and single peaked preferences with peak at cardinality equal to n and every player in N^- has anonymous and single peaked preferences with peak at cardinality equal to 1.

Many of the above mentioned restrictions (such as the single peakedness, the tree intermediate preferences, the descending separable preferences, or the top coalition property) are imposed on the whole preference profile. Alcalde and Romero-Medina (2000) argue that the decision problem whether this sort of conditions are satisfied or not can be as difficult to solve as the problem of finding a core stable partition directly. Therefore they proposed other restrictions, which are imposed on each i ndividual's preferences separately. Their conditions are called *the Union Responsiveness Condition, the Intersection Responsiveness Condition* and *the Essentiality*. Alcalde and Romero-Medina showed that these three conditions are independent and each one of them alone guarantees the existence of a core stable partition.

Perhaps the clearest and the most intuitive condition is that of essentiality (for the remaining two definitions see Alcalde and Romero-Medina (2000)):

Definition 15 *We say that a* **coalition** $S \in \mathcal{N}_i$ **is essential** *for player* i *if*

(1) $\{i\} \succ_i T$*, for any coalition* T *with* $S \not\subseteq T$*; and*

(2) for any two coalitions T *and* T'*, if* $S \subseteq T \subset T'$*, then* $T \succ_i T'$*.*

Player i*'s preferences satisfy* **the Essentiality** *if there exists a coalition* $S \in \mathcal{N}_i$ *that is essential for him.*

Alcalde and Revilla (2000) described another restriction imposed on each individual's preferences, called the Top Responsiveness Condition. For simplicity, let us denote by $Ch_i(S)$ a subcoalition $S' \subseteq S$ such that $S' \succeq_i S''$ for any other subcoalition $S'' \subseteq S$. Thus $Ch_i(S)$ represents the choice set of player $i \in N$ of a given coalition $S \in \mathcal{N}_i$.

Definition 16 *Player* i*'s preferences satisfy the* **Top Responsiveness Condition** *(TRC for short) if for any two coalitions* $S, T \in \mathcal{N}_i$ *the following two* *conditions are fulfilled:* *(1) if $Ch_i(S) \succ_i Ch_i(T)$, then $S \succ_i T$; and*

(2) if $S \subset T$ and $Ch_i(T) \subseteq Ch_i(S)$, then $S \succ_i T$.

Alcalde and Revilla showed that for any preference profile satisfying TRC there exists a core stable partition. They also provided an algorithm, called the Tops Covering Algorithm, which finds a core stable partition for any profile satisfying TRC. In addition, they proved that if players are restricted to have preferences satisfying TRC, then their algorithm is the only one that produces a stable partition and it is strategy-proof in the sense that no player can profitably misrepresent his true preferences to obtain a better outcome.

4 Algorithms and their complexity

In the majority of the reviewed papers on coalition formation games authors focus on possible restrictions of players' preferences ensuring the existence of a stable partition. However, in many situations, it is not sufficient to declare that the game admits a stable partition but we need to identify one such partition. Or, in other situations, we can have a concrete proposed partition for our game and we need to decide whether this partition is stable or not.

Thus, for many practical applications it can be useful to describe efficient algorithms, which solve the given computational or decision problem. In particular, we are looking for an efficient algorithm, which for any game $(N, \mathcal{P})$ finds a stable partition, and for an algorithm, which for any game $(N, \mathcal{P})$ and a partition $\mathcal{M}$ decides whether or not the partition $\mathcal{M}$ is stable with respect to profile $\mathcal{P}$.

Since for a given problem different algorithms can be proposed, we need some criteria for evaluating and comparing them. A very natural criterion is the amount of computations or time used by the algorithm. Each algorithm A

can be associated with a function $f_A(n)$ expressing that for any input of size n the algorithm A uses no more than $f_A(n)$ time units (seconds, instructions, steps, etc.). The function $f_A(n)$ is called computational complexity of algorithm A.

Typical complexity functions are $log\ n, n, n^k, 2^n$, etc. Depending on particular algorithm's complexity we then talk about logarithmic, linear, polynomial or exponential algorithms. Clearly, logarithmic algorithms are faster (for sufficiently large inputs) than polynomial algorithms and those are faster (and thus more desired) than exponential ones.

