Vertex-distinguishing proper edge colourings of some regular graphs

Janka Rudašová and Roman Soták

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Institute of Mathematics, Faculty of Science
University of P. J. Šafárik
Jesenná 5, 041 54 Košice, Slovak Republic
rudasova@science.upjs.sk, sotak@science.upjs.sk

Abstract

The vertex-distinguishing index $\chi'_s(G)$ of a graph G is the minimum number of colours required to properly colour the edges of G in such a way that any two vertices are incident with different sets of colours. We consider this parameter for some regular graphs. Moreover, we prove that for any graph, the value of this invariant is not changed if the colouring above is, in addition, required to be equitable.
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1 Introduction

Let G be a simple graph. For $d \geq 0$, let V_d be the set of all vertices of degree d in G and let $n_d = |V_d|$ be the number of these vertices. Let $\chi'(G)$ be the minimum number of colours required for a proper edge colouring of G. Given such a proper colouring with colours $\{1, \ldots, k\}$ (a k-colouring in the sequel) and a vertex v of G, denote by $S(v)$ the set of colours used to colour the edges incident to v. The set of edges of G coloured by colour a is denoted by E_a. A proper edge k-colouring of G is called *equitable* if $||E_a| - |E_b|| \leq 1$ for all $a, b \in \{1, \ldots, k\}$.

A proper edge colouring of a graph is said to be *vertex-distinguishing* if each pair of vertices is incident to a different set of colours, that is, if $S(u) \neq S(v)$ for all vertices $u \neq v$. A vertex-distinguishing proper edge colouring will also be called a *strong* colouring. A graph has a strong colouring if and only if it has not more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec*-graph. The minimum number of colours required for a strong colouring of a *vdec*-graph G is denoted by $\chi'_s(G)$. This definition is given by [3].

The concept of vertex-distinguishing colouring was introduced independently by Aigner, Triesch and Tuza [1], by Burris and Schelp [6] and by Horňák and Soták [9], and has been considered in several papers [3, 5, 7, 8, 10]. In [6] Burris and Schelp posed the following conjecture:

Conjecture 1.1 *Let G be a vdec-graph and k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) \in \{k, k+1\}$.*

This paper contains two separate results. The first one proves the above conjecture for some r-regular graphs with small components; this result is a special case of a more general statement that shows that the conjecture is true for sufficiently many copies of an arbitrary 1-factorizable graph. The second one concerns the existence of equitable strong colourings; we prove that for any graph G there exists an equitable strong k-colouring for any $k \geq \chi'_s(G)$.

2 Strong colourings of regular graphs

For a graph G, let $lb(G) = \min\{k : \binom{k}{d} \geq n_d, \delta(G) \leq d \leq \Delta(G)\}$. Thus, $lb(G)$ is a lower bound for $\chi'_s(G)$.

Balister et al. [4] proved that if $\Delta(G) \geq \sqrt{2|V(G)|} + 4$ and $\delta(G) \geq 5$ then $\chi'_s(G) \leq lb(G) + 1$. This result covers a relatively large class of graphs; on the other hand, for the graph nG (consisting of n disjoint copies of G) the assumptions are not fulfilled (it is enough to take $n > \frac{(\Delta(G)-4)^2}{|V(G)|}$). Further, it is also interesting to study regular graphs of small degree. The solution for the case of 2-regular graphs is given by the following theorem (which is an immediate corollary of the result of Balister [2]):

Theorem 2.1 *Let G be an union of cycles $C_{m_1}, \ldots, C_{m_t}$ and let $L = \sum_{i=1}^{t} m_i$, $m_i \geq 3$ for $i = 1, \ldots, t$. Then $\chi'_s(G) \leq k$ if and only if either*

1. *k is odd, $L = \binom{k}{2}$ or $L \leq \binom{k}{2} - 3$, or*
2. *k is even, $L \leq \binom{k}{2} - \frac{k}{2}$.*

For 3-regular graphs, the following results are known:

Theorem 2.2 ([11]) *Let L_n be a graph of n-sided prism. Then $\chi'_s(L_n) \leq lb(L_n) + 1$.*

Theorem 2.3 ([11]) $\chi'_s(nK_4) \le lb(nK_4) + 1$.

