On complete tripartite graphs arbitrarily decomposable into closed trails *

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Abstract

In the paper it is proved that any complete tripartite graph $K_{r,r,r}$, where $r = 5 \cdot 2^n$ and n is a nonnegative integer, has the following property: Whenever $(l_1, \dots, l_p)$ is a sequence of integers ≥ 3 adding up to $|E(K_{r,r,r})|$, there is a sequence $(T_1, \dots, T_p)$ of edge-disjoint closed trails in $K_{r,r,r}$ such that T_i is of length l_i, $i = 1, \dots, p$.

1 Introduction

In any simple finite nonoriented graph G with $\delta(G) \geq 2$ there is a cycle. Therefore, if G is *even* (if all vertices of G are of even degrees), it is an edge-disjoint union of cycles. Several authors investigated edge decompositions of complete multipartite graphs into cycles of equal lengths. The bipartite case has been completely solved by Sotteau [8]. For complete tripartite graphs some partial results are known, see Billington and Cavenagh [3], [4], Cavenagh [5]. In the general case a reader can consult Cockayne and Hartnell [6].

A connected edge-disjoint union of cycles is an Eulerian graph and has a closed Eulerian trail. So, an even graph can be expressed as an edge-disjoint union of closed trails, and there are many possibilities how to do it. Balister [1] has proved that if n is odd and $l_1, \dots, l_p$ are integers ≥ 3 adding up to $|E(K_n)|$, there are edge-disjoint closed trails $T_1, \dots, T_p$ in K_n such that T_i is

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of length l_i, $i = 1, \dots, p$. In the same paper a similar result has been reached for the graph $K_n - M_n$, where n is even and M_n is a perfect matching in K_n. Balister [2] has shown that there are positive constants n and ε such that any even graph G with $|V(G)| \geq n$ and $\delta(G) \geq (1-\varepsilon)|V(G)|$ can be (edge-)decomposed in the above manner. Another graphs with analogous properties concerning closed trails are complete bipartite graphs $K_{m,n}$ with m, n even, as proved by Horňák and Woźniak [7] (note that in that case all l_i's have to be even). In this paper we do concentrate on complete tripartite graphs.

Let us now precise the problem we are going to deal with. For integers p, q we use the notation $[p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : p \leq z\}$. The *concatenation* of finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB = (a_1, \dots, a_m, b_1, \dots, b_n)$. The concatenation is associative, and so we can use the symbol $\prod_{i=1}^{k} A_i$ for the result of concatenation of finite sequences $A_1, \dots, A_k$ (independently from the order in which "partial" concatenations are realised). If $k \in [0, \infty)$ and $A_i = A$ for any $i \in [1, k]$, we write A^k instead of $\prod_{i=1}^{k} A_i$ (so that $A^0 = (\ )$ is the empty sequence).

A *closed trail* of length $n \in [3, |E(G)|]$ (an *n-trail* for short) in a graph G is a sequence $T = \prod_{i=0}^{n}(v_i)$ of vertices of G such that $v_0 = v_n$, $v_i v_{i+1} \in E(G)$ and $v_i v_{i+1} \neq v_j v_{j+1}$ for any $i, j \in [0, n-1]$, $i \neq j$. The set $\{v_i v_{i+1} : i \in [0, n-1]\}$ of edges of T induces an Eulerian subgraph of G and throughout the whole paper we shall identify T with this subgraph. Deleting the edges of a closed trail from an even graph G results in an even graph with a smaller number of edges. This process can be continued until the edgeless graph $(V(G), \emptyset)$ is reached. If $(T_1, \dots, T_p)$ is the sequence of (closed) trails occurring in the process, then $1 \leq p \leq \lfloor |E(G)|/3 \rfloor$, $\{E(T_i) : i \in [1, p]\}$ is a decomposition of $E(G)$ and $\sum_{i=1}^{p} |E(T_i)| = |E(G)|$. Let $\mathrm{Lct}(G)$ be the set of all lengths of closed trails in G and let

$$\mathrm{Sct}(G) := \bigcup_{p=1}^{\lfloor |E(G)|/3 \rfloor} \{(l_1, \dots, l_p) \in (\mathrm{Lct}(G))^p : \sum_{i=1}^{p} l_i = |E(G)|\}.$$

A sequence $L = (l_1, \dots, l_p) \in \mathrm{Sct}(G)$ is said to be *G-realisable* if there is a *G-realisation* of L, a sequence $(T_1, \dots, T_p)$ of edge-disjoint closed trails in G (so that $\bigcup_{i=1}^{p} E(T_i) = E(G)$). A graph G is said to be *arbitrarily decomposable into closed trails* (ADCT for short) if $\mathrm{Sct}(G) \neq \emptyset$ and any sequence from $\mathrm{Sct}(G)$ is G-realisable. Evidently, if a graph is ADCT, it is even.

A sequence $A = (a_1, \dots, a_m)$ is said to be *changeable* to a sequence $B = (b_1, \dots, b_m)$ if there is a bijection $\pi : [1, m] \to [1, m]$ such that $b_i = a_{\pi(i)}$ for any $i \in [1, m]$; if A is changeable to B, we write $A \sim B$. For $I \subseteq [1, m]$

let $A\langle I\rangle$ be the subsequence of A created by deleting from A all a_i's with $i \in [1, m] - I$. If $A \in \mathbb{R}^m$ and $r \in \mathbb{R}$, we denote by $\mathrm{nd}(A)$ the (unique) nondecreasing sequence that is changeable to A and by $f_r(A)$ the *frequency* of r in A. For $l \in \mathbb{Z}$ let $(l)_4$ be the unique $m \in [0, 3]$ such that $l \equiv m \pmod 4$.

Let G, H be isomorphic graphs and let $\varphi : V(G) \to V(H)$ be an isomorphism from G onto H. If $T = \prod_{i=0}^{n}(v_i)$ is a closed trail in G, then $\varphi(T) := \prod_{i=0}^{n}(\varphi(v_i))$ is a closed trail in H. Further, if $\mathcal{T} = \prod_{i=1}^{q}(T_i)$ is a sequence of edge-disjoint closed trails in G, then $\varphi(\mathcal{T}) := \prod_{i=1}^{q}(\varphi(T_i))$ is a sequence of edge-disjoint closed trails in H.

Consider edge-disjoint closed trails T_1, T_2 in a graph G and let $T_1 + T_2$ denote the set of all closed trails T in G with $E(T) = E(T_1) \cup E(T_2)$. Clearly, $T_1 + T_2$ is nonempty if and only if $V(T_1) \cap V(T_2) \neq \emptyset$.

A sequence is said to be *simple* if no two its terms at distinct positions are the same. Let G be a graph and let $m \in [1, |V(G)|]$. A simple sequence $(v_1, \dots, v_m)$ of vertices of G is *similar* to a simple sequence $(w_1, \dots, w_m)$ of vertices of G provided that there is an automorphism φ of G, such that $\varphi(v_i) = w_i$ for any $i \in [1, m]$; if $m = 1$, we say for short that a vertex v_1 is similar to a vertex w_1. The relation of similarity of vertices of a graph G is an equivalence and a *similarity class* of G is a class of this equivalence.

A set S of edges of a graph G is said to be *complementary bipartite* in G provided that the graph $G - S$ is bipartite. Let $\mathrm{Cb}(G)$ denote the system of all sets $S \subseteq E(G)$ that are complementary bipartite in G and let $\mathrm{mcb}(G)$ be the minimum cardinality of a set $S \in \mathrm{Cb}(G)$.

Let $r \in [1, \infty)$ and $(n_1, \dots, n_r) \in [1, \infty)^r$. Throughout the whole paper we shall suppose that the r-partition of the complete r-partite graph $K_{n_1,\dots,n_r}$ is $\{\{v_{i,j} : i \in [1, n_j]\} : j \in [1, r]\}$. If $(n_1, \dots, n_r) = (n)^r$, we write for short $K_{(n)r}$ instead of $K_{n_1,\dots,n_r}$.

2 Some preparatory results

Lemma 1 *If G is a graph, $L_1, L_2 \in \mathrm{Sct}(G)$ and $L_1 \sim L_2$, then L_1 is G-realisable if and only if L_2 is G-realisable.*

Proof. If $L_1 = (l_1, \dots, l_p)$ has a G-realisation $(T_1, \dots, T_p)$ and $\pi : [1, p] \to [1, p]$ is such a bijection that $L_2 = (l_{\pi(1)}, \dots, l_{\pi(p)})$, then $(T_{\pi(1)}, \dots, T_{\pi(p)})$ is a G-realisation of L_2. ■

Proposition 2 *If G is a graph, $S \in \mathrm{Cb}(G)$ and T is a closed trail in G of an odd length, then $E(T) \cap S \neq \emptyset$.*

Proof. T is a non-bipartite graph, so it cannot be a subgraph of the bipartite graph $G - S$. ■

Proposition 3 *If G is a graph and a sequence $L = (l_1, \dots, l_p) \in \mathrm{Sct}(G)$ is G-realisable, then L contains at most $\mathrm{mcb}(G)$ odd terms.*

Proof. Let $S \in \mathrm{Cb}(G)$. Suppose that $(T_1, \dots, T_p)$ is a G-realisation of the sequence L in G and put $I := \{i \in [1,p] : l_i \equiv 1 (\mathrm{mod}\ 2)\}$. If $i \in I$, by Proposition 2 there exists $e_i \in E(T_i) \cap S$. Since trails in $(T_1, \dots, T_p)$ are edge-disjoint, we have $|I| = |\bigcup_{i\in I}\{e_i\}| \le |\bigcup_{i\in I}(E(T_i) \cap S)| \le |S| = \mathrm{mcb}(G)$. ∎

Let $\mathrm{Sct}^*(G)$ be the subset of $\mathrm{Sct}(G)$ consisting of sequences with at most $\mathrm{mcb}(G)$ odd terms. From Proposition 3 it follows that if a sequence $L \in \mathrm{Sct}(G)$ is G-realisable, then $L \in \mathrm{Sct}^*(G)$. So, if $\mathrm{Sct}(G) - \mathrm{Sct}^*(G) \neq \emptyset$, the graph G is not ADCT.