The gap between an exponential function and a polynomial function is quite staggering: For example, given an input of size 50, an algorithm with exponential complexity 2^n would require 2^{50} instructions, what takes (on a computer executing 10^6 instructions per second) more than 30 years, while an algorithm with polynomial complexity n^2 would require for the same input only 50^2 instructions, what takes on the same computer less than a second.

This unbridgeable gap between polynomial and exponential functions translates into a clear distinction between algorithms with polynomial complexity and those with exponential complexity. A polynomial algorithm is regarded efficient, and a problem, for which a polynomial algorithm exists, is considered tractable. On the other hand, a problem, for which only exponential algorithms exist, is considered intractable, as it is practically unsolvable (except for small values of n).

The issue of problems' tractability is systematically treated in the computational complexity theory. A nice introduction to this theory can be found in the book of Garey and Johnson (1979). The complexity theory typically deals with so-called decision problems. A characteristic feature of a decision problem is that it requires a simple answer "yes" or "no". For example, in CFG we deal with the following decision problems:

1) "Given a game $(N, \mathcal{P})$, does there exists a stable partition for this game?"

2) "Given a game $(N, \mathcal{P})$ and a partition $\mathcal{M}$, is $\mathcal{M}$ stable with respect to $\mathcal{P}$?"

3) "Given a game $(N, \mathcal{P})$, a partition $\mathcal{M}$ and a coalition S, does S block $\mathcal{M}$?"

Although many practical problems are not decision problems (rather they are computational problems: e.g. we want to compute a stable partition, we want to find a blocking set), many of them have their decision counterparts. Usually, a computational problem is not much harder to solve than its decision counterpart and it can be solved by a polynomial number of calls to the algorithm for its decision counterpart. We say that a computational problem and its decision counterpart are polynomially equivalent. Thus it is sufficient to study tractability of decision problems.

Depending on problems' complexity all decision problems are classified to complexity classes. The two most important classes are P and NP. The class P corresponds to those decision problems that are tractable when computed in the fundamental mode, where at each time moment a uniquely determined instruction is executed. The class NP, on the other hand, corresponds to those decision problems that are tractable when computed in the nondeterministic mode, where at each time moment a set of next instructions is executable and the machine can choose to execute any one of these instructions. Clearly $P \subseteq NP$, the P versus NP (or $P \neq NP$) question asks whether this inclusion is proper.

Since most real world computers operate in the fundamental mode, P is clearly very important. The practical significance of NP is not immediately evident. In 1971, Cook proved an interesting connection between P and NP. He showed that there exists a decision problem, which is one of the hardest problems in the class NP, in the sense that if this problem is in P then $P = NP$. Formal definition of the hardest problem in NP (called NP-complete problem) is based on the concept of polynomial transformability, which is defined as follows:

Definition 17 *A decision problem R_1 is* **polynomially transformable** *to another decision problem R_2 iff there exists a transformation f, which given any instance x of R_1 constructs in polynomial time (depending on size of x) an instance y of R_2 such that:*

$$x \text{ is a "yes" instance of } R_1 \Longleftrightarrow y \text{ is a "yes" instance of } R_2$$

Definition 18 *A decision problem R is NP*-**hard** *iff every decision problem from the class NP is polynomially transformable to R.*
A decision problem R is NP-**complete** *iff R is NP-hard and $R \in NP$.*

Thus, to show that a decision problem R is NP-complete one needs to prove two facts: 1) that R belongs to NP and 2) that every problem in NP is polynomially transformable to R. The historically first problem shown to be NP-complete was SAT, i.e. the problem of determining whether a Boolean formula in conjunctive normal form is satisfiable (Cook (1971)).

Since the relation of polynomial transformability is transitive, to establish NP-completeness of a problem R it suffices to verify R's membership in NP and to describe a polynomial transformation from some known NP-complete problem to R. This indirect approach is particularly effective if one chooses the NP-complete problem rather similar to R: the list of over 300 NP-complete problems in Garey and Johnson (1979) is a good source when making this choice.

The relation of polynomial transformability has another nice property: if a problem R_1 is polynomially transformable to a problem R_2 and R_2 belongs to the class P, then also R_1 belongs to P. Thus the question whether $P = NP$ is reduced to the fundamental tractability of any NP-complete problem. In fact, most researchers believe that $P \neq NP$. Because of this belief, a proof that a certain problem is NP-complete is taken as a warning signal that one should not expect to find an efficient (i.e. polynomial) algorithm solving the problem.