In this section we generalize the method used in [11] to show that $\chi'_s(nG) \le lb(nG) + 1$ when G is a 3-regular 1-factorizable graph with $|V(G)| \le 12$, or $G = K_{t,t}$ for $3 \le t \le 7$, or $G = K_6$. For each of these three cases, the proof is by induction on the number of colours; in the induction step, the new colour $k+1$ induces a perfect matching in a certain number of copies of G.

Let $n(G,k) = \max\{n : \chi'_s(nG) \le k\}$ denote the greatest number n such that nG has a strong colouring using at most k colours.

Lemma 2.4 *If G has a perfect matching M, then $n(G, k+1) \ge n(G,k) + n(G-M,k)$.*

Proof: We take a strong colouring of $n(G,k)$ copies of G using at most k colours. The new colour $k+1$ is used for colouring the perfect matching in $n(G-M,k)$ copies of G, where vertices of these copies are distinguished by a strong colouring of these copies that uses at most k colours. In this way we obtain a strong colouring of $n(G,k) + n(G-M,k)$ copies of G that uses at most $k+1$ colours. □

Theorem 2.1 gives the following

Corollary 2.5 *Let G be 2-regular graph. Then*

a) if k is even, then $n(G,k) = \left\lfloor \frac{\binom{k}{2} - \frac{k}{2}}{|V(G)|} \right\rfloor$

b) if k is odd and $\binom{k}{2} \equiv 0 \ (mod\ |V(G)|)$ *then* $n(G,k) = \frac{\binom{k}{2}}{|V(G)|}$

c) if k is odd and $\binom{k}{2} \not\equiv 0 \ (mod\ |V(G)|)$ *then* $n(G,k) = \left\lfloor \frac{\binom{k}{2} - 3}{|V(G)|} \right\rfloor$.

To confirm Conjecture 1.1 for an arbitrary number of copies of an r-regular graph G on p vertices, it is sufficient to prove that, for all $k > r$, $n(G,k) \ge \left\lfloor \frac{\binom{k-1}{r}}{p} \right\rfloor$ holds. Then, by definition,

$$\chi'_s(nG) = k \text{ for } n(G, k-1) < n \le n(G,k)$$

and, moreover,

$$\chi'_s(nG) = k \text{ for } \frac{\binom{k-1}{r}}{p} < n \le n(G,k)$$

$$\chi'_s(nG) = k+1 \text{ for } n(G,k) < n \le \frac{\binom{k}{r}}{p}$$

Therefore, for n satisfying $\binom{k-1}{r} < np \leq \binom{k}{r}$ we have $\chi'_s(nG) \leq k+1$, which confirms the conjecture.

Note that in some proofs that follow, the inequality $n(G,k) \geq \left\lfloor \frac{\binom{k-1}{r}}{p} \right\rfloor$ will be used (and proved) without the floor function, or, even, we will use and prove sharp inequality.

Theorem 2.6 *There exists a function $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for each 1-factorizable r-regular graph G ($r \geq 2$) on p vertices and for each $k \geq g(p,r)$,*

$$n(G,k) > \frac{\binom{k-1}{r}}{p}.$$

Proof: By induction (first by r, then by k).

Firstly, let $r=2$. We will show that it is enough to put $g(p,2) = 2p+2$. Since Corollary 2.5 gives in this case the exact value of $n(G,k)$, we have (for $k \geq 6$)

$$n(G,k) \geq \left\lfloor \frac{\binom{k}{2} - \frac{k}{2}}{p} \right\rfloor > \frac{\binom{k}{2} - \frac{k}{2}}{p} - 1 = \frac{\binom{k-1}{2} + \binom{k-1}{1} - \frac{k}{2}}{p} - 1 =$$

$$= \frac{\binom{k-1}{2}}{p} + \frac{\frac{k}{2} - 1 - p}{p} \geq \frac{\binom{k-1}{2}}{p} + \frac{0}{p} = \frac{\binom{k-1}{2}}{p}.$$

Now suppose that theorem is valid for each $r' < r$; we will show that $g(p,r)$ exists.