Proposition 4 *If $n \in [1, \infty)$, $\{G^{(i)} : i \in [1,n]\}$ is a set of pairwise edge-disjoint graphs and $G = \bigcup_{i=1}^n G^{(i)}$, then $\mathrm{mcb}(G) \ge \sum_{i=1}^n \mathrm{mcb}(G^{(i)})$.*

Proof. Suppose that $S \subseteq E(G)$ and put $S^{(i)} := S \cap E(G^{(i)})$ for $i \in [1,n]$. If $|S| < s := \sum_{i=1}^n \mathrm{mcb}(G^{(i)})$, there is $j \in [1,n]$ such that $|S^{(j)}| < \mathrm{mcb}(G^{(j)})$. The graph $G - S$ is a supergraph of the graph $G^{(j)} - S^{(j)}$ that is not bipartite, hence $S \notin \mathrm{Cb}(G)$. Thus, $\mathrm{mcb}(G)$ cannot be smaller then s. ∎

Proposition 5 *If a sequence $(n_1, n_2, n_3) \in [1,\infty)^3$ is nondecreasing, then $\mathrm{mcb}(K_{n_1,n_2,n_3}) = n_1 n_2$.*

Proof. Let V_1, V_2, V_3 be parts of $G := K_{n_1,n_2,n_3}$ with $|V_i| = n_i$, $i = 1,2,3$. Then $E(G) = E_{1,2} \cup E_{2,3} \cup E_{3,1}$, where $E_{i,j} := \{xy : x \in V_i, y \in V_j\}$, $i,j \in [1,3]$, $i \neq j$. The graph $G - E_{1,2}$ is bipartite, hence $\mathrm{mcb}(G) \le |E_{1,2}| = n_1 n_2$. Assume there is $S \in \mathrm{Cb}(G)$ with $|S| < n_1 n_2$. Then $S = S_{1,2} \cup S_{2,3} \cup S_{3,1}$, where $S_{i,j} := S \cap E_{i,j}$. If $i,j,k \in [1,3]$ and $\{i,j,k\} = [1,3]$, then the deletion of $e \in S_{i,j}$ from G destroys exactly n_k triangles of G. Therefore the deletion of S from G destroys at most $|S_{1,2}|n_3 + |S_{2,3}|n_1 + |S_{3,1}|n_2 \le (|S_{1,2}| + |S_{2,3}| + |S_{3,1}|)n_3 = |S|n_3 < n_1 n_2 n_3$ triangles of G and $G - S$ has a triangle in contradiction with the fact that $G - S$ is bipartite. ∎

Let $p_{(n)r} : V(K_{(n)r}) \to [1,r]$ be the function defined by $p_{(n)r}(v_{i,j}) = j$ for any $i \in [1,n]$ and $j \in [1,r]$ (a vertex x of $K_{(n)r}$ is assigned the number of the part containing x).

Proposition 6 *Let $n, r \in [1,\infty)$, $m \in [1, rn]$ and let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m)$ be simple sequences of vertices of $K_{(n)r}$ such that there is a permutation $\pi : [1,r] \to [1,r]$ satisfying $\pi(p_{(n)r}(v_i)) = p_{(n)r}(w_i)$ for any $i \in [1,m]$. Then v is similar to w.*

Proof. Consider a bijection $\varphi : V(K_{(n)r}) \to V(K_{(n)r})$ such that $\varphi(\{v_{i,j} : i \in [1,n]\}) = \{v_{i,\pi(j)} : i \in [1,n]\}$ for any $j \in [1,r]$ and $\varphi(v_i) = w_i$ for any $i \in [1,m]$. Clearly, φ is an automorphism of $K_{(n)r}$. ∎

Theorem 7 *If a graph K_{n_1,n_2,n_3} with $n_1 \le n_2 \le n_3$ is* ADCT*, then either $(n_1, n_2, n_3) \in \{(1,1,3), (1,1,5)\}$ or $n_1 = n_2 = n_3$.*

Proof. Let $G := K_{n_1,n_2,n_3}$ and $e := |E(G)|$. Vertices of G are of even degrees $n_1 + n_2$, $n_2 + n_3$ and $n_3 + n_1$, hence n_1, n_2 and n_3 are of the same parity.

(1) Suppose that $n_1 = n_2 = 1$ and $n_3 = 6n + k$, where $n \geq 1$ and $k \in \{1, 3, 5\}$. The set $\mathrm{Lct}(G)$ contains $3, 4, 8$, since the following sequences are closed trails in G: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1})$ and $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1}, v_{3,3}, v_{1,2}, v_{4,3}, v_{1,1})$. As $e = 2(6n + k) + 1 = 3 \cdot 4n + 2k + 1$ and $2k + 1 \in \{3, 7, 11\}$, the set $\mathrm{Sct}(G)$ contains one of the sequences $(3)^{4n+1}$, $(3)^{4n+1}(4)$, $(3)^{4n+1}(8)$, having $4n + 1 \geq 5$ odd terms. By Propositions 3 and 5 then G is not ADCT.

(2) Suppose $2 \leq n_1 < n_3$. The set $\mathrm{Lct}(G)$ contains $3, 4, 5$, since the following sequences are closed trails in G: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,2}, v_{2,1}, v_{1,3}, v_{1,1})$ and $(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,3}, v_{1,1})$. If $k := \lfloor \frac{e-3}{3} \rfloor$, then $e = 3k + r$, where $r \in [3, 5]$, and the sequence $(3)^k(r) \in \mathrm{Sct}(G)$ has at least k odd terms. Since $\mathrm{mcb}(G) = n_1n_2$ (Proposition 5), the inequality $n_1n_2 < k$ implies that G is not ADCT. Let us show that $n_1n_2 < k$. Indeed, this inequality is a consequence of the first from the following two equivalent inequalities: $3n_1n_2 + 5 < e = n_1n_2 + n_1n_3 + n_2n_3$, $5 < n_1(n_3 - n_2) + n_2(n_3 - n_1)$. Now, since $n_3 - n_1 \geq 2$ and $n_3 - n_2 \geq 0$, the latter inequality is true; note that if $n_2 = 2$, then $n_1 = 2$ and on the righthand side we have $4(n_3 - n_1) \geq 8$. ■

Lemma 8 *If G is an even graph, then* $\mathrm{Lct}(G) \subseteq [3, |E(G)| - 3] \cup \{|E(G)|\}$.

Proof. Let T be a closed trail in G. Clearly, T has at least three edges (a subgraph of G induced by m edges, $m \in [1, 2]$, has at least two vertices of degree one). If $E(T) \neq E(G)$, then $G - T$ is a nonempty even subgraph of G. As any component of $G - T$ has at least three edges, we have $|E(T)| = |E(G)| - |E(G - T)| \leq |E(G)| - 3$. ■

Proposition 9 *Let G be a tripartite graph with tripartition $\{V_1, V_2, V_3\}$ and let T be a closed trail in G such that there are $i, j \in [1, 3]$, $i \neq j$, and $x, y \in V(T)$, satisfying $V(T) \cap V_i = \{x\}$ and $V(T) \cap V_j = \{y\}$. Then either $(|E(T)|)_4 = 3$ and $xy \in E(T)$ or $(|E(T)|)_4 = 0$.*

Proof. With $k := 6 - i - j$ we have $\{i, j, k\} = [1, 3]$. If $z \in V(T) \cap V_k$, then $xz, yz \in E(T)$.

(1) If $xy \notin E(T)$, then T is a closed trail in the bipartite graph with bipartition $\{\{x, y\}, V_k\}$ and $(|E(T)|)_4 = 0$ since the vertices x and y must alternate in T.

(2) If $xy \in E(T)$, then $T' := T - \{xy, yz, zx\}$ is an even graph and either $E(T') = \emptyset$ or T' is connected. In the latter case we can proceed as in (1) with T' instead of T. Therefore, in both cases $(|E(T')|)_4 = 0$ and $|E(T)| = |E(T')| + 3 \equiv 3 \pmod 4$. ■

Proposition 10 *The graphs $K_{1,1,3}$ and $K_{1,1,5}$ are* ADCT.

Proof. Let $n \in \{3, 5\}$ and let V_1, V_2, V_3 be parts of $K_{1,1,n}$ with $|V_1| = |V_2| = 1$. If T is a closed trail in $K_{1,1,n}$, then clearly $V(T) \cap V_i = V_i$, $i = 1, 2$. Therefore, by Proposition 9, $(|E(T)|)_4 \in \{0, 3\}$. Further, if T' is a closed trail in $K_{1,1,n}$ that is edge-disjoint with T, then $T + T' \neq \emptyset$. Thus, using edge-disjoint closed trails $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{2,3}, v_{1,2}, v_{3,3}, v_{1,1})$ in $K_{1,1,3} \subseteq K_{1,1,5}$ and $(v_{1,1}, v_{4,3}, v_{1,2}, v_{5,3}, v_{1,1})$ in $K_{1,1,5}$, we can see that $\mathrm{Lct}(K_{1,1,3}) = \{3, 4, 7\}$, $\mathrm{Lct}(K_{1,1,5}) = \{3, 4, 7, 8, 11\}$ and, for both $n = 3, 5$, any sequence from $\mathrm{Sct}(K_{1,1,n})$ is $K_{1,1,n}$-realisable. ■

3 The graph $K_{5,5,5}$ is ADCT

In the main theorem of our paper we shall prove by induction on n that the graph $G_n := K_{(5\cdot 2^n)3}$, $n \in [0, \infty)$, is ADCT. For brevity we denote the part $\{v_{i,j} : i \in [1, 5 \cdot 2^n]\}$, $j \in [1, 3]$, of $K_{(5\cdot 2^n)3}$, by $V_{n,j}$. Consider three mutually isomorphic 16-edge graphs F_1 (full lines in Fig. 1), F_2 (short-dashed lines), F_3 (long-dashed lines), and put $F^j := \bigcup_{i=1}^{j} F_i$, $j = 1, 2, 3$; then $F^3 = K_{5,5,5} - K_{3,3,3}$. Let $i, j \in [1, 3]$. The tripartition of F_i is $\{V_i^3, V_{i+1}^1 \cup V_{i+1}^2, V_{i+2}^2\}$, where lower indices are taken modulo 3 in the set $[1, 3]$ (such a convention will be used throughout the whole paper without explicitly mentioning it), $V_i^1 \subseteq V_i^3$, $V_i^2 \cup V_i^3 = V_{0,i}$ and $|V_i^j| = j$. The mapping $\iota_i : V(F_1) \to V(F_i)$, determined by $\iota_i(v_{j,k}) = v_{j,k-1+i}$ is a natural isomorphism from F_1 onto F_i with $\iota_i(V_l^m) = V_{l-1+i}^m$ (for all four meaningful pairs (l, m)).

Proposition 11 *If* $j \in [1, 3]$, *then* $\mathrm{mcb}(F^j) = 4j$.