The importance of considering also computational complexity questions in cooperative game theory is being gradually recognized by general audience. Deng and Papadimitriou (1994) argue that computational complexity is a suitable notion for capturing bounded rationality and they study the computational complexity of several games based on graphs. Rather naturally, there are plenty of computational complexity studies for cooperative games resulting from combinatorial optimisation problems (e.g. games on graphs, networks or matroids). From some recent papers let us mention at least Faigle et al. (1997, 1998), Nagamochi et al. (1997) or Fang et al. (2002).

Special cases of CFG are matching problems derived from the famous stable marriage problem and its one-sided generalisation the stable roommates problem (for details see monograph Gusfield and Irving (1989) and references therein). For these two problems and their variants a deep research of computational complexity questions has been done. As a rule, there exist polynomial algorithms for deciding the existence of a stable matching in the case when players' preferences are strict, while the existence problems in the case with indifferences are NP-complete (Irving et al. (1999), Irving (1985, 1994), Ronn (1990)). Here, let us note that very similar results have been obtained for special CFG with preferences over sets derived from the preferences over individuals (for a detailed survey see Section 6).

Computational complexity of general CFG with arbitrary preferences over coalitions has been studied in Ballester (2003), who proved that all the problems of deciding the existence of a core stable, a Nash stable and an individually stable partition are NP-hard. Furthermore, he showed that these decision problems remain NP-hard even if players' preferences are strictly anonymous. Dimitrov et al. (2004) study the computational complexity of CFG with two special classes of separable preferences (see Section 3) and they show that a strongly core stable partition under friends appreciation can be found in polynomial time, while finding a core stable partition under enemies aversion is NP-hard.

5 Extending preferences

In the majority of the literature from the area of CFG, players are supposed to have preferences over all coalitions to which they could belong. Except of computational problems connected with even scanning such preference lists, it is also hard to imagine a rational person able to formulate a complete ranking of coalitions, since its length grows exponentially with the number of players. One possibility of avoiding great amount of computations is to start with players having preferences over individuals, and then consider possible extensions of these preferences to preferences over groups of players.

The problem of extending an order on a set of alternatives to an order on its power set has gained a great attention in literature independently from coalition formation. In this section we survey some well-known methods of extension. We denote set of alternatives by N and its power set by 2^N. Throughout the section we will use the symbol $\succeq$ to denote preferences over alternatives as well as preferences over sets of alternatives. Thus notation $x \succeq y$ means that alternative x is preferred to alternative y, while notation $S \succeq T$ means that the set of alternatives S is preferred to the set of alternatives T (the meaning will be clear from the context).

Naturally, preference relation $\succeq$ over sets of alternatives is considered to be an extension of preference relation $\succeq$ over alternatives if for any two alternatives $x, y \in N, \{x\} \succeq \{y\}$ if and only if $x \succeq y$.

A meaningful ordering over sets of alternatives should have some plausible properties, these are usually formulated in the related literature as various axioms (for a list of considered axioms see Barberà, Bossert and Pattanaik (2004) or Packard (1979)). The two most appealing axioms are dominance and independence (Barberà, Bossert and Pattanaik (2004)). In some papers, dominance is referred to as the Gärdenfors principle, and independence is called the monotonicity property (Kannai and Peleg (1984)).

Definition 19 *A preference relation* $\succeq$ *over sets of alternatives satisfies* **dominance** *if for any set of alternatives* $S \in 2^N$ *and any alternative* $x \in N/S$*:*

$$\text{if } x \succ y \text{ for all } y \in S \text{ then } S \cup \{x\} \succ S$$
$$\text{if } y \succ x \text{ for all } y \in S \text{ then } S \succ S \cup \{x\}$$

Definition 20 *A preference relation* $\succeq$ *over sets of alternatives satisfies* **independence** *if for any two sets of alternatives* $S, T \in 2^N$ *and any alternative* $x \in N/(S \cup T)$*:* *if* $S \succ T$ *then* $S \cup \{x\} \succeq T \cup \{x\}$.

These two axioms are very mild. Dominance requires that adding an alternative which is better (worse) than all alternatives in a given set S leads to a set that is better (worse) than S. Independence, on the other hand, requires that if a set S is strictly preferred to a set T, then adding the same new alternative to both sets does not reverse this ranking. Surprisingly, it turns out that dominance and independence are incompatible. Kannai and Peleg (1984) have shown that for a set of alternatives N with more than six elements there exists no preference relation $\succeq$ over sets of alternatives satisfying dominance and independence.