Using the induction assumption in the induction proofs by k and r, we have $n(G,k+1) \geq n(G,k) + n(G-M,k) > \frac{\binom{k-1}{r}}{p} + \frac{\binom{k-1}{r-1}}{p} = \frac{\binom{k}{r}}{p}$. Then it is enough to choose $g(p,r) \geq g(p,r-1)$ such that $n(G, g(p,r)) > \frac{\binom{g(p,r)-1}{r}}{p}$. We will not state the minimal possible value of $g(p,r)$; instead of this, we will show how this value may be found when $g' = g(p,r-1)$ is known.

The construction of a strong colouring for the desired number of copies proceeds as follows: for $i = g', g'+1, \ldots, g-1$ (where $g = g(p,r)$) we colour subsequently $n(G-M,i)$ copies of $G-M$ using colours $1,\ldots,i$ and the missing perfect matching M is coloured by the colour $i+1$. The colouring obtained in this way is a strong colouring of $\sum\limits_{i=g'}^{g-1} n(G-M,i)$ copies of G using colours $1,\ldots,g$. Since $\sum\limits_{i=g'}^{g-1} n(G-M,i) \geq \sum\limits_{i=g'}^{g-1} \left(\frac{1}{p} + \frac{\binom{i-1}{r-1}}{p} \right) = \frac{g-g'}{p} + \frac{1}{p} \sum\limits_{i=g'}^{g-1} \binom{i-1}{r-1}$, it is enough to choose g such that $\frac{g-g'}{p} + \frac{1}{p} \sum\limits_{i=g'}^{g-1} \binom{i-1}{r-1} > \frac{1}{p}\binom{g-1}{r}$. But, as $\binom{g-1}{r} = \binom{g-2}{r-1} + \binom{g-2}{r} =$

$\binom{g-2}{r-1}+\binom{g-3}{r-1}+\binom{g-3}{r}=\cdots=\sum_{i=g'}^{g-1}\binom{i-1}{r-1}+\binom{g'-1}{r}$, it is enough to choose g such that $\frac{g-g'}{p}>\frac{1}{p}\binom{g'-1}{r}$ (for example, any $g>g'+\binom{g'-1}{r}$). □

Note that instead of taking $n(G,k)$ copies of the same graph G, it is even possible to combine distinct graphs, the only thing that has to be fulfilled is that each such graph is an r-regular 1-factorizable graph on p vertices.

Corollary 2.7 *Let G be disjoint union of sufficiently many r-regular 1-factorizable graphs on p vertices. Then $\chi'_s(G)\le lb(G)+1$.*

Now we apply this result on some regular graphs with small number of vertices.

Theorem 2.8 *Let G be 3-regular graph with 1-factor on at most 12 vertices. Then, for each positive integer n, $\chi'_s(nG)\le lb(nG)+1$.*

Proof: We show that $n(G,k)\ge\left\lfloor\frac{\binom{k-1}{3}}{|V(G)|}\right\rfloor$ for $k\ge 4$, which implies the theorem. Let $p=|V(G)|$.

We know (by Corollary 2.5) that $n(G-M,k)\ge\left\lfloor\frac{\binom{k}{2}-\frac{k}{2}}{p}\right\rfloor$ for $k\ge 6$. We proceed by induction. Suppose that the theorem holds for $k=4p$. For $k>4p$, we have

$$n(G,k+1)\ge n(G,k)+n(G-M,k)\ge\left\lfloor\frac{\binom{k-1}{3}}{p}\right\rfloor+\left\lfloor\frac{\binom{k}{2}-\frac{k}{2}}{p}\right\rfloor=$$

$$=\frac{\binom{k-1}{3}-a}{p}+\frac{\binom{k}{2}-\frac{k}{2}-b}{p}=\frac{\binom{k}{3}}{p}+\frac{\frac{k}{2}-1-a-b}{p}>\frac{\binom{k}{3}}{p}\ge\left\lfloor\frac{\binom{k}{3}}{p}\right\rfloor,$$

since $a,b<p$.

For small values of $k\le 4p$ we summarize the lower bounds for $n(G,k)$ by constructing tables as follows (for p=8, see Table 1):

k	4	5	6	7	8	9	10	11	12	13	14	15	...
$n(G-M,k)$	0	0	1	2	3	4	5	6	7	9	10	12	...
$n(G,k)\ge$	0	0	0	1	3	6	10	15	21	28	37	47	...
$\left\lfloor\frac{\binom{k-1}{3}}{p}\right\rfloor$	0	0	1	2	4	7	10	15	20	27	35	45	...
deficit			1	1	1	1							...

Table 1: The values for $p=8,k\le 4p=32$

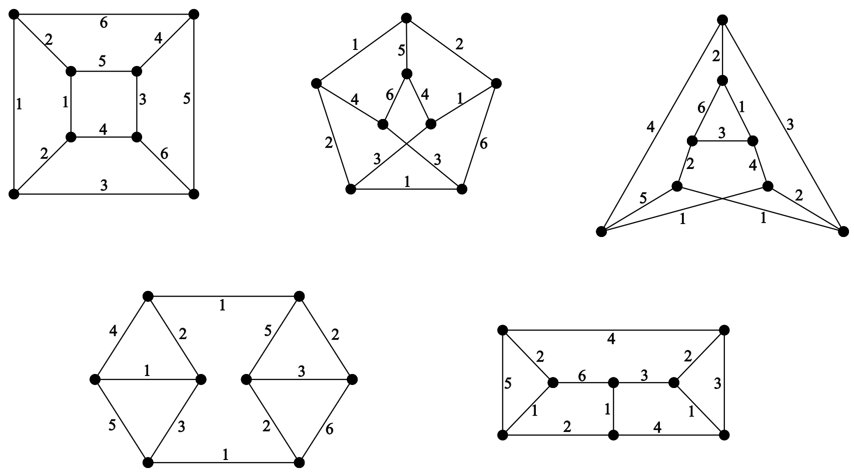

Figure 1: 3-regular graphs on 8 vertices

We use only the fact that $n(G,4)=0$ (which is fulfilled even for $G=K_4$) and the formula $n(G,k+1)\geq n(G,k)+n(G-M,k)$. For each $p\in\{4,6,8,10,12\}$, in the corresponding tables, the estimated $n(G,k)$ is at least $\left\lfloor\frac{\binom{k-1}{3}}{p}\right\rfloor$ except of the case $p=8$ and $k=6,7,8,9$ where the deficit (that is, the positive difference between $\left\lfloor\frac{\binom{k-1}{3}}{p}\right\rfloor$ and the estimation of $n(G,k)$) is equal to 1. This problem may be fixed by showing that $n(G,6)\geq 1$ for each cubic graph G on 8 vertices. However, in these cases, it is not hard to find a particular strong colouring that uses 6 colours (note that there are $\binom{6}{3}=20$ colour sets available while only 8 vertices have to be distinguished, see the Figure 1). □

In the following, we need several easy results:

Lemma 2.9 *Let G be 2-regular graph on p vertices. Then for each $k\geq 2p$,*

$$n(G,k)\geq\frac{\binom{k-1}{2}}{p}.$$

Proof: If k is even, then $n(G,k)=\left\lfloor\frac{\binom{k}{2}-\frac{k}{2}}{p}\right\rfloor=\frac{\binom{k}{2}-\frac{k}{2}-a}{p}=\frac{\binom{k-1}{2}}{p}+\frac{\frac{k}{2}-1-a}{p}\geq\frac{\binom{k-1}{2}}{p}+\frac{0}{p}$ because $0\leq a\leq p-1$. In the case of k being odd, we have $n(G,k)\geq\left\lfloor\frac{\binom{k}{2}-3}{p}\right\rfloor=\frac{\binom{k}{2}-3-a}{p}=\frac{\binom{k-1}{2}}{p}+\frac{k-4-a}{p}>\frac{\binom{k-1}{2}}{p}$. □

Lemma 2.10 *Let G be an r-regular graph on p vertices with a 1-factor M. Let there exist an integer g' such that, for each $k\geq g'$, $n(G-M,k)\geq\frac{\binom{k-1}{r-1}}{p}$, and let* *there exist an integer* $g \geq g'$ *such that* $n(G,g) \geq \frac{\binom{g-1}{r}}{p}$. *Then, for each* $k \geq g$,

$$n(G,k) \geq \frac{\binom{k-1}{r}}{p}.$$

Proof: By induction on k. For $k = g$, we have $n(G,k) \geq \frac{\binom{k-1}{r}}{p}$ by assumptions. For $k+1$, $n(G,k+1) \geq n(G,k) + n(G-M,k) \geq \frac{\binom{k-1}{r}}{p} + \frac{\binom{k-1}{r-1}}{p} = \frac{\binom{k}{r}}{p}$. □

Finally, we prove the following theorem showing the validity of the main conjecture for several other regular graphs:

Theorem 2.11 *Let* $G \in \{K_{4,4}, K_{5,5}, K_{6,6,}, K_{7,7}, K_6\}$. *Then, for each integer* n, $\chi'_s(nG) \leq lb(nG) + 1$.

Proof: For the proof for each of these graphs, we construct the table with the estimations of $n(G-tM,k)$ for $t = \Delta(G)-2, \Delta(G)-3, \ldots, 0$. By Lemmas 2.9 and 2.10, for the estimations for $k = 2|V(G)|$ we need to examine just the values $n(G,k)$ for $k \leq 2|V(G)|$ (note that in this case we can replace g and g' in Lemma 2.10 by $2|V(G)|$). Each table is constructed from the exact values of $n(G-tM,k)$ for $t = \Delta(G)-2$, exact values of $n(G-tM,4)$ for $t = \Delta(G)-2,\ldots,0$ (this value is 0) and using the inequality $n(G-tM,k+1) \geq n(G-tM,k)+n(G-(t+1)M,k)$; just the estimations of $n(G-tM,k)$ are replaced by the values in Figure 2.

We give here only the first rows of the table for $G = K_{6,6}$; the details are left to the reader. □

k	$n(G-4M,k)$	$n(G-3M,k) \geq$	$n(G-2M,k) \geq$	$n(G-M,k) \geq$	$n(G,k) \geq$	$\left\lceil \frac{\binom{k-1}{6}}{p} \right\rceil$
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	1	1*	1*	0	0	0
7	1	2	2	1	0	0
8	2	3	4	3	1	0
9	3	5	7	7	4	2
10	3	8	12	14	11	7

∗ these values are estimated by the strong colourings in Figure 2

Table 2: The values for $G = K_{6,6}$

Note that this proof contains also a proof of the conjecture for other regular graphs of the type nH, where $H = G - tM$; but, using the strong colourings in Figure 2, one cannot consider the arbitrary matchings M.

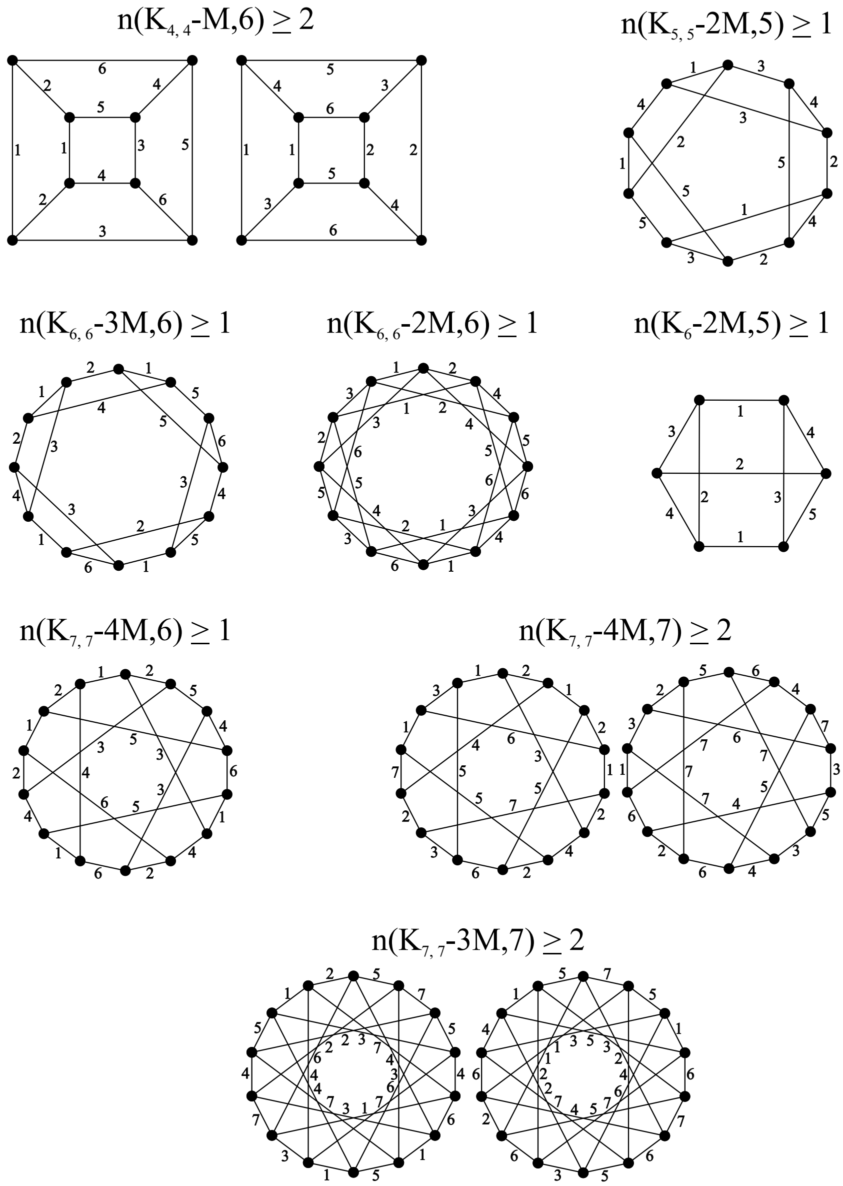

Figure 2: Estimations of $n(G - tM, k)$

3 Equitable strong colourings

Theorem 3.1 *Let G be vdec-graph and $k \geq \chi'_s(G)$. Then there exists an equitable strong k-colouring of G.*

Proof: Since $k \geq \chi'_s(G)$, there exists a strong k-colouring of G. From all such colourings, choose the colouring ψ for which the sum $\sum_{x,y} ||E_x| - |E_y||$ is minimal. In ψ, consider the pair a, b of colours such that $|E_a| - |E_b|$ is the maximal. Now, assume that ψ is not equitable; then $|E_a| - |E_b| \geq 2$. Obviously, for $c \in \{1, \ldots, k\}$, $|E_b| \leq |E_c| \leq |E_a|$, otherwise we get a contradiction with the choice of a, b. We will show that it is possible to interchange the colours a, b at certain edges of G such that the new colouring ψ' is again strong, but with the smaller value of $\sum_{x,y} ||E_x| - |E_y||$ than ψ has, a contradiction.

Let $G_{a,b}$ be the spanning subgraph of G induced by the set $E_a \cup E_b$. Then the components of $G_{a,b}$ are even cycles or paths. But, since the colours a and b alternate at the edges of the components of $G_{a,b}$ and $n_{a,b} = |E_a| - |E_b| \geq 2$, we obtain that in $G_{a,b}$ there are at least $n_{a,b}$ paths of odd length such that their end-edges are coloured by a; more precisely, if we denote by p_a (p_b, respectively) the number of odd length paths with end-edges coloured by a (b respectively), then $n_{a,b} = p_a - p_b$.

Now define the multigraph $H_{a,b}$ in the following way: $V(H_{a,b}) = V_a \cup V_b$ where $V_a = \{u \in V(G) : \ S(u) \cap \{a, b\} = \{a\}\}$, $V_b = \{u \in V(G) : \ S(u) \cap \{a, b\} = \{b\}\}$; the edge set of $H_{a,b}$ is $E(H_{a,b}) = E^*_p \cup E^*_q$ where $E^*_p = \{uv : u, v \in V_a \cup V_b$ and there is $u - v$ path in $G_{a,b}\}$, $E^*_q = \{uv : \ (\exists S \subseteq \{1, \ldots, k\} \setminus \{a, b\})$ such that $S(u) = S \cup \{a\} \wedge S(v) = S \cup \{b\}\}$. In other words, the vertices of $H_{a,b}$ are the ends of the above mentioned paths in $G_{a,b}$ and the edges of E^*_q join the vertices for which the colour sets are mutually exchanged by the interchange of the colours a, b. The set E^*_p is a perfect matching such that the number of edges between vertices of V_a is p_a and the number of edges between vertices of V_b is p_b; the remaining edges are between vertices of V_a and V_b. The set E^*_q is also a matching and all its edges are between vertices of V_a and V_b. Like for the graph $G_{a,b}$, for $H_{a,b}$ we have $\Delta(H_{a,b}) \leq 2$, thus, $H_{a,b}$ consists of even cycles (a 2-cycle is allowed) or paths.

Consider a component K of $H_{a,b}$ and put $g(K) = |E(K) \cap E(\langle V_a \rangle_{H_{a,b}})| - |E(K) \cap E(\langle V_b \rangle_{H_{a,b}})|$. This value counts how many more edges connect just vertices of V_a compared to the number of edges that connect just the vertices of V_b. Since the edges of E^*_p and E^*_q alternate in the paths and cycles of $H_{a,b}$ and the edges of E^*_q are just between vetices of V_a and V_b, we have $g(K) \in \{-1, 0, 1\}$. Note that $\sum_K g(K) = p_a - p_b = n_{a,b} \geq 2$. From this we obtain that, in $H_{a,b}$, there is a component K (more precisely, at least $n_{a,b}$ such components) for which $g(K) = 1$.

Now, construct the new colouring ψ' as follows: in the original graph G, interchange the colours a, b at all edges belonging to paths of $G_{a,b}$ that correspond to edges of $E(K) \cap E_p^*$. From the construction of $G_{a,b}$ and $H_{a,b}$ we have that, after this interchange, the number of occurrences of the colour a decreases by 1 and the number of occurrences of the colour b increases by 1. Furthermore, for some vertices of G, their colour sets are modified, but, according to the edges of E_q^* in $H_{a,b}$, the described interchange results just in the vanishing of two colour sets containing colour a and not containing colour b, as these two sets are replaced with the new ones that were missing in ψ.

Finally, when counting the sum $\sum_{x,y} ||E_x| - |E_y||$ for the colouring ψ', then, comparing with the original sum for ψ, the following terms are changed:

- $||E_a| - |E_b|| + ||E_b| - |E_a||$ where in the new sum this term is decreased by 4,
- $||E_a| - |E_c|| + ||E_b| - |E_c|| + ||E_c| - |E_a|| + ||E_c| - |E_b||$ where in the new sum this term is decreased by 4 if $|E_b| < |E_c| < |E_a|$ and it is not changed if $|E_c| \in \{|E_a|, |E_b|\}$.

Hence, the total sum is decreased by at least 4, which is a contradiction with the choice of ψ. □

This result may be helpful for determining the values of $\chi'_s(G)$, since, for showing that $\chi'_s(G) > lb(G)$, it is enough to show that there is no equitable strong colouring of G using $lb(G)$ colours. Similar methods were used in the research of balanced colourings, see [4].

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