Proof. Putting $E_i := \{xy \in E(F_i) : x \in V_{i+1}^2, y \in V_{i+2}^2\}$, $i = 1, 2, 3$, it is easy to see that $\bigcup_{k=1}^{j} E_k \in \mathrm{Cb}(F^j)$, and so $\mathrm{mcb}(F^j) \leq 4j$. On the other hand, the sets $\{v_{2,1}, v_{4,2}, v_{4,3}\}$, $\{v_{2,1}, v_{5,2}, v_{5,3}\}$, $\{v_{3,1}, v_{4,2}, v_{5,3}\}$ and $\{v_{3,1}, v_{4,3}, v_{5,2}\}$ induce in F_1 four pairwise edge-disjoint K_3's. Therefore, by Proposition 4, $\mathrm{mcb}(F^j) \geq j\mathrm{mcb}(F_1) \geq j \cdot 4\mathrm{mcb}(K_3) = 4j$. ■

A closed trail T in F^3 is said to be F^3-*extendable* if $V(T) \cap V^3 \neq \emptyset$, where $V^3 := V_1^3 \cup V_2^3 \cup V_3^3$. If $i \in [1, 3]$, the graph $F_i - V^3$ is isomorphic to C_4, hence any closed trail in F_i of length $\neq 4$ is F^3-extendable. An F^3-extendable closed trail T is said to be F^3-*good* if $V(T) \cap V_{0,j} \neq \emptyset$, $j = 1, 2, 3$. Since a graph, induced in G_0 by two of its parts $V_{0,1}, V_{0,2}, V_{0,3}$, is bipartite, a closed trail in F^3 of an odd length is F^3-good. Moreover, from the structure of the graph F_i, $i \in [1, 3]$, it is easy to see that any closed trail in F_i of length $\neq 4$ is F^3-good.

For $i \in [1, 3]$, a closed trail T in F_i is said to be F_i-*good* if $V(T) \cap V_i^3 \neq \emptyset$, $V(T) \cap V_{i+1}^2 \neq \emptyset$ and $V(T) \cap V_{i+2}^2 \neq \emptyset$. Evidently, an F_i-good closed trail in F_i is also F^3-good.

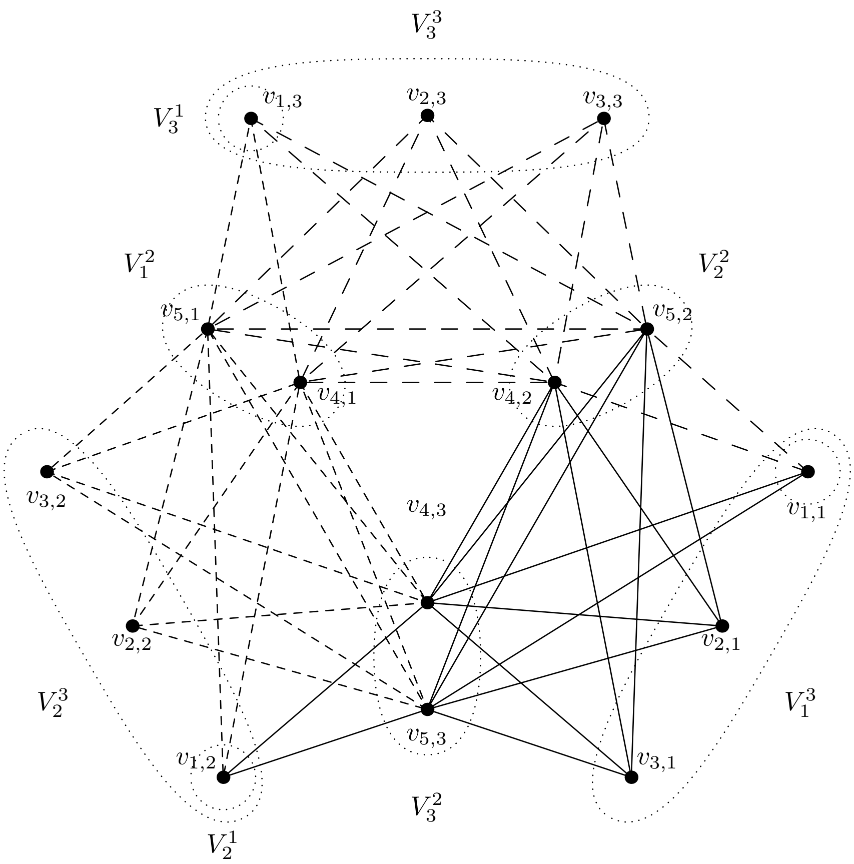

Figure 1: The graph F^3

Let $i \in [1,3]$, $l \in \mathrm{Lct}(F^3)$ and let $\mathcal{T}$ be a sequence of closed trails in F^3. Consider the following five conditions:

(C1.i) Any trail of $\mathcal{T}$ length $\neq 4$ is F_i-good.

(C2) Any trail of $\mathcal{T}$ is F^3-extendable.

(C3.$i.l$) $\mathcal{T}$ has an l-trail T such that $V(T) \cap V_{i+1}^1 \neq \emptyset$, $V(T) \cap V_{i+1}^2 \neq \emptyset$ and $V(T) \cap V_{i+2}^2 \neq \emptyset$.

(C4) Any trail of $\mathcal{T}$ of length $\neq 4$ is F^3-good.

(C5.l) $\mathcal{T}$ has an l-trail T such that either $V(T) \cap V_1^2 \neq \emptyset$ and $V(T) \cap V_3^1 \neq \emptyset$ or $V(T) \cap V_2^2 \neq \emptyset$.

For $i \in [1,3]$, an F_i-realisation $\mathcal{T}$ of a sequence $L \in \mathrm{Sct}(F_i)$ is said to be *good* [*l-good*] if $\mathcal{T}$ satisfies (C1.i) and (C2) [(C1.i), (C2) and (C3.$i.l$)].

Theorem 12 *Let $i \in [1,3]$, $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(F_i)$ and $j \in [1,p]$. Then the following hold:*

1. *There is a good F_i-realisation of L.*

2. *If* $\mathrm{nd}(L) \neq (3)^4(4)$ *and* $l_j \geq 4$*, there is an* l_j*-good* F_i*-realisation of* L*.*

Proof. We are able to prove the statement of our Theorem for any sequence $L = (l_1, \ldots, l_p)$ of integers from $[3, 13] \cup \{16\}$ such that $\sum_{k=1}^{p} l_k = 16$. This in turn implies that $\mathrm{Lct}(F_i) = [3, 13] \cup \{16\}$. We proceed in this way throughout tho whole paper: when working with a graph G, we do not take care of the structure of $\mathrm{Lct}(G)$, but in all cases it turns out that $\mathrm{Lct}(G)$ is of maximal extent, i.e., $\mathrm{Lct}(G) = [3, |E(G)| - 3] \cup \{|E(G)|\}$ (cf. Lemma 8).

Using Lemma 1 we may suppose without loss of generality that L is non-decreasing (i.e., $\mathrm{nd}(L) = L$). Examples of appropriate F_1-realisations of L are presented at http://umv.science.upjs.sk/adct and can be seen by clicking on Graph F_1. To pass to F_i-realisations, $i \in [2, 3]$, consider the isomorphism ι_i. ■

The next proposition shows that the exclusion of the sequence $(3)^4(4)$ in Theorem 12.2 is unavoidable.

Proposition 13 *If* $i \in [1, 3]$ *and* T *is a* 4*-trail of an* F_i*-realisation of* $(3)^4(4)$*, then* $V(T) = V_i^1 \cup V_{i+1}^1 \cup V_{i+2}^2$*.*

Proof. The statement follows from the fact that both $v_{1,i}$ and $v_{1,i+1}$ belong in F_i only to trails of length ≥ 4. ■

Let $L \in \mathrm{Sct}^*(F^2)$ and let l be a term of L. An F^2-realisation $\mathcal{T}$ of L is said to be *l-good* if $\mathcal{T}$ satisfies (C2), (C4) and (C5.l).

Theorem 14 *If* $L = (l_1, \ldots, l_p) \in \mathrm{Sct}^*(F^2)$ *and* $i \in [1, p]$*, there exists an* l_i*-good* F^2*-realisation of* L*.*

Proof. (1) Assume there is a decomposition $\{I^1, I^2\}$ of $[1, p]$ such that $\sum_{k \in I^j} l_k = 16$, $j = 1, 2$. We may suppose without loss of generality that $i \in I^1 \Leftrightarrow l_i = 3$. Let $\mathcal{T}^1$ be a good F_1-realisation of $L\langle I^1 \rangle$ and, provided that $l_i \geq 4$ and $\mathrm{nd}(L\langle I^1 \rangle) \neq (3)^4(4)$ [$l_i = 3$ or $\mathrm{nd}(L\langle I^1 \rangle) = (3)^4(4)$], let $\mathcal{T}^2$ be an l_i-good [a good] F_2-realisation of $L\langle I^2 \rangle$; both realisations do exist by Theorem 12. Then $\mathcal{T} := \mathcal{T}^1\mathcal{T}^2$ is an l_i-good F^2-realisation of L. First note that any trail of $\mathcal{T}^j$, $j = 1, 2$, satisfies (C2) and (C4) (as a consequence of (C1.j)). Further, $\mathcal{T}$ has an l_i-trail T such that either $V(T) \cap V_1^2 \neq \emptyset$ and $V(T) \cap V_3^1 \neq \emptyset$ (if $l_i \geq 4$, see (C3.2.l_i) or Proposition 13) or $V(T) \cap V_2^2 \neq \emptyset$ (if $l_i = 3$, see (C1.1)).

(2) Now assume there is a decomposition $\{\{r\}, I^1, I^2\}$ of $[1, p]$ such that $\sum_{k \in I^j} l_k \leq 13$, $j = 1, 2$. Putting $l_r^j := 16 - \sum_{k \in I^j} l_k$ and $L^j := L\langle I^j \rangle(l_r^j)$, we have $L^j \in \mathrm{Sct}(F_j)$, $j = 1, 2$, and $l_r^1 + l_r^2 = 32 - \sum_{k \in I^1 \cup I^2} l_k = l_r$. We may suppose without loss of generality that $i \neq r \Rightarrow (i \in I^1 \Leftrightarrow l_i = 3)$.

If $l_r^1 \geq 4$ and $\mathrm{nd}(L^1) \neq (3)^4(4)$ [$l_r^1 = 3$ or $\mathrm{nd}(L^1) = (3)^4(4)$], let $\mathcal{T}^1$ be an l_r^1-good [a good] F_1-realisation of L^1.

If $i \in I^2$ and $\mathrm{nd}(L^2) \neq (3)^4(4)$ [$i = r$, $l_r^2 \geq 4$ and $\mathrm{nd}(L^2) \neq (3)^4(4)$] {otherwise}, let $\mathcal{T}^2$ be an l_i-good [an l_r^2-good] {a good} F_2-realisation of L^2.

If $i \neq r$ and $l_i = 3$, let T_i be any l_i-trail of $\mathcal{T}^1$; it satisfies $V(T_i) \cap V_2^2 \neq \emptyset$ (see (C1.1)). If $i \neq r$ and $l_i \geq 4$, let T_i be an l_i-trail of $\mathcal{T}^2$ satisfying $V(T_i) \cap V_1^2 \neq \emptyset$ and $V(T_i) \cap V_3^1 \neq \emptyset$ (see (C3.2.l_i) or Proposition 13). Further, $\mathcal{T}^j$ contains an l_r^j-trail $T_r^j \neq T_i$ (if T_i is defined at all, i.e., if $i \neq r$), $j = 1, 2$, with the following two properties: either $V(T_r^1) \cap V_2^1 \neq \emptyset$ and $V(T_r^1) \cap V_3^2 \neq \emptyset$ (if $l_r^1 \geq 4$, see (C3.1.l_r^1) or Proposition 13) or $V(T_r^1) \cap V_3^2 \neq \emptyset$ (if $l_r^1 = 3$, see (C1.1)); either $V(T_r^2) \cap V_2^1 \neq \emptyset$ (if $\mathrm{nd}(L^2) = (3)^4(4)$, see Proposition 13) or $V(T_r^2) \cap V_3^2 \neq \emptyset$ (otherwise, see (C1.2) or (C3.2.l_r^2)).