Barberà and Pattanaik (1984) have provided a similar impossibility result. Here independence is strengthened to strict independence and dominance is weakened to simple dominance (see the definitions bellow). Barberà and Pattanaik have proved that for N containing at least three elements there is no ordering $\succeq$ on the power set of N, which satisfies simple dominance and strict independence.

Definition 21 *A preference relation* $\succeq$ *over sets of alternatives satisfies* **simple dominance** *if for any two alternatives* $x, y \in N$*, if* $x \succ y$ *then* $\{x\} \succ \{x, y\}$ *and* $\{x, y\} \succ \{y\}$.

Definition 22 *A preference relation* $\succeq$ *over sets of alternatives satisfies* **strict independence** *if for any two sets of alternatives* $S, T \in 2^N$ *and any alternative* $x \in N/(S \cup T)$*:* *if* $S \succ T$ *then* $S \cup \{x\} \succ T \cup \{x\}$.

Fortunately, there are also some positive results. Barberà, Barrett and Pattanaik (1984) have shown that if only simple dominance and independence are required, then for any set of alternatives N there exists a preference relation $\succeq$ over sets of alternatives satisfying these two axioms. Moreover, if simple dominance and independence are satisfied, then any nonempty subset S of N must be indifferent to the two-element set consisting of the best and the worst alternatives of S only (Barberà, Barrett and Pattanaik (1984), Bossert, Pattanaik and Xu (2000)). Hence an ordering over all singletons and two-element sets can represent the whole ordering over sets of alternatives.

There are several papers concentrating on providing axiomatic charaterizations of some natural extension rules. A very nice survey of these results can be found in Barberà, Bossert and Pattanaik (2004). In many proposed extensions the best and/or the worst alternatives play a crucial role. In this section we denote the best and the worst alternative of a set S by $B(S)$ and $W(S)$, respectively.

Perhaps the simplest extension rules are called maxi-max and maxi-min orderings, studied in Packard (1979). The **maxi-max** relation is based solely on the best alternative of a set in the sense that $S \succeq T$ if and only if $B(S) \succeq B(T)$. The **maxi-min** relation, on the other hand, is based solely on the worst alternative in a set, thus $S \succeq T$ if and only if $W(S) \succeq W(T)$.

Arlegi (2001) characterizes maxmin and minmax orderings, which use the best as well as the worst alternative as the criteria for ranking sets of alternatives. Under the **maxmin** relation, $S \succ T$ if either $B(S) \succ B(T)$ or $B(S) \sim B(T)$ and $W(S) \succ W(T)$. Thus maxmin ordering uses the best alternative as the primary criterion and the worst alternative plays the role of a tiebreaker. Under the **minmax** relation, roles of the best and the worst alternative are switched, i.e. $S \succ T$ if either $W(S) \succ W(T)$ or $W(S) \sim W(T)$ and $B(S) \succ B(T)$.

Bossert (1989) describes a quasi-ordering, where a set S is preferred to a set T if and only if the best alternative of S is preferred to the best alternative of T and simultaneously the worst alternative of S is preferred to the worst alternative of T. Note that this extension differs from the previous ones since it leads to incomparable sets of alternatives.

Pattanaik and Peleg (1984) provide refinements of maxi-max and maxi-min preference relations by considering their lexicographical extensions. The **lexicographic maxi-max** ordering considers first the best alternatives of two sets S and T to be compared. If $B(S) \succ B(T)$ then $S \succ T$. However, if $B(S) \sim B(T)$, the best alternatives are eliminated from both sets and the remaining sets $S' = S/\{B(S)\}$ and $T' = T/\{B(T)\}$ are considered. Now, if $B(S') \succ B(T')$ then S is declared to be better than T ($S \succ T$), but if again $B(S') \sim B(T')$, the best alternatives are removed, the reduced sets are considered, and so on. If after a number of successive eliminations one of the original sets (say S) is reduced to the empty set but the reduction of the second one (i.e. T) is nonempty, then $S \succ T$. The **lexicographic maxi-min** ordering is dual to the lexicographic maxi-max ordering. So here, first the worst alternatives are compared and in the case of indifference between them they are eliminated and the remaining reduced sets are considered. Now, if after a number of eliminations the set S is reduced to the empty set while the corresponding reduction of set T is nonempty, then $T \succ S$.

As an alternative to the maxmin and minmax orderings, Bossert, Pattanaik and Xu (2000) define lexicographic maxmin and lexicographic minmax extension rules. The **lexicographic maxmin** relation uses repetitively the maxmin rule in the following way. If a set S is declared to be better than a set T according to the maxmin rule, then this strict preference is respected also by the lexicographic maxmin rule. However, if S and T are indifferent according to the maxmin, i.e. $B(S) \sim B(T)$ and simultaneously $W(S) \sim W(T)$, then the best as well as the worst alternatives are removed from both sets and the reduced sets are again compared according to the maxmin rule, and so on. If this procedure leads to a situation where one set (say S) is reduced to the empty set but the other (i.e. T) is not, then $S \succ T$. The **lexicographic minmax** relation is defined analogously. For further details see Bossert, Pattanai and Xu (2000).

We conclude this review with three different extensions where also other alternatives play a role in establishing the resulting ranking of sets. An interesting extension rule studied in Nitzan and Pattanaik (1984) is, for instance, based on median alternatives. For any set of alternatives $S = \{x_1, x_2, \ldots, x_s\}$, such that $x_i \succ x_{i+1}$ for each $i \in \{1, 2, \ldots, s-1\}$, we define the median set of set S as $med(S) = \{x_{(s+1)/2}\}$ if s is odd, and $med(S) = \{x_{s/2}, x_{s/2+1}\}$ if s is even. Nitzan and Pattanaik suppose that an ordering over all singletons and two-element sets is given, and they define the **median-based** extension rule as one under which $S \sim med(S)$ for each set of alternatives $S \in 2^N$.

There are also certain extension rules, which take into account all the alternatives in the compared sets. It is supposed that a utility function $u : N \rightarrow R$, satisfying $u(x) > u(y)$ if and only if $x \succ y$, is given. Then, when comparing two sets of alternatives S and T we can compare sums of utilities or the average utility in the compared sets. The first approach leads to the **additively separable preferences** (Bogomolnaia and Jackson (2002), Banerjee, Konishi and

Sönmez (2001)), where $S \succ T$ if and only if $\sum_{x\in S} u(x) > \sum_{y\in T} u(y)$. The second approach leads to **averaging preference relation** studied in Packard (1979), where, on the other hand, $S \succ T$ if and only if $\sum_{x\in S} u(x)/|S| > \sum_{y\in T} u(y)/|T|$.

6 Stable partitions with preferences derived from the best and/or the worst player in a coalition

From a computational point of view, there exists a rather extensive study for CFG, where players' preferences over coalitions are derived from their preferences over individuals on the basis of the best and/or the worst player. As we have seen in the previous section, the best and the worst player of a coalition are of a great importance also from a utility theory point of view. Throughout this section, we denote the best and the worst player of a coalition S according to player i's individual preferences by $\mathcal{B}_i(S)$ and $\mathcal{W}_i(S)$, respectively.

The study of CFG with preferences over coalitions derived from these extreme players was started in Cechlárová and Romero-Medina (2001). The authors consider two extensions of preferences, which are almost identical with Packard's maxi-max and maxi-min relations.

Definition 23 *A player $i \in N$* **strictly $\mathcal{B}$-prefers** *a coalition S to a coalition T (with $S, T \in \mathcal{N}_i$) if either he strictly prefers player $\mathcal{B}_i(S)$ to player $\mathcal{B}_i(T)$ or he is indifferent between $\mathcal{B}_i(S)$ and $\mathcal{B}_i(T)$ and coalition S contains smaller number of players than coalition T.*

Definition 24 *A player $\in N$* **strictly $\mathcal{W}$-prefers** *a coalition S to a coalition T (with $S, T \in \mathcal{N}_i$) if he strictly prefers player $\mathcal{W}_i(S)$ to player $\mathcal{W}_i(T)$.*

The first extension represents in some sense an "overoptimistic" behaviour of players, because each player looks only at the best player of a given coalition and does not care about the rest of players explicitly, only through the size of the coalition. (Let us remark, that without taking into account the cardinality of compared coalitions as a tie-breaker, the grand coalition N would be a strongly core stable partition for any preference profile $\mathcal{P}$.)

On the other hand, the second extension represents in some sense an "over-pessimistic" behaviour of players, where players are trying to avoid those players whom they do not like. (Since this players' aversion alone leads to a formation of really small sets, taking into account also the cardinality of compared sets was unnecessary in this extension.)

For simplicity, the set of all core stable partitions in a game $(N, \mathcal{P})$ with players' preferences over sets being $\mathcal{B}$-preferences ($\mathcal{W}$-preferences) will be denoted by $C_{\mathcal{B}}(\mathcal{P})$ (and $C_{\mathcal{W}}(\mathcal{P})$, respectively). Similarly, notations $SC_{\mathcal{B}}(\mathcal{P})$ and $SC_{\mathcal{W}}(\mathcal{P})$ denote the corresponding sets of all strongly core stable partitions in these games.

Cechlárová and Romero-Medina (2001) have shown that in the case when the original players' preferences over individuals are strict, for any game with $\mathcal{B}$-preferences sets $C_{\mathcal{B}}(\mathcal{P})$ and $SC_{\mathcal{B}}(\mathcal{P})$ are nonempty and one strongly core stable partition can be found by a simple polynomial algorithm, called BSTABLE. However, this algorithm need not obtain a correct solution in the case when players' preferences over individuals contain indifferences. Moreover, in the presence of indifferences we can guarantee the nonemptyness of neither $C_{\mathcal{B}}(\mathcal{P})$ nor $SC_{\mathcal{B}}(\mathcal{P})$. (See Cechlárová and Hajduková (2002), who provided an example of a game with $\mathcal{B}$-preferences containing five players and just one tie in original preference profile admitting no core stable partition.)

As there exist preference profiles with $C_{\mathcal{B}}(\mathcal{P}) = \emptyset$, it is relevant to consider also the computational complexity of corresponding decision problems: "Given a preference profile and type of preferences, is $C_{\mathcal{B}}(\mathcal{P}) = \emptyset$?". Cechlárová and Hajduková (1999) showed that both decision problems belong to the class NP. In particular, they described polynomial algorithms for testing whether or not a given partition belongs to $C_{\mathcal{B}}(\mathcal{P})$ or $SC_{\mathcal{B}}(\mathcal{P})$, respectively. Later, Cechlárová and Hajduková (2002) constructed polynomial transformations from the known NP-complete problem R-3-SAT to the studied decision problems. Thus, it is an NP-complete problem to decide the nonemptyness of $C_{\mathcal{B}}(\mathcal{P})$ and $SC_{\mathcal{B}}(\mathcal{P})$ in case of indifferences.

Cechlárová and Romero-Medina (2001) studied the game with $\mathcal{W}$-preferences assuming that all players' preferences over individuals are strict. They showed that the CFG with $\mathcal{W}$-preferences is very similar to the stable roommates problem (extensively studied by Gusfield and Irving (1989)) in the sense that all the solutions of the stable roommates problem with respect to the given preference profile over individuals are strongly core stable (and hence also core stable) partitions of coalition formation game with $\mathcal{W}$-preferences. However, this does not hold conversely, i.e. there exist preference profiles admitting a (strongly) core stable partition and yet no stable roommates solution.

A more complex study of the model with $\mathcal{W}$-preferences can be found in Cechlárová and Hajduková (2004). They show that in the case of strict preferences, stable partitions cannot contain very large sets. More precisely, core stable partitions can contain only singletons, two-element and three-element sets, while strongly core stable partitions can contain even only singletons and two-element sets. Further, they proved that any partition, that is not (strongly) core stable, is (weakly) blocked by a singleton or a two-element set. Thus, there exist simple polynomial algorithms for testing the membership of a given partition in $C_{\mathcal{W}}(\mathcal{P})$ and $SC_{\mathcal{W}}(\mathcal{P})$.

Unlike the game with $\mathcal{B}$-preferences, where the existence of a strongly core stable partition is guaranteed at least for profiles with strict preferences, there exist games with strict $\mathcal{W}$-preferences admitting no (strongly) core stable partition. Hence again, it is relevant to consider the corresponding problems of deciding whether for a given game $(N, \mathcal{P})$ with $\mathcal{W}$-preferences a (strongly) core stable partition exists. For the games with $\mathcal{W}$-preferences without ties in players' preferences over individuals Cechlárová and Hajduková (2004) propose polynomial algorithms to decide about the nonemptyness of $C_{\mathcal{W}}(\mathcal{P})$ and $SC_{\mathcal{W}}(P)$. Both proposed algorithms are just slight modifications of Irving's stable roommates algorithm described in Irving (1985).

However again, in the presence of indifferences, the studied problems become more complicated. In particular, problem of deciding the nonemptyness of $C_{\mathcal{W}}(\mathcal{P})$ is NP-complete, while the existence of a polynomial algorithm for testing the nonemptyness of $SC_{\mathcal{W}}(\mathcal{P})$ is still an opened question.

Cechlárová and Hajduková (2003) studied the stable partition problem in which all players have strict preferences over other individuals and when comparing coalitions they take into account the best as well as the worst player of a coalition.

Definition 25 *A player $i \in N$* **strictly $\mathcal{BW}$-prefers** *a coalition S to a coalition T ($S, T \in \mathcal{N}_i$) if either he strictly prefers player $\mathcal{B}_i(S)$ to player $\mathcal{B}_i(T)$ or he is indifferent between $\mathcal{B}_i(S)$ and $\mathcal{B}_i(T)$ but he strictly prefers player $\mathcal{W}_i(S)$ to player $\mathcal{W}_i(T)$.*

Definition 26 *A player $i \in N$* **strictly $\mathcal{WB}$-prefers** *a coalition S to a coalition T ($S, T \in \mathcal{N}_i$) if either he strictly prefers player $\mathcal{W}_i(S)$ to player $\mathcal{W}_i(T)$ or he is indifferent between $\mathcal{W}_i(S)$ and $\mathcal{W}_i(T)$ but he strictly prefers player $\mathcal{B}_i(S)$ to player $\mathcal{B}_i(T)$.*

Notice, that $\mathcal{BW}$ and $\mathcal{WB}$-preferences are in fact identical with Arlegi's maxmin and minmax preference relation, respectively. Apparently, $\mathcal{BW}$-preferences are closely related to $\mathcal{B}$-preferences and $\mathcal{WB}$-preferences to $\mathcal{W}$-preferences. However, the connection between games with $\mathcal{W}$ and $\mathcal{WB}$-preferences is much stronger than the relation between games with $\mathcal{B}$ and $\mathcal{BW}$-preferences. Again, the set of all (strongly) core stable partitions in studied games with $\mathcal{BW}$-preferences and $\mathcal{WB}$-preferences will be denoted by $C_{\mathcal{BW}}(\mathcal{P}), C_{\mathcal{WB}}(\mathcal{P}), SC_{\mathcal{BW}}(\mathcal{P})$ and $SC_{\mathcal{WB}}(\mathcal{P})$, respectively.

Cechlárová and Hajduková (2003) proved that $C_{\mathcal{W}}(\mathcal{P}) = C_{\mathcal{WB}}(\mathcal{P}) = SC_{\mathcal{WB}}(\mathcal{P})$ for any profile $\mathcal{P}$ of strict players' preferences over individuals. As a consequence, one can use the polynomial algorithms derived in Cechlárová and Hajduková (2001) to test whether $C_{\mathcal{WB}}(\mathcal{P})$ and $SC_{\mathcal{WB}}(\mathcal{P})$ are nonempty and if this is the case then to find one (strongly) core stable partition.

On the other hand, the authors constructed a game showing that neither $C_{\mathcal{B}}(\mathcal{P}) \subseteq C_{\mathcal{BW}}(\mathcal{P})$ nor $C_{\mathcal{BW}}(\mathcal{P}) \subseteq C_{\mathcal{B}}(\mathcal{P})$. Further, they proved that the partition obtained by the algorithm BSTABLE belongs to $C_{\mathcal{BW}}(\mathcal{P})$, and thus the existence of a core stable partition is guaranteed also for games with $\mathcal{BW}$-preferences. However, this partition need not belong to $SC_{\mathcal{BW}}(\mathcal{P})$. Moreover, there exist games $(N, \mathcal{P})$ with an empty set $SC_{\mathcal{BW}}(\mathcal{P})$ and the corresponding decision problem is NP-hard. For the time being, there exists no efficient algorithm for testing whether a given partition belongs to $SC_{\mathcal{BW}}(\mathcal{P})$, and so the membership of the corresponding decision problem in the class NP remains an open question.

For various extensions of players' preferences over individuals and two stability concepts *Table 1* summarizes the known complexity results as well as open cases for the core nonemptyness problem, while *Table 2* does it for the checking stability problem.

Table 1: The existence problem for stable partitions

	STRICT PREFERENCES		**INDIFFERENCES**	
	stable	**strongly stable**	**stable**	**strongly stable**
$\mathcal{B}$-PREFERENCES	polynomial (algorithm BSTABLE)	polynomial (algorithm BSTABLE)	NP-compl.	NP-compl.
$\mathcal{BW}$-PREFERENCES	polynomial (algorithm BSTABLE)	NP-hard	???	NP-hard
$\mathcal{W}$-PREFERENCES	polynomial (modif. Irving's algorithm)	polynomial (Irving's algorithm)	NP-compl.	???
$\mathcal{WB}$-PREFERENCES	polynomial (modif.Irving's algorithm)	polynomial (modif. Irving's algorithm)	???	???

Table 2: Testing stability of a given partition

$\mathcal{B}$-PREFERENCES	polynomial (blocking sets equivalent to cycles in special digraphs)
$\mathcal{W}$-PREFERENCES **$\mathcal{WB}$-PREFERENCES**	polynomial (blocking sets of sizes at most two)
$\mathcal{BW}$-PREFERENCES	???

Besides the questions depicted in *Tables 1* and *2*, a number of other interesting open problems remain. These include:

- For the moment, there are some results on complexity of CFG with $\mathcal{B}$, $\mathcal{W}$, $\mathcal{BW}$ and $\mathcal{WB}$-preferences, but we know nothing about a structure of all stable partitions. For a game with $\mathcal{W}$-preferences, one can take an inspiration from results obtained for the stable roommates problem, which has been shown to be very similar to such a game. In particular, for a stable roommates problem, the structure of all stable matchings was shown to be a semilattice. Moreover, in Gusfield and Irving (1989), an efficient algorithm to construct the set of all stable matchings was described.

- It would be interesting to study the computational complexity questions also for problems of deciding the existence of a Nash stable, an individually stable and a contractually individually stable partition in games with $\mathcal{B}$, $\mathcal{W}$, $\mathcal{BW}$ and $\mathcal{WB}$-preferences.

- All four studied extensions of players' preferences over individuals are based solely on the extreme (the best and/or the worst) players of a coalition. Could we obtain some interesting results also for CFG with median-based preferences?

7 Concluding remarks

In this paper we attempt to provide a survey of recent approaches to the problem of coalition formation. In particular, we concentrate on hedonic coalition formation games. The essential concept in CFG is the concept of stability. Thus, we survey definitions of several stability concepts, describe situations where a particular stability concept is the most appropriate and discuss relationships between these concepts. As there exist games admitting no stable partition, researchers focused to describe various models, where players' preferences over coalitions are suitably restricted in order to guarantee the existence of a stable partition. In our survey we present a summary of results obtained for different restrictions of players' preferences.

Further, we stress the importance of devising efficient algorithms for solving CFG and the necessity of considering also the computational complexity questions of studied problems. From the computational point of view, the main difficulty of CFG models is that players are supposed to submit their preferences over all coalitions they could belong to, and therefore already the input of a game (consisting of players' preference lists) has an exponential size. Apparently, too long input causes a great amount of computations and thus exponential complexity of considered problems. An interesting idea is to represent players' preferences over sets by their preferences over individuals and a particular extension rule. We emphasize the result from a utility theory, that under an extension satisfying two mildest axioms (simple dominance and independence), every nonempty set S must be indifferent to the two-element set consisting of the best and the worst alternatives of S. Thus the best and the worst alternative of a set can play a crucial role in many reasonable extensions.

In this paper, we give an overview of the known algorithmic results obtained for coalition formation games with $\mathcal{B}$, $\mathcal{W}$, $\mathcal{BW}$ and $\mathcal{WB}$-preferences, where players' preferences over sets are derived from the players' preferences over individuals on the basis of the best and/or the worst player. For the most considered problems, in the case of strict preferences, a (strongly) core stable partition can be found in polynomial time, while in the case with indifferences the corresponding decision core nonemptyness problems become NP-complete. We conclude this review with a list of open problems that are worth of a further study.

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