The kidney exchange game*

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Abstract. The most effective treatment for kidney failure that is currently known is transplantation. As the number of cadaveric donors is not sufficient and kidneys from living donors are often not suitable for immunological reasons, there are attempts to organize exchanges between patient-donor pairs. In this paper we model this situation as a cooperative game and propose some algorithms for finding a solution.

Keywords. Kidney transplantation, cooperative game, Pareto optimal solution, core, algorithm. **Mathematical subjects classification.** 91B68, 68Q25

1 Introduction

Renal failure is a very serious illness for which the most effective treatment that is currently known is kidney transplantation. Ideally, a kidney from a deceased donor could be used, but the supply of those in spite of joint efforts of national and even international organisations (for example Eurotransplant Foundation [19] and the United Network for Organ Sharing in the USA [20]) is not sufficient for the growing demand and the waiting time of a patient is unpredictable. As the operation techniques improved and the risk for a living donor of a kidney (a genetic or an emotional relative of the patient) was minimized, the number of live-donor transplantations increased. Moreover, some studies [18] show that grafts from living donors have a higher survival rate.

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For a transplantation to be successful, some immunological requirements must be fulfilled. Basically, ABO incompatibility and a positive cross-match are an absolute contraindication, moreover, the greater the number of HLA mismatches between the donor and the recipient, the greater the chance of rejection [9]. Hence, it often happens that a willing donor cannot donate his/her kidney to the intended recipient. Therefore in several countries systematic kidney exchange programs have been established: in Romania [11], the Netherlands [10], USA [14, 15, 16]; in other cases there are isolated examples (in the Middle East [7]).

Kidney exchange (KE for short) is still a controversial issue, however the aim of this paper is not to discuss the ethical and legal aspects of this concept. In spite of some pessimistic expectations (the British Transplantation Society estimated potential benefits from living donors' exchanges to be around 3% [3]), in an institute of Romania the monthly mean number of transplantations increased from 4.2 to 6.1 since the KE program started [11] and a simulation study using real USA statistical data revealed that the utilization of kidneys from living donors without exchanges was around 55%, whilst with exchanges it increased to 91% when the size of the simulated population was 300 [14].

We follow the approach started in [14] and [15], in that we represent KE as a cooperative game, in which patient-donor pairs seek cyclic exchanges of kidneys. Since all operations on a cycle should be performed simultaneously (to avoid the risk that one of the donors will withdraw his or her commitment after the others have undergone nephrectomy [7]), cycles should be as short as possible for logistical reasons. Therefore we directly incorporate cycle lengths into preference models (notice that in [14] the obtained cycle lengths were just observed after simulations of the algorithm used, and in [16] the number of matched patients was derived as a function of the maximum allowed cycle length). Further, besides Pareto optimal solutions we also consider the core and the strict core of the KE game. We conclude with computational complexity considerations of the proposed algorithms and some open questions.

2 The KE game

An instance of the KE game is represented by a directed graph $G = (V, A)$ where each vertex $v \in V$ corresponds to a patient and his intended (incompatible) donor (or donors). A pair $(i, j) \in A$ if the patient corresponding to vertex i can accept a kidney from a donor corresponding to vertex j. (Hence, if patient i has a compatible donor, $(i, i) \in A$.) Moreover, for each vertex i there is a linear ordering $\leq_i$ on the set of endvertices of arcs incident from i. This ordering is represented by a preference list of i and we assume that i is the last entry in each preference list. If $j \leq_i k$ and simultaneously $k \leq_i j$, we say that i is *indifferent* between j and k, or that j and k are *tied* in i's preference list, and write $j \sim_i k$; if $j \leq_i k$ but not $k \leq_i j$, then i *strictly prefers* j to k, and we write $j <_i k$. Hence, in general, preference lists may contain ties.

Definition 1 *A* kidney exchange *game (KE game for short) is a triple* $\Gamma = (V, G, \mathcal{O})$*, where* V *is the set of players,* G *is a digraph with a vertex set* V *and* $\mathcal{O} = \{\leq_i; i \in V\}$.

Definition 2 *A* solution *of the KE game* $\Gamma = (V, G, \mathcal{O})$ *is a permutation* π *of* V *such that* $i \neq \pi(i)$ *implies* $(i, \pi(i)) \in A$ *for each* $i \in V$. *If* $(i, \pi(i)) \in A$*, we say that* i *is* covered *by* π*, otherwise* i *is* uncovered. *A player* i *is assigned in a solution* π *the pair* $(\pi(i), C^\pi(i))$*, where* $C^\pi(i)$ *denotes the cycle of* π *containing* i.

A player evaluates a permutation not only according to the player he is assigned to, but he also takes into account the cycle length. This is expressed by the following extension of preferences from $\mathcal{O}$ to preferences over (player,cycle) pairs. The same symbol is used for preferences over players as well as over pairs and permutations.

Definition 3 *A player* i *prefers pair* (j, M) *to pair* (k, N) *if*
(i) $j <_i k$ *or*
(ii) $j \sim_i k$ *and* $|M| \leq |N|$.

Definition 4 *A coalition* $S \subseteq V$ weakly blocks *a solution* π *if there exists a permutation* σ *of* S *such that*
(i) $(\sigma(i), C^\sigma(i)) \leq_i (\pi(i), C^\pi(i))$ *for each* $i \in S$ *and*
(ii) $(\sigma(j), C^\sigma(j)) <_j (\pi(j), C^\pi(j))$ *for at least one* $j \in S$.
A coalition $S \subseteq V$ blocks *a solution* π *if there exists a permutation* σ *of* S *such that* $(\sigma(i), C^\sigma(i)) <_i (\pi(i), C^\pi(i))$ *for each* $i \in S$.

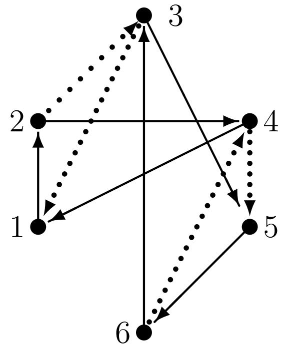
Now we define the studied solution concepts.

Definition 5 *A permutation* π *is* Pareto optimal *for game* Γ *(belongs to* $PO(\Gamma)$ *for short) if the grand coalition* V *does not block; it is* strongly Pareto optimal *(*$\pi \in SPO(\Gamma)$ *for short) if* V *does not weakly block.*

Definition 6 *A permutation* π *is in the* core $C(\Gamma)$ *of game* Γ *if no coalition blocks* π *and* π *is the* strong core $SC(\Gamma)$ *of* Γ *if no coalition weakly blocks.*

Notice that our concepts of *strongly Pareto optimal* and *Pareto optimal* solutions correspond to what are usually called *Pareto optimal* and *weakly Pareto optimal* solutions respectively [14]. However, we use the first two terms to make the terminology consistent within this paper.

As each blocking coalition is also weakly blocking, we have $SC(\Gamma) \subseteq C(\Gamma) \subseteq PO(\Gamma)$ and $SC(\Gamma) \subseteq SPO(\Gamma) \subseteq PO(\Gamma)$ in general, but the inclusion $C(\Gamma) \subseteq SPO(\Gamma)$ is not always true even in the case without indifferences.



Consider the KE game Γ given by the digraph to the left, where the first entries in the preference lists are denoted by solid lines, while the second entries correspond to dotted lines. Permutation $\pi = (1,2,3)(4,5,6)$ is in $C(\Gamma)$, but $\pi \notin SPO(\Gamma)$, as the grand coalition weakly blocks π by considering permutation $\sigma = (1,2,4)(5,6,3)$. However, in the practical kidney exchange application it is not clear which solution concept is the most suitable one.

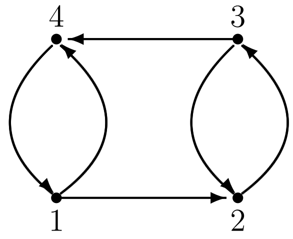
3 The case with preferences

With no indifferences, the famous Top Trading Cycles (TTC for short) algorithm (Figure 1) can be used for the KE game. The TTC algorithm was originally proposed by Gale in [17] for housing markets, where cycle lengths were not taken into account. It was shown that the TTC algorithm outputs a permutation in the core of the housing market also in the case with indifferences (ties are broken arbitarily). In [13] Roth and Postlewaite proved that if there are no indifferences, the strong core of the housing market is nonempty and contains a unique permutation. Further, Roth [12] proved that the TTC algorithm is strategy-proof. However, a detailed consideration of algorithmic questions connected with the TTC algorithm is quite recent; in [1] its implementation with $O(m)$ time complexity was proposed, where m is the number of arcs in G. In [6], the TTC algorithm is called Algorithm B-stable and it was shown that in the case with no indifferences, its output is in $SC(\Gamma)$.

Input. KE game $\Gamma = (V, G, \mathcal{O})$. **Output.** A permutation $\pi = C_1 C_2 \dots C_r$ of V.
Step 0. $N := V$, round $r := 0$.
Step 1. Choose an arbitrary player i_0.
Step 2. Player i_0 points to his favourite i_1 in N. i_1 points to his favourite i_2 in N etc. A cycle arises or player i_k cannot point.
Step 3. $r := r+1$. If a cycle C was obtained, then $C_r := C$, otherwise $C_r = (i_k)$. $N := N - C_r$.
Step 4. If $N \neq \emptyset$, go to Step 1, otherwise end.

Figure 1: **The Top Trading Cycles (TTC) Algorithm.**

However, the TTC algorithm may fail to find a permutation in the $C(\Gamma)$ and also in $PO(\Gamma)$ if indifferences are present. In the example in the next page, the TTC algorithm may, depending on how the ties in vertices 1 and 3 are broken, output the permutation $(1,2,3,4)$, while here $SPO(\Gamma) = PO(\Gamma) = C(\Gamma) = SC(\Gamma)$ and they contain a unique permutation $(1,4)(2,3)$.



Moreover, in [5] it was proved that in the case with indifferences it is NP-complete to decide whether $C(\Gamma) \neq \emptyset$, and it is also NP-complete to decide whether $SC(\Gamma) \neq \emptyset$. Notice that it is always the case that $SPO(\Gamma) \neq \emptyset$: if we define a partial order on the set of all permutations of V by setting $\sigma \leq \pi$ if $(\sigma(i), C^{\sigma}(i)) \leq_i (\pi(i), C^{\pi}(i))$ for each player, then the minimal permutations in this partial order are strongly Pareto optimal (and hence also Pareto optimal). We show later, however, that it is NP-hard to find a permutation in $SPO(\Gamma)$ (Theorem 5).

4 The Simple KE game

Let us now suppose that all compatible kidneys are equally suitable for transplantation, as suggested in [15] and [16]. We will say that now the players have *dichotomous* preferences (we follow the terminology of [2]), or that all acceptable vertices are tied. Hence patients with a compatible donor are not considered and the KE game is identified simply with a digraph $G = (V, A)$ without loops.

Definition 7 *A simple KE game is the pair* $\Gamma = (V, G)$*, where* V *is the set of players and* $G = (V, A)$ *is a digraph without loops.*

Similarly as for the KE game from Definition 1, the solution of a simple KE game is a permutation π of V such that $i \neq \pi(i)$ implies $(i, \pi(i)) \in A$ for each $i \in V$ and the notation from Definition 2 can be used. Further, preferences of players over (player, cycle) pairs now reduce to saying that a player prefers to be covered to being uncovered, and if covered, he prefers to be in a shorter cycle.

In [15], where dichotomous preferences were considered, the authors allowed only cycles of length 2. Hence the KE game was represented by an undirected graph $G = (V, E)$ with $ij \in E$ if patient i can accept kidney from donor j and also conversely. A strongly Pareto optimal solution was identified as a maximum cardinality matching.

Even without the restriction on cycle lengths, we are able to formulate a necessary and sufficient condition for a permutation to be in $C(\Gamma)$.

Theorem 1 $\pi \in C(\Gamma)$ *if and only if* G *contains no cycle* $C = (i_1, i_2, \ldots, i_k)$ *such that* $|C^{\pi}(i_j)| > k$ *or* $\pi(i_j) = i_j$ *for each* $j = 1, 2, \ldots, k$.

Proof. Such a cycle clearly blocks π. On the other hand, if $\pi \notin C(\Gamma)$ then a blocking coalition has a form of C. ∎

Theorem 1 suggests an iterative approach for finding a permutation in the core of the simple KE game. The algorithm is depicted in Figure 2.

Theorem 2 *Algorithm CoreSimpleGame correctly finds a permutation in* $C(\Gamma)$ *of the simple KE game in polynomial time.*

Input: A simple KE game $\Gamma = (V, G)$. **Output:** A permutation π of V.
Step 1. Set $k := 2$ and create the 2-reduced graph $G_2 = (V_2, E_2)$ of digraph G by letting:
$V_2 = \{v \in V : v \text{ lies on a 2-cycle in } G\},\ E_2 = \{e = ij : (i,j) \in A \ \&\ (j,i) \in A\}$.
Find a maximal matching M_2 in G_2 and set $\pi(i) = j$ for each $ij \in M_2$.
Delete from G all vertices covered by M_2.
Step 2. $k := k + 1$.
Step k. Create the k-reduced digraph $G_k = (V_k, A_k)$ of G by letting:
$V_k = \{v \in V : v \text{ lies on a } k\text{-cycle in } G\}$, $A_k = \{e \in A : e \text{ lies on a } k\text{-cycle in } G\}$.
Find a maximal k-cycle packing M_k of G_k and set $\pi(i) = j$ for each arc $(i,j) \in M_k$.
Delete from G all vertices covered by M_k. If G contains no cycle of length $k+1$, end. Otherwise go to Step 2.

Figure 2: **Algorithm CoreSimpleGame**

For the strong core, Theorem 1 can be easily modified:

Theorem 3 *$\pi \in SC(\Gamma)$ if and only if G contains no cycle $C = (i_1, i_2, \ldots, i_k)$ such that either $|C^\pi(i_j)| > k$ or $\pi(i_j) = i_j$ for at least one $j \in \{1, 2, \ldots, k\}$, and either $|C^\pi(i_l)| \geq k$ or $\pi(i_l) = i_l$ for the remaining vertices i_l of cycle C.*

Corollary 1 *If $\pi \in SC(\Gamma)$ then π restricted to the 2-reduced graph G_2 is a perfect matching of V_2 and for each $k > 2$, π restricted to the k-reduced digraph G_k is a partition of V_k into directed k-cycles. Consequently, if either G_2 admits no perfect matching or G_k admits no partition of V_k into directed k-cycles for some k, then $SC(\Gamma) = \emptyset$.*

Hence Algorithm CoreSimpleGame could be used to find a permutation in $SC(\Gamma)$, if in Step 2 a perfect matching and later a partition of V_k into directed k-cycles can be found. However, this problem already becomes difficult.

Theorem 4 *The problem of deciding whether $SC(\Gamma) = \emptyset$ for a simple KE game is NP-complete.*

Proof. Membership in the class NP follows from Theorem 3. In [8, Theorem 3.7] the NP-completeness of UNDIRECTED TRIANGLE PACKING is established. It is easily possible to obtain NP-completeness for DIRECTED TRIANGLE PACKING in graphs with no bidirected arcs, where each vertex lies on a directed triangle, by considering the reduction constructed there and directing appropriately the arcs in the graph shown in [8, Figure 3.8].

An instance $G = (V, A)$ of DIRECTED TRIANGLE PACKING gives rise to an instance Γ of a simple KE game. It is then straightforward to verify that G admits a directed triangle packing if and only if $SC(\Gamma) \neq \emptyset$. ■

However, in spite of the fact that $SPO(\Gamma) \neq \emptyset$ for all KE games, we have:

Theorem 5 *It is NP-hard to find a permutation in $SPO(\Gamma)$ for a simple KE game.*

Proof. Suppose we have a polynomial-time algorithm $\mathcal{A}$ for finding a permutation in $SPO(\Gamma)$. Now let $G = (V, A)$ be a directed graph with no bidirected arcs (given as an instance of DIRECTED TRIANGLE PACKING). Run $\mathcal{A}$ on G to find $\pi \in SPO(\Gamma)$. If π contains only 3-cycles then clearly π gives a directed triangle packing for G. Now suppose that it is not the case that π contains only 3-cycles. Suppose for a contradiction that G admits a directed triangle packing. This defines a permutation σ. Consider any vertex $v \in V$ that does not belong to a 3-cycle in π (there is at least one such vertex). If v is uncovered by π then v improves in σ, since v is covered by σ. Otherwise v belongs to a cycle of length > 3 in π and v also improves in σ, since v belongs to a 3-cycle in σ. Hence V weakly blocks π via σ, contradicting the assumption that $\pi \in SPO(\Gamma)$. ∎

5 Conclusion and open questions

The research on efficient algorithms for kidney exchange problems has just started. So there are still many questions to be answered and many problems to be solved. Let us mention just a few directions for possible further research.

- As the basic priority is to treat as many patients as possible, find a solution of the KE game with the maximum number of covered verties.
- As patients enter and leave the waiting lines unpredictably, explore the possibilities of on-line algorithms for kidney exchange.

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