We may suppose without loss of generality that $l_r^1 = 3 \Rightarrow \mathrm{nd}(L^2) \neq (3)^4(4)$ (otherwise, if $m \in I^2$ and $l_m = 3$, then $l_m + \sum_{k \in I^1} l_k = 16$ and the case (1) applies). Therefore, we have either $V(T_r^1) \cap V(T_r^2) \supseteq V_2^1 \neq \emptyset$ or $V(T_r^j) \cap V_3^2 \neq \emptyset$, $j = 1, 2$. In the latter case, since V_3^2 is a similarity class in F_1, we may suppose without loss of generality that $V(T_r^1) \cap V(T_r^2) \neq \emptyset$. Thus, in both cases there is a trail $T_r \in T_r^1 + T_r^2$.

Denote as $\hat{\mathcal{T}}^j$ the sequence obtained by deleting T_r^j from $\mathcal{T}^j$, $j = 1, 2$. Then $\mathcal{T} := \hat{\mathcal{T}}^1 \hat{\mathcal{T}}^2 (T_r)$ is an l_i-good F^2-realisation of $L\langle I^1 \rangle L\langle I^2 \rangle (l_r) \sim L$. First, the only trail of $\mathcal{T}$, that is neither in $\mathcal{T}^1$ nor in $\mathcal{T}^2$, is an F^3-extendable trail T_r (of length ≥ 6) with $V(T_r) \supseteq V(T_r^j)$, $j = 1, 2$. If T_r^1 is F_1-good, it is also F^3-good, hence so is T_r. In the opposite case $l_r^1 = 4$, $\mathrm{nd}(L^1) = (3)^4(4)$ and, by Proposition 13, both T_r^1 and T_r are F^3-good. If $i \neq r$, the l_i-trail T_i satisfies (C5.l_i). Finally, if $i = r$, the l_r-trail T_r satisfies (C5.l_r): with $l_r^2 \geq 4$ we have $V(T_r) \cap V_1^2 \supseteq V(T_r^2) \cap V_1^2 \neq \emptyset$ and $V(T_r) \cap V_3^1 \supseteq V(T_r^2) \cap V_3^1 \neq \emptyset$ (by (C3.2.l_r^2) or Proposition 13), while $l_r^2 = 3$ implies $V(T_r) \cap V_2^2 \supseteq V(T_r^1) \cap V_2^2 \neq \emptyset$ (by (C1.1) or (C3.l_r^1); note that here we may suppose without loss of generality that $\mathrm{nd}(L^1) \neq (3)^4(4)$).

(3) In what follows we suppose that the sequence L is nondecreasing and the assumptions of (1) and (2) are not fulfilled. Let $q \in [1, p]$ be such that $\sum_{i=1}^{q-1} l_i \leq 13$ and $\sum_{i=1}^{q} l_i > 13$. Then $\sum_{i=1}^{q} l_i \in \{14, 15, 17, 18\}$ (if $\sum_{i=1}^{q} l_i \geq 19$ then $\sum_{i=q+1}^{p} l_i \leq 13$ and (2) is fulfilled with $I^1 := [1, q-1]$ and $I^2 := [q+1, p]$). Let M_k be the set of all nondecreasing sequences with terms from $[3, \infty)$ adding up to k and let S_k be the set of all nondecreasing sequences $L = (l_1, \dots, l_p) \in \mathrm{Sct}^*(F^2)$ such that $\sum_{i=1}^{q} l_i = k$ for some $q \in [1, p-1]$ and L violates the assumptions of both (1) and (2). We are going to determine the structure of S_k, $k = 14, 15, 17, 18$.

(31) If $L \in S_{14}$ then $l_j \in \bigcap_{i=1}^{q} \{l_i, l_i+1, l_i+3, l_i+4\}$ for any $j \in [q+1, p] \neq \emptyset$. Indeed, if $l_j = l_i + 2$ for some $i \in [1, q]$ and $j \in [q+1, p]$, then (1) is fulfilled with $I^1 := [1, q] - \{i\} \cup \{j\}$ and $I^2 := [q+1, p] - \{j\} \cup \{i\}$. If $l_j \geq l_i + 5$ for some $i \in [1, q]$ and $j \in [q+1, p]$, then with $I^1 := [2, q]$, $I^2 := \{1\} \cup [q+1, p] - \{j\}$, we have $\sum_{k \in I^1} l_k \leq 11$ and $\sum_{k \in I^2} l_k = 18 - (l_j - l_1) \leq 13$. By analysing all sequences $(l_1, \dots, l_q)$ from M_{14} such that $l_q - l_1 \leq 4$ (the above

intersection must be nonempty), we obtain $S_{14} = \{(3)^3(5)(6)^3,\ (3)^2(4)^3(7)^2,\ (3,5)(6)^4,\ (5)(9)^3,\ (6,8)(9)^2,\ (7)^3(11),\ (7)^2(8,10),\ (14,18)\}$. As (somewhat typical) examples of this analysis consider the following:

(a) If $(l_1, l_2, l_3, l_4) = (3)^3(5)$, then $l_j \in \{3,4,6,7\} \cap \{5,6,8,9\} = \{6\}$ for any $j \in [5,p]$, hence $p = 7$ and $L = (3)^3(5)(6)^3$.

(b) If $(l_1, l_2, l_3) = (4)^2(6)$, then $l_j \in \{4,5,7,8\} \cap \{6,7,9,10\} = \{7\}$ for any $j \in [4,p]$, and, since $7 \nmid 18$, there is no such L.

(c) If $(l_1, l_2) = (7)^2$, then $l_j \in \{7,8,10,11\}$ for any $j \in [3,p]$, hence $p = 4$ and (l_3, l_4) is either $(7,11)$ or $(8,10)$, which yields $L = (7)^3(11)$ or $L = (7)^2(8,10)$.

(32) If $L \in S_{15}$, then $l_j \in \bigcap_{i=1}^{q}\{l_i, l_i+2, l_i+3\}$ for any $j \in [q+1,p]$, hence $l_q - l_1 \le 3$. In the same way as above we obtain the set $S_{15} = \{(3)^7(5,6),\ (3)^5(5)(6)^2,\ (5)^5(7),\ (15,17)\}$; notice that $(3)^9(5) \notin \mathrm{Sct}^*(F^2)$ and, consequently, $(3)^9(5) \notin S_{15}$.

(33) If $L \in S_{17}$, then $l_q = 17 - \sum_{i=1}^{q-1} l_i \ge 4$. Further, $l_j \in \{l_q, l_q+1\}$ for any $j \in [q+1,p]$, otherwise with $I^1 := [1,q-1]$ and $I^2 := [q,p] - \{j\}$ we have $\sum_{i\in I^1} l_i \le 13$ and $\sum_{i \in I^2} l_i = 15-(l_j-l_q) \le 13$. We can easily find that $S_{17} = \{(3)^4(5)^4,\ (3)^2(4)(7)^2(8),\ (3,4)(5)^5,\ (4)^3(5)^4,\ (3)(7)^3(8),\ (4,6)(7)^2(8),\ (5)^2(7)^2(8),\ (3,14,15)\}$.

(34) If $L \in S_{18}$, then $l_q \ge 5$ and $l_j = l_q$ for any $j \in [q+1,p]$, for otherwise $\sum_{i=q}^{p-1} l_i \le 13$. Thus $S_{18} = \{(3)^2(5)(7)^3,\ (3)(4)^2(7)^3,\ (4)(7)^4,\ (5,6)(7)^3,\ (4)(14)^2\}$.

Examples of l_i-good F^2-realisations of sequences $L \in S_{14} \cup S_{15} \cup S_{17} \cup S_{18}$ can be seen by clicking on Graph F^2 (see the address presented in the proof of Theorem 12). ■

An F^3-realisation $\mathcal{T}$ of a sequence $L \in \mathrm{Sct}^*(F^3)$ is said to be *good* if $\mathcal{T}$ satisfies (C2) and (C4).

Theorem 15 *If $L \in \mathrm{Sct}^*(F^3)$, there exists a good F^3-realisation of L.*

Proof. Let $L = (l_1, \dots, l_p)$.

(1) Assume there is a decomposition $\{I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^1} l_i = 16$, $\sum_{i\in I^2} l_i = 32$, and $\mathrm{nd}(L\langle I^2\rangle) \ne (3)^9(5)$ (so that $L\langle I^2\rangle \in \mathrm{Sct}^*(F^2)$). Consider an arbitrary $i \in I^2$. By Theorems 12 and 14 there exists a good F_3-realisation $\mathcal{T}^1$ of $L\langle I^1\rangle$ and an l_i-good F^2-realisation $\mathcal{T}^2$ of $L\langle I^2\rangle$. Then $\mathcal{T}^1\mathcal{T}^2$ is a good F^3-realisation of $L\langle I^1\rangle L\langle I^2\rangle \sim L$.

(2) Let there exist a decomposition $\{\{r\}, I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^1} l_i \le 13$, $\sum_{i\in I^2} l_i \le 29$ and $\mathrm{nd}(L\langle I^2\rangle) \notin \{(3)^8(5), (3)^9\}$. If $l_r^j := 16j - \sum_{i\in I^j} l_i$, $j = 1,2$, then $l_r^1 + l_r^2 = 48 - \sum_{i\in I^1\cup I^2} l_i = l_r$. Put $L^j := L\langle I^j\rangle(l_r^j)$, $j = 1,2$. If $l_r^1 \ge 4$ and $\mathrm{nd}(L^1) \ne (3)^4(4)$ [$l_r^1 = 3$ or $\mathrm{nd}(L^1) = (3)^4(4)$], let $\mathcal{T}^1$ be an l_r^1-good [a good] F_3-realisation of L^1. Then $\mathcal{T}^1$ has an l_r^1-trail T_r^1 such that $V(T_r^1) \cap V_2^2 \ne \emptyset$ and either $V(T_r^1) \cap V_3^1 \ne \emptyset$ (if $\mathrm{nd}(L^1) = (3)^4(4)$) or

$V(T_r^1) \cap V_1^2 \neq \emptyset$ (otherwise). Further, let $\mathcal{T}^2$ be an l_r^2-good F^2-realisation of L^2 and let T_r^2 be an l_r^2-trail of $\mathcal{T}^2$ satisfying (C5.l_r^2). Then either $V(T_r^1) \cap V(T_r^2) \supseteq V_3^1 \neq \emptyset$ or there is $k \in [1,2]$ such that $V(T_r^j) \cap V_k^2 \neq \emptyset$, $j = 1,2$. In the latter case, as both V_1^2 and V_2^2 are similarity classes in F_3, we may suppose without loss of generality that $V(T_r^1) \cap V(T_r^2) \neq \emptyset$. Thus, in any case there is a trail $T_r \in T_r^1 + T_r^2$. If $\hat{\mathcal{T}}^j$ results from $\mathcal{T}^j$ by deleting T_r^j, $j = 1,2$, then $\hat{\mathcal{T}}^1\hat{\mathcal{T}}^2(T_r)$ is a good F^3-realisation of $L\langle I^1\rangle L\langle I^2\rangle(l_r) \sim L$.

(3) Suppose that L is nondecreasing and violates the assumptions of both (1) and (2). Let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leq 13$ and $s := \sum_{i=1}^{q} l_i > 13$. Then $s \neq 16$, for otherwise $l_i \geq l_q \geq 4$ for any $i \in [q+1,p]$, and with $I^1 := [1,q]$ and $I^2 := [q+1,p]$ we are in the case (1). Also, $s \leq 18$, since $s \geq 19$ leads to $l_i \geq l_q \geq 6$ for any $i \in [q+1,p]$, and then with $I_1 := [1,q-1]$ and $I_2 := [q+1,p]$ we are in the case (2). Thus, $s \in \{14,15,17,18\}$ and, similarly as in the proof of Theorem 14, we conclude that L is one of the following sequences: $(3)^{12}(6)^2$, $(3)^{10}(6)^3$, $(3)^8(6)^4$, $(3)^6(6)^5$, $(3)^4(6)^6$, $(3)^2(6)^7$, $(3)(5)^9$, $(4)^2(5)^8$, $(5)^8(8)$, $(6)^8$, $(14)(17)^2$; for good F^3-realisations of these sequences a reader is referred to Graph F^3. ■

Put $P := \{(0,0), (0,3), (3,0)\}$.

Proposition 16 *Let $n \in [2,\infty)$ and let T_1, T_2 be edge-disjoint closed trails in the graph $K_{n,n,n}$ with the tripartition $\{V_1,V_2,V_3\}$. Then the following hold:*

1. *There is $p \in [1,3]$ such that $\{V(T_1),V(T_2)\}$ has a system of distinct representatives in V_p.*

2. *If there is $m \in [1,2]$ such that $V(T_m) \cap V_i \neq \emptyset$, $i = 1,2,3$, and $((|E(T_1)|)_4, (|E(T_2)|)_4) \notin P$, then there are $q,r \in [1,3]$, $q \neq r$, such that $\{V(T_1),V(T_2)\}$ has a system of distinct representatives in both V_q and V_r.*

Proof. Set $c_i := (|E(T_i)|)_4$ and let $\rho_i : [1,3] \to [1,3]$ be a bijection satisfying $x(i,1) \geq x(i,2) \geq x(i,3)$, where $x(i,j) := |V(T_i) \cap V_{\rho_i(j)}|$, $i = 1,2$, $j = 1,2,3$. Clearly, $x(i,2) \geq 1$, $i = 1,2$.

1. If there is $i \in [1,2]$ such that $x(i,2) \geq 2$, then there exists $j \in [1,2]$ such that $V(T_{3-i}) \cap V_{\rho_i(j)} \neq \emptyset$, and we are done with $p := \rho_i(j)$. In the opposite case $x(i,2) = x(i,3) = 1$, $i = 1,2$. Let $i \in [1,2]$ be such that $x(i,1) = \max(x(1,1), x(2,1))$. If $x(i,1) \geq 2$, then, since $V(T_{3-i}) \cap V_{\rho_i(1)} \neq \emptyset$, we can take $p := \rho_i(1)$. So, suppose that $x(k,j) = 1$, $k = 1,2$, $j = 1,2,3$. As $E(T_1) \cap E(T_2) = \emptyset$, we have $V(T_1) \neq V(T_2)$, and the existence of p follows.

2. Here we have $x(m,3) \geq 1$. If $x(3-m,3) = 0$, then T_{3-m} is a bipartite graph, $x(3-m,2) \geq 2$ and we are done with $q := \rho_{3-m}(1)$ and $r := \rho_{3-m}(2)$. So, suppose that $x(3-m,3) \geq 1$ and let $\rho : [1,3] \to [1,3]$ be a bijection satisfying $x(1) \geq x(2) \geq x(3)$, where $x(j) := |X_j|$ and $X_j := (V(T_1) \cup V(T_2)) \cap V_{\rho(j)}$, $j = 1,2,3$. If $x(2) = x(3) = 1$, then $X_j = V(T_1) \cap V_{\rho(j)} =$

$V(T_2) \cap V_{\rho(j)} = \{x_j\}$, $j = 2, 3$. From Proposition 9 it follows that $c_i \in \{0, 3\}$ and $c_i = 3 \Rightarrow x_2x_3 \in E(T_i)$, $i = 1, 2$. Since $E(T_1) \cap E(T_2) = \emptyset$, we obtain $(c_1, c_2) \in P$, a contradiction. Thus, $x(2) \geq 2$, and it suffices to take $q := \rho(1)$ and $r := \rho(2)$. ■

If $l \in [3, 5]$, then any closed trail of length l is in fact a cycle. Let us identify all $2l$ closed trails of length l in $K_{3,3,3}$ having the same edge set and denote by $\mathcal{C}_l$ the set of all (representatives of) closed trails of length l in $K_{3,3,3}$. Then $c_l := |\mathcal{C}_l|$ is equal to the number of l-element subsets of $E(K_{3,3,3})$ inducing a cycle of length l.

Let T be a closed trail of length $l \in [3, 5]$, let $V_i := V(T) \cap \{v_{1,i}, v_{2,i}, v_{3,i}\}$ and $v_i := |V_i|$, $i = 1, 2, 3$. Let $i, j, k \in [1, 3]$ be such that $\{i, j, k\} = [1, 3]$ and $v_i \leq v_j \leq v_k$. If $l \in \{3, 5\}$, T is a non-bipartite subgraph of $K_{3,3,3}$, therefore $v_i \geq 1$. If $l = 3$, then $v_1 = v_2 = v_3 = 1$, and so $c_3 = \binom{3}{1}^3 = 27$. If $l = 4$, then either $v_i = 0$ and $v_j = v_k = 2$ or $v_i = v_j = 1$ and $v_k = 2$, so that $c_4 = 3 \cdot \binom{3}{2}^2 + 3 \cdot \binom{3}{1}^2 \cdot \binom{3}{2} = 108$. If $l = 5$, then from Proposition 9 it follows that $v_i = 1$, $v_j = v_k = 2$. The edge set of T is uniquely determined by a simple sequence $\prod_{i=1}^{5}(x_i)$ such that $x_1, x_3 \in V_j$, $x_2, x_4 \in V_k$ and $x_5 \in V_i$. Therefore, $c_5 = 3 \cdot (3 \cdot 3 \cdot 2 \cdot 2 \cdot 3) = 324$.

A $K_{3,3,3}$-realisation $(T_1, \ldots, T_p)$ of a sequence $(l_1, \ldots, l_p) \in \mathrm{Sct}(K_{3,3,3})$ is said to be *good* if, for any $i \in [1, p]$, $E(T_i)$ is a union of edge sets of some (edge-disjoint) trails from $\mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$.

Theorem 17 *If $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(K_{3,3,3})$, there exists a good $K_{3,3,3}$-realisation of L.*

Proof. Good $K_{3,3,3}$-realisations of sequences from $\mathrm{Sct}(K_{3,3,3})$ have been found by a computer; to see them, click on Graph $K_{3,3,3}$. ■

If $n \in [0, \infty)$, we have $e_n := |E(G_n)| = 3 \cdot (5 \cdot 2^n)^2$. A closed trail T in G_n is said to be G_n-*good* if $V(T) \cap V_{n,i} \neq \emptyset$, $i = 1, 2, 3$. Clearly, if T is of an odd length, it is G_n-good. A G_n-realisation $\mathcal{T}$ of a sequence from $\mathrm{Sct}(G_n)$ is said to be *good* if any trail of $\mathcal{T}$ of length $\not\equiv 0 \pmod 4$ is G_n-good. The graph G_n is said to be *strongly* ADCT provided that (i) for any $L \in \mathrm{Sct}(G_n)$ there is a good G_n-realisation of L and (ii) if $(t, 1)$, $t \in [1, 5]$, is a position of Table 1 (regarded as a 5×2 matrix) containing a sequence $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(G_n)$ (n has to be in a specified congruence class modulo 3 such that the exponent of (7) is an integer), there is a good G_n-realisation $(T_1, \ldots, T_p)$ of L satisfying the conditions presented in the position $(t, 2)$ of Table 1.

Theorem 18 *The graph G_0 is strongly* ADCT.

Proof. Consider a sequence $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(G_0)$.

(1) Suppose there is a decomposition $\{I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i = 48$ and $\sum_{i \in I^2} l_i = 27$. If, moreover, $L\langle I^1 \rangle \in \mathrm{Sct}^*(F^3)$, then, by Theorem 15,

$(3)^4(7)^{\frac{e_n-12}{7}}$	$v_{i,1} \in V(T_i)$, $i = 1,2,3$
$(3)^3(4)(7)^{\frac{e_n-13}{7}}$	$v_{i,1} \in V(T_i)$, $i = 1,2,3$
$(4)^6(7)^{\frac{e_n-24}{7}}$	$v_{i,1} \in V(T_i)$, $i = 1,2,3$, $v_{i,2} \in V(T_{3+i})$, $i = 1,2$, $v_{1,3} \in V(T_6)$
$(4)^5(7)^{\frac{e_n-20}{7}}$	$v_{i,1} \in V(T_i)$, $i = 1,2,3$, $v_{i,2} \in V(T_{3+i})$, $i = 1,2$
$(4)^3(7)^{\frac{e_n-12}{7}}$	$v_{i,1} \in V(T_i)$, $i = 1,2,3$

Table 1: Required properties of G_n-realisations

there exists a good F^3-realisation $\mathcal{T}^1$ of $L\langle I^1\rangle$ and, by Theorem 17, there exists a good $K_{3,3,3}$-realisation $\mathcal{T}^2$ of $L\langle I^2\rangle$. The above additional assumption is true, if $f_3(L) \le 8$, since in such a case $L\langle I^1\rangle$ contains at most twelve odd terms (note that $\lfloor \frac{48-8\cdot 3}{5} \rfloor = 4$ and, by Proposition 11, $\mathrm{mcb}(F^3) = 12$). Then $\mathcal{T}^1\mathcal{T}^2$ is a good G_0-realisation of the sequence $L\langle I^1\rangle L\langle I^2\rangle$. Indeed, trails of $\mathcal{T}^1$ of length $\ne 4$ are G_0-good since they are F^3-good, and trails of $\mathcal{T}^2$ of length $\not\equiv 0 \pmod 4$ are G_0-good as they are composed of at least one trail of length 3 or 5. So, we may suppose that $f_3(L) \ge 9$ and choose I^2 so that $L\langle I^2\rangle = (3)^9$. If, additionally $L\langle I^1\rangle \in \mathrm{Sct}(F_3) - \mathrm{Sct}^*(F_3)$, then $L\langle I^1\rangle$ has more than 12 odd terms and it is easy to see that for $\mathrm{nd}(L)$ we have only six possibilities, namely $(3)^{25}$, $(3)^{23}(6)$, $(3)^{22}(4,5)$, $(3)^{22}(9)$, $(3)^{21}(5,7)$ and $(3)^{20}(5)^3$. Examples of corresponding good G_0-realisations are available by clicking on Graph G_0.

(2) Let there be a decomposition $\{\{r\}, I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^1} l_i \le 45$, $\sum_{i\in I^2} l_i \le 24$. If $l_r^1 := 48 - \sum_{i\in I^1} l_i$ and $l_r^2 := 27 - \sum_{i\in I^2} l_i$, then $l_r = l_r^1 + l_r^2$. We may suppose that $f_3(L) \le 8$ (otherwise we are in the case (1)), and then $L^1 := L\langle I^1\rangle(l_r^1) \in \mathrm{Sct}^*(F^3)$. By Theorem 15 there exists a good F^3-realisation $\mathcal{T}^1(T_r^1)$ of L^1 with $V(T_r^1) \cap V(K_{3,3,3}) = V(T_r^1) \cap V^3 \ne \emptyset$ (T_r^1 is F^3-extendable). Further, by Theorem 17 there is a $K_{3,3,3}$-realisation $\mathcal{T}^2(T_r^2)$ of the sequence $L^2 := \langle I^2\rangle(l_r^2) \in \mathrm{Sct}(K_{3,3,3})$. By Proposition 6 we may suppose without loss of generality that $V(T_r^1) \cap V(T_r^2) \ne \emptyset$, hence there is a trail $T_r \in T_r^1 + T_r^2$. If $(l_r)_4 \ne 0$, then $(l_r^j)_4 \ne 0$ for some $j \in [1,2]$, and so $V(T_r) \cap V_{0,i} \supseteq V(T_r^j) \cap V_{0,i} \ne \emptyset$, $i = 1,2,3$. Thus, $\mathcal{T}^1\mathcal{T}^2(T_r)$ is a good G_0-realisation of $L\langle I^1\rangle L\langle I^2\rangle(l_r) \sim L$.

(3) Now suppose that L is a nondecreasing sequence violating the assumptions of both (1) and (2). Let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \le 45$ and $\sum_{i=1}^{q} l_i > 45$. Then $\sum_{i=1}^{q} l_i \in \{46,47,49,50\}$. Proceeding analogously as in the proof of Theorem 14 we exhibit five possibilities for L, namely $(4)^3(7)^9$, $(4)(7)^9(8)$, $(5)^{15}$, $(5)(7)^{10}$, $(25)^3$. For examples of remaining appropriate G_0-realisations see again Graph G_0 (note that for $n = 0$ the

“real” sequences of Table 1 are $(4)^3(7)^9$ and $(3)^4(7)^9$). ■

4 The main theorem

Our main Theorem will be proved by induction on n. Therefore, we need to know how to construct G_{n+1} from G_n. Consider two copies G_n^1, G_n^2 of the graph G_n such that the tripartition of G_n^k is $\{V_{n,j}^k : j \in [1,3]\}$ with $V_{n,j}^k = \{v_{i,j}^k : i \in [1, 5 \cdot 2^n]\}$, where $v_{i,1}^1 = v_{i,1}^2$ for any $i \in [1, 5 \cdot 2^n]$ and $V_{n,j_1}^1 \cap V_{n,j_2}^2 = \emptyset$ for any $j_1, j_2 \in [2,3]$. Then $\varphi_n^k : V(G_n) \to V(G_n^k)$, with $\varphi_n^k(v_{i,j}) = v_{i,j}^k$ for any $i \in [1, 5 \cdot 2^n]$ and $k \in [1,3]$, is a natural isomorphism from G_n onto G_n^k, $k = 1, 2$. Let $H_n := G_n^1 \cup G_n^2$ (see Fig. 2); the subsets $V_{n,j}^k, j \in [2,3], k \in [1,2]$, are called *eccentric parts* of H_n, while $V_{n,1}^1 = V_{n,1}^2$ is the *central part* of H_n. The graph H_n is tripartite with one possible tripartition $\{W_{n,1}, W_{n,2}, W_{n,3}\}$, where $W_{n,1} := V_{n,1}^1$ and $W_{n,j} := V_{n,j}^1 \cup V_{n,j}^2$, $j = 2, 3$.

Suppose that $L = (l_1, \dots, l_p) \in \mathrm{Sct}(H_n)$ is an H_n-realisable sequence and $\mathcal{T} = (T_1, \dots, T_p)$ is an H_n-realisation of L. Consider two terms l, l' of L. The H_n-realisation $\mathcal{T}$ of L is said to be (l, l')-*global* if there is an l-trail T_i and an l'-trail T_j of $\mathcal{T}$, $i \neq j$, such that $V(T_i) \cap (V_{n,2}^k \cup V_{n,3}^k) \neq \emptyset$ and $V(T_j) \cap (V_{n,2}^{3-k} \cup V_{n,3}^{3-k}) \neq \emptyset$ for some $k \in [1,2]$ (or, equivalently, $V(T_i) \cup V(T_j) \not\subseteq V(G_n^k)$, $k = 1, 2$).

Proposition 19 *Let $n \in [0, \infty)$, $m \in [1, 5 \cdot 2^n]$, and let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m)$ be simple sequences of vertices of the graph H_n. Then v is similar to w whenever one of the following conditions is fulfilled:*

1. *$\{v_i : i \in [1, m]\} \subseteq V_{n,j_1}^{k_1}$ and $\{w_i : i \in [1, m]\} \subseteq V_{n,j_2}^{k_2}$ for some $j_1, j_2 \in [2,3]$ and $k_1, k_2 \in [1,2]$;*
2. *$m = 2$, $v_1 \in V_{n,j_1}^{k_1}$, $v_2 \in V_{n,5-j_1}^{k_1}$, $w_1 \in V_{n,j_2}^{k_2}$ and $w_2 \in V_{n,5-j_2}^{k_2}$ for some $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$;*
3. *$m = 2$, $v_1 \in V_{n,j_1}^{k_1}$, $v_2 \in V_{n,j_1}^{3-k_1}$, $w_1 \in V_{n,j_2}^{k_2}$ and $w_2 \in V_{n,j_2}^{3-k_2}$ for some $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$;*
4. *$m = 2$, $v_1 \in V_{n,j_1}^{k_1}$, $v_2 \in V_{n,5-j_1}^{3-k_1}$, $w_1 \in V_{n,j_2}^{k_2}$ and $w_2 \in V_{n,5-j_2}^{3-k_2}$ for some $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$.*

Proof. There is a bijection $\varphi : V(H_n) \to V(H_n)$ such that $\varphi(V_{n,1}^1) = V_{n,1}^1$, any eccentric part of H_n is mapped under φ to an eccentric part of H_n, $\varphi(V_{n,2}^k \cup V_{n,3}^k) \in \{V_{n,2}^1 \cup V_{n,3}^1, V_{n,2}^2 \cup V_{n,3}^2\}$, $k = 1, 2$, and $\varphi(v_i) = w_i$ for any $i \in [1, m]$; clearly, φ is an automorphism of H. ■

Now consider two copies H_n^1, H_n^2 of the graph H_n such that, for both $l = 1, 2$, H_n^l has parts $V_{n,j}^{k,l} = \{v_{i,j}^{k,l} : i \in [1, 5 \cdot 2^n]\}$, $j = 1, 2, 3$, $k = 1, 2$, where $V_{n,1}^{1,l} = V_{n,1}^{2,l}$, with $v_{i,1}^{1,l} = v_{i,1}^{2,l}$ for any $i \in [1, 5 \cdot 2^n]$, is the central part, $V_{n,1}^{1,1} \cap V_{n,1}^{1,2} = \emptyset$, and eccentric parts are chosen so that $v_{i,2}^{k,1} = v_{i,2}^{k,2}$ and

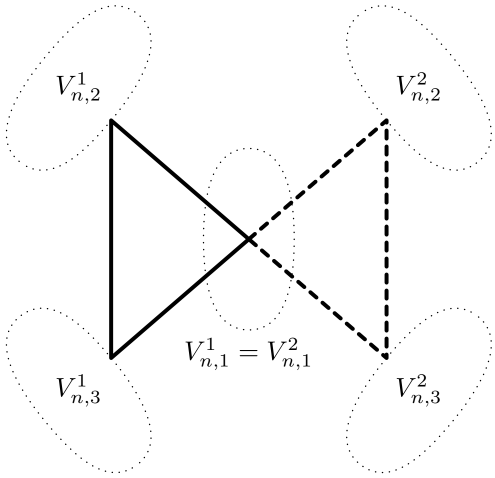

Figure 2: The graph H_n

$v^{k,1}_{i,3} = v^{3-k,2}_{i,3}$ for any $i \in [1, 5 \cdot 2^n]$ and $k \in [1, 2]$. Then $\psi^l_n : V(H_n) \to V(H^l_n)$, with $\psi^l_n(v^k_{i,j}) = v^{k,l}_{i,j}$ for any $i \in [1, 5 \cdot 2^n]$, $j \in [1, 3]$ and $k \in [1, 2]$, is a natural isomorphism from H_n onto H^l_n, $l = 1, 2$. In the graph H^l_n the eccentric part $V^{k,l}_{n,2}$ is "joined" to the eccentric part $V^{k,l}_{n,3}$, $k = 1, 2$ (see Fig. 3). Clearly, the graph $H^1_n \cup H^2_n$ is isomorphic to G_{n+1}. We shall suppose that, if $i \in [1, 5.2^n]$, then $v^{1,1}_{i,j} = v_{i,j}$ (recall the notation of vertices of the graph $G_n = K_{(5 \cdot 2^n)3}$) for any $j \in [1, 3]$, $v^{1,2}_{i,1} = v_{5 \cdot 2^n + i,1}$ and $v^{2,1}_{i,j} = v_{5 \cdot 2^n + i,j}$ for any $j \in [2, 3]$.

Let $n \in [0, \infty)$. A closed trail T in H_n is said to be H_n-*good* if $V(T) \cap W_{n,i} \neq \emptyset$, $i = 1, 2, 3$. Evidently, if T is of an odd length, it is H_n-good. An H_n-realisation $\mathcal{T}$ of a sequence from $\mathrm{Sct}(H_n)$ is said to be *good* if any trail of $\mathcal{T}$ of length $\not\equiv 0 \pmod 4$ is H_n-good. The graph H_n is said to be *strongly* ADCT provided that (i) for any $L \in \mathrm{Sct}(H_n)$ there is a good H_n-realisation of L and (ii) if $t \in [1, 8]$ and $L = (l_1, \dots, l_p) \in \mathrm{Sct}(H_n)$ is "a real $(t, 1)$-sequence" of Table 2 (so that the exponent of (7) is an integer), there is a good H_n-realisation $(T_1, \dots, T_p)$ of L satisfying "$(t, 2)$-conditions" of Table 2.

Theorem 20 *For any $n \in [0, \infty)$, the graph G_n is strongly* ADCT.

Proof. For $n \in [0, \infty)$ and $X \in \{G, H\}$ let $S(X_n)$ denote the following statement: The graph X_n is strongly ADCT. We are going to prove the statement $\forall n \in [0, \infty)$ $S(G_n)$ by induction on n. By Theorem 18, $S(G_0)$ is true.

So, suppose that $n \in [0, \infty)$ and $S(G_n)$ is true. We prove that the implication $S(G_n) \Rightarrow S(G_{n+1})$ is true by proving that both implications $S(G_n) \Rightarrow S(H_n)$ and $(S(G_n) \wedge S(H_n)) \Rightarrow S(G_{n+1})$ are true.

$(3)^4(7)^{\frac{2e_n-12}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2, 3$
$(3)^2(4)(7)^{\frac{2e_n-10}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2$
$(3)^2(6)(7)^{\frac{2e_n-12}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2$
$(3)^2(7)^{\frac{2e_n-6}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2$
$(4)^6(7)^{\frac{2e_n-24}{7}}$	$v^1_{i,2} \in V(T_i)$, $v^2_{i,3} \in V(T_{3+i})$, $i = 1, 2, 3$
$(4)^5(7)^{\frac{2e_n-20}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2, 3$, $v^1_{i,3} \in V(T_{3+i})$, $i = 1, 2$
$(4)^3(7)^{\frac{2e_n-12}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2, 3$
$(4)^3(7)^{\frac{2e_n-12}{7}}$	$v^1_{i,2} \in V(T_i)$, $i = 1, 2$, $v^1_{1,1} \in V(T_3)$

Table 2: Required properties of H_n-realisations

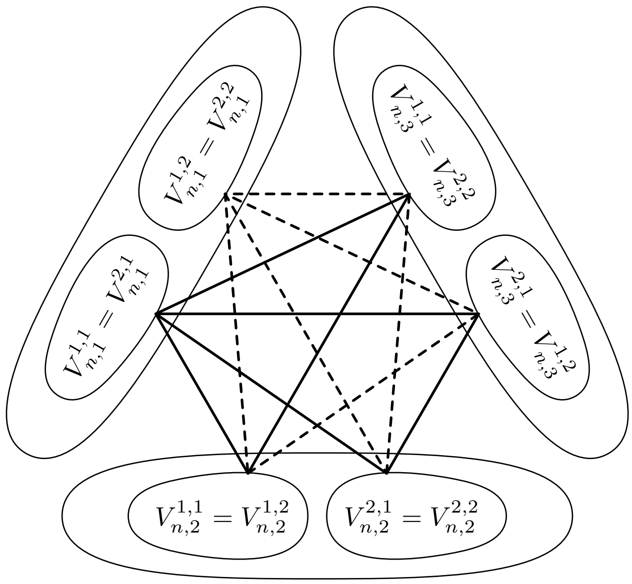

Figure 3: The graph G_n

(a) $S(G_n) \Rightarrow S(H_n)$

Claim 1 *A sequence $L = (l_1, \dots, l_p) \in \mathrm{Sct}(H_n)$ has a good H_n-realisation whenever one of the following conditions is fulfilled:*

1. *There is a decomposition $\{I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^1} l_i = e_n$.*

2. *There is $m \in [1,p]$ and a decomposition $\{\{i_1,\dots,i_m\}, I^1, I^2\}$ of $[1,p]$ such that, for both $j = 1,2$, there is a sequence $L^j = (l^j_{i_1},\dots,l^j_{i_m})$ such that $L^jL\langle I^j\rangle \in \mathrm{Sct}(G_n)$ and a good G_n-realisation $(T^j_{i_1},\dots,T^j_{i_m})\mathcal{T}^j$ of $L^jL\langle I^j\rangle$ satisfying $l_{i_k} = l^1_{i_k} + l^2_{i_k}$ and $V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1} \neq \emptyset$ for any $k \in [1,m]$.*

Proof. 1. With $I^2 := [1,p] - I^1$ we have $\sum_{i\in I^2} l_i = e_n$. By $S(G_n)$ there is a good G_n-realisation $\mathcal{T}^j$ of $L\langle I^j\rangle$, $j = 1,2$. If T is a trail of $\mathcal{T}^j$, $j \in [1,2]$, and $i \in [1,3]$, then $V(\varphi^j_n(T)) \cap W_{n,i} \supseteq V(\varphi^j_n(T)) \cap V^j_{n,i} = \varphi^j_n(V(T) \cap V_{n,i})$. As $Y \neq \emptyset \Rightarrow \varphi^j_n(Y) \neq \emptyset$ for any $Y \subseteq V(G_n)$, it is clear that $\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good H_n-realisation of the sequence $L\langle I^1\rangle L\langle I^2\rangle \sim L$.

2. We have $V(\varphi^1_n(T^1_{i_k})) \cap V(\varphi^2_n(T^2_{i_k})) = \varphi^1_n(V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1}) \neq \emptyset$, hence there is a trail $T_{i_k} \in \varphi^1_n(T^1_{i_k}) + \varphi^2_n(T^2_{i_k})$ for any $k \in [1,m]$. If $(l_{i_k})_4 \neq 0$, there exists $j \in [1,2]$ such that $(l^j_{i_k})_4 \neq 0$, and then $V(T_{i_k}) \cap W_{n,i} \supseteq V(T^j_{i_k}) \cap V^j_{n,i} = \varphi^j_n(V(T^j_{i_k}) \cap V_{n,i}) \neq \emptyset$ (as $T^j_{i_k}$ is G_n-good), $i = 1,2,3$, so that T_{i_k} is H_n-good. Therefore, $(T_{i_1},\dots,T_{i_m})\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good H_n-realisation of the sequence $(l_{i_1},\dots,l_{i_m})L\langle I^1\rangle L\langle I^2\rangle \sim L$. ■

Claim 2 *A sequence $(l_1,\dots,l_p) \in \mathrm{Sct}(H_n)$ has a good H_n-realisation whenever one of the following conditions is fulfilled:*

1. *There is a decomposition $\{\{i_1\}, I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^j} l_i \le e_n - 3$, $j = 1,2$.*

2. *There is a decomposition $\{\{i_1,i_2\}, I^1, I^2\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$, such that $l^1_{i_1} + l^1_{i_2} + \sum_{i\in I^1} l_i = e_n$.*

Proof. 1. Put $l^j_{i_1} := e_n - \sum_{i\in I^j} l_i \in [3, l_{i_1} - 3]$, $j = 1,2$. By $S(G_n)$ there is a good G_n-realisation $(T^j_{i_1})\mathcal{T}^j$ of $(l^j_{i_1})L\langle I^j\rangle$, $j = 1,2$. By Proposition 6 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_1}) \cap V_{n,1} \neq \emptyset$, and then it suffices to use Claim 1.2.

2. If $l^2_{i_k} := l_{i_k} - l^1_{i_k}$, then $l^2_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$, and $l^2_{i_1} + l^2_{i_2} + \sum_{i\in I^2} l_i = e_n$. By $S(G_n)$ there is a good G_n-realisation $(T^j_{i_1}, T^j_{i_2})\mathcal{T}^j$ of the sequence $(l^j_{i_1}, l^j_{i_2})L\langle I^j\rangle$, $j = 1,2$. Because of Propositions 6 and 16.1 we may suppose without loss of generality that $V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1} \neq \emptyset$, $k = 1,2$, and we are done by Claim 1.2 again. ■

Consider $L \in \mathrm{Sct}(H_n)$ and suppose that $\mathrm{nd}(L) = (l_1,\dots,l_p)$. Let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \le e_n - 3$ and $\sum_{i=1}^{q-1} l_i + l_p > e_n - 3$.

Since $e_n = 75 \cdot 4^n$, the following statements are easy to be checked:

$$\forall m \in \{3,5,6\}\ e_n \equiv 0 \pmod m, \tag{G1}$$

$$n = 0 \Rightarrow e_n \equiv 3 \pmod 4, \tag{G2}$$

$$n \in [1,\infty) \Rightarrow e_n \equiv 0 \pmod 4, \tag{G3}$$

$$\exists r \in \{3,5,6\}\ e_n \equiv r \pmod 7. \tag{G4}$$

(1) If $\sum_{i=1}^{q-1} l_i + l_p = e_n$, use Claim 1.1. (In order to simplify the presentation, $\rightarrow i.j$ will mean that Claim $i.j$ guarantees the existence of a good H_n-realisation of a sequence $\sim L$. Moreover, we write for short f_i instead of $f_i(L)$.)

(2) If $\sum_{i=1}^{q-1} l_i + l_p \geq e_n + 3$, then $\sum_{i=q}^{p-1} l_i = 2e_n - (\sum_{i=1}^{q-1} l_i + l_p) \leq e_n - 3 \rightarrow$ 2.1 (with $I^1 := [1, q-1]$ and $I^2 := [q, p-1]$).

(3) $\exists \delta \in \{-2, -1, 1, 2\}$, $\sum_{i=1}^{q-1} l_i + l_p = e_n + \delta$

(31) If $l_{p-1} \geq 8$, then, with $m := \max(3, 3-\delta)$, we have $m + (l_p - m - \delta) + \sum_{i=1}^{q-1} l_i = e_n = (l_{p-1} - m) + (m + \delta) + \sum_{i=q}^{p-2} l_i \rightarrow 2.2$.

(32) If $l_{p-1} \leq 7$, let $r \in [q, p]$ be defined by the inequalities $\sum_{i=1}^{r-1} l_i \leq e_n - 3$ and $\sum_{i=1}^{r} l_i > e_n - 3$.

(321) $\sum_{i=1}^{r} l_i = e_n \rightarrow 1.1$.

(322) If $\sum_{i=1}^{r} l_i \geq e_n + 3$, then $\sum_{i=r+1}^{p} l_i \leq e_n - 3 \rightarrow 2.1$.

(323) $\sum_{i=1}^{r} l_i = e_n + \varepsilon$, $\varepsilon \in \{-2, -1, 1, 2\}$

(3231) $\varepsilon \in [-2, -1]$

(32311) If $l_p \geq l_1 + 3 - \varepsilon$, then $\sum_{i=2}^{r} l_i = e_n + \varepsilon - l_1 \leq e_n - 4$ and $l_1 + \sum_{i=r+1}^{p-1} l_i = l_1 + e_n - \varepsilon - l_p \leq e_n - 3 \rightarrow 2.1$.

(32312) $l_p \leq l_1 + 2 - \varepsilon$

(323121) If there is $j \in [1, r]$ and $k \in [r+1, p]$ such that $l_k = l_j - \varepsilon$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i + l_k = e_n \rightarrow 1.1$.

(323122) $\forall j \in [1, r]$ $\forall k \in [r+1, p]$ $l_k \neq l_j - \varepsilon$

(3231221) If $\varepsilon = -2$, then $l_i \in [l_1, l_1 + 4]$ for any $i \in [1, p]$.

(32312211) If $l_r = l_1$, then $e_n - 2 \equiv 0 \pmod{l_1}$ in contradiction with $l_1 \leq 7$ and (G1)–(G4).

(32312212) If $l_r = l_1 + 1$, then $f_{l_1+2} = f_{l_1+3} = 0$.

(323122121) If $l_{p-1} = l_1 + 1$, then $e_n + 2 \leq (p-r)(l_1+1) + 3$, $p - r \geq \frac{e_n - 1}{l_1 + 1} \geq \lfloor \frac{74}{7} \rfloor = 10$ and $l_{r+1} = l_{r+2} = l_1 + 1$.

(3231221211) If $l_2 = l_1$, then $\sum_{i=3}^{r+2} l_i = e_n \rightarrow 1.1$.

(3231221212) If $l_2 = l_1 + 1$, then $e_n - 2 = r(l_1 + 1) - 1$ and $e_n \equiv 1 \pmod{l_1 + 1}$ in contradiction with (G1)–(G4).

(323122122) If $l_{p-1} = l_1 + 4$, then $l_1 = 3$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1,r] - I} l_i + l_{p-1} + l_p = e_n \rightarrow 1.1$.

(32312213) If $l_r = l_1 + 2$, then $l_k = l_1 + 3$ for any $k \in [r+1, p]$, $e_n + 2 \equiv 0 \pmod{l_1 + 3}$, hence, because of $l_1 \leq 4$ and (G1), $l_1 = 4$. As $f_5 = 0$ (a consequence of $l_p = 7$), from $\sum_{i=1}^{r} l_i = e_n - 2 \geq 73$ it follows that there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$. Therefore, $\sum_{i \in [1, r+2] - I} l_i = e_n \rightarrow 1.1$.

(32312214) $l_r = l_1 + 3$

(323122141) If $l_{p-1} = l_1 + 3$, there is $m \in [0, 1]$ such that $e_n + 2 \equiv m \pmod{l_1 + 3}$, and then $l_1 \leq 4$ together with (G1) imply $l_1 = 4$. We have also $f_5 = 0$ and $77 \leq e_n + 2 = 7(p - r - 1) + l_p$, so that $l_{r+1} = l_{r+2} = 7$.

(3231221411) If $f_4 \geq 3$ or $f_6 \geq 2$, there is $I \subseteq [1,r]$ such that $\sum_{i\in[1,r+2]-I} l_i = e_n \to 1.1$.

(3231221412) $f_4 \leq 2 \wedge f_6 \leq 1$

(32312214121) If $l_p = 7$, then $e_n + 2 = 7(p-r)$, $e_n - 2 \equiv 3 \pmod 7$, $\mathrm{nd}(L) = (4,6)(7)^{2r-2}$, $l_i^1 := 3 \leq l_i - 3$, $i = 3,4$, and $3+3+6+7(r-2) = e_n \to 2.2$ (with $i_1 := 3$, $i_2 := 4$, $I^1 := \{2\} \cup [5, r+2]$ and $I^2 := \{1\} \cup [r+3, 2r]$).

(32312214122) If $l_p = 8$, then $e_n + 2 = 7(p-r)+1$, $e_n - 2 \equiv 4 \pmod 7$, $\mathrm{nd}(L) = (4)(7)^{2r-2}(8)$, $l_p^1 := 5 \leq l_p - 3$, $l_2^1 := 4 \leq l_2 - 3$ and $5+4+4+7(r-2) = e_n \to 2.2$.

(323122142) If $l_{p-1} = l_1 + 4$, then $l_1 = 3$, $f_5 = 0$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r]-I} l_i + l_{p-1} + l_p = e_n \to 1.1$.

(32312215) If $l_r = l_1 + 4$, then $l_1 = 3$, $f_5 = 0$, $e_n + 2 = 7(p-r)$ and $e_n - 2 \equiv 3 \pmod 7$.

(323122151) If $f_3 \geq 4$, $f_4 \geq 3$ or $f_6 \geq 2$, there is $I \subseteq [1,r]$ such that $\sum_{i\in[1,r+2]-I} l_i = e_n \to 1.1$.

(323122152) $f_3 \leq 3 \wedge f_4 \leq 2 \wedge f_6 \leq 1$

(3231221521) If $f_4 = 2$ and $l_j = l_{j+1} = 4$ for some $j \in [1, r-1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+1} l_i = e_n - 3$ and $l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = e_n - 4 \to 2.1$.

(3231221522) $f_4 = 1$

(32312215221) If $\mathrm{nd}(L) = (3)^2(4)(7)^{2r-4}$, then $l_i^1 := 4 \leq l_i - 3$, $i = 4,5$, and $4+4+4+7(r-2) = e_n \to 2.2$.

(32312215222) If $\mathrm{nd}(L) = (4,6)(7)^{2r-4}$, then $l_i^1 := 3 \leq l_i - 3$, $i = 3,4$, and $3+3+6+7(r-2) = e_n \to 2.2$.

(3231221523) If $f_4 = 0$, then $\mathrm{nd}(L) = (3)(7)^{\frac{2e_n-3}{7}}$ and we can use Claim 1.2 with $m := 3$, $i_k := k+1$, $l_{i_k}^1 = 3$, $l_{i_k}^2 = 4$, $k = 1,2,3$, $I^1 := \{1\} \cup [5, r+2]$ and $I^2 := [r+3, 2r]$ (notice that by $S(G_n)$ there are good G_n-realisations of the sequences $(3)^4(7)^{\frac{e_n-12}{7}}$ and $(4)^3(7)^{\frac{e_n-12}{7}}$ corresponding to the rows 1 and 5 of Table 1).

(3231222) If $\varepsilon = -1$, then $l_i \in [l_1, l_1+3]$ for any $i \in [1,p]$.

(32312221) If $l_r = l_1$, then $e_n - 1 = rl_1$ in contradiction with (G1)–(G4).

(32312222) If $l_r = l_1 + 1$, then $l_k \notin [l_1+1, l_1+2]$ for any $k \in [r+1, p]$, $e_n + 1 = (p-r)(l_1+3)$, $l_1 \leq 4$ and, because of (G1), $l_1 = 4$. Since $4f_4 + 5f_5 = e_n \geq 75$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 20$, and then $\sum_{i\in[1,r+3]-I} l_i = e_n \to 1.1$.

(32312223) If $l_r = l_1 + 2$, then $f_{l_1+3} = 0$, hence $e_n + 1 = (p-r)(l_1+2)$, $l_1 \leq 5$ and, because of (G1), $l_1 = 5$ and $e_n - 1 \equiv 5 \pmod 7$.

(323122231) If $f_5 \geq 4$, then $\sum_{i=5}^{r+3} l_i = e_n \to 1.1$.

(323122232) If $f_5 \leq 3$, then $\mathrm{nd}(L) = (5)(7)^{2r-1}$, $l_i^1 := 3 \leq l_i - 3$, $i = 2,3$, and $3+3+7(r-1) = e_n \to 2.2$.

(32312224) If $l_r = l_1 + 3$, then $e_n + 1 = (p - r)(l_1 + 3)$, $l_1 \le 4$ and, using (G1), $l_1 = 4$, so that $f_6 = 0$ and $e_n - 1 \equiv 5 \pmod 7$.

(323122241) $\exists j \in [2, r-1]\ l_j = 5$

(3231222411) If $f_4 \ge 2$, then $\sum_{i=3}^{j-1} l_i + \sum_{i=j+1}^{r+2} l_i = e_n \to 1.1$.

(3231222412) $f_4 = 1$

(32312224121) If $f_5 \ge 4$, then $l_1 + \sum_{i=6}^{r+3} l_i = e_n \to 1.1$.

(32312224122) If $f_5 \le 3$, then $\mathrm{nd}(L) = (4)(5)^3(7)^{2r-5}$, $l_1 + l_2 + \sum_{i=5}^{r+1} l_i = e_n - 4$ and $l_3 + l_4 + \sum_{i=r+2}^{p-1} l_i = e_n - 3 \to 2.1$.

(323122242) $f_5 = 0$

(3231222421) If $f_4 \ge 5$, then $\sum_{i=6}^{r+3} l_i = e_n \to 1.1$.

(3231222422) If $f_4 \le 4$, then $\mathrm{nd}(L) = (4)^3(7)^{\frac{2e_n-12}{7}}$. By $S(G_n)$ there exists a good G_n-realisation $(T_1^1, T_2^1, T_3^1, T_4^1, T_5^1)\mathcal{T}^1$ of the sequence $(4)^5(7)^{\frac{e_n-20}{7}}$ such that $v_{i,1} \in V(T_i^1)$, $i = 1,2,3$ and $v_{i,2} \in V(T_{3+i}^1)$, $i = 1,2$ (see the row 4 of Table 1). Therefore, by Proposition 6, there is also a good G_n-realisation $(\bar{T}_1^1, \bar{T}_2^1, \bar{T}_3^1, \bar{T}_4^1, \bar{T}_5^1)\bar{\mathcal{T}}^1$ of $(4)^5(7)^{\frac{e_n-20}{7}}$ such that $v_{i,1} \in V(\bar{T}_i^1)$, $i = 1,2$, and $v_{i,2} \in V(\bar{T}_{2+i}^1)$, $i = 1,2,3$. Further, by $S(G_n)$ and Propositions 6 and 16.1, there exist good G_n-realisations $(T_1^2, T_2^2)\mathcal{T}^2$ and $(\bar{T}_1^2, \bar{T}_2^2)\bar{\mathcal{T}}^2$ of the sequence $(3)^2(7)^{\frac{e_n-6}{7}}$ such that $v_{i+1,1} \in V(T_i^2)$ and $v_{i,1} \in V(\bar{T}_i^2)$, $i = 1,2$. Since $V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_{i-1}^2)) \supseteq \{v_{i,1}^1\}$, $i = 2,3$, and $V(\varphi_n^1(\bar{T}_i^1)) \cap V(\varphi_n^2(\bar{T}_i^2)) \supseteq \{v_{i,1}^1\}$, $i = 1,2$, there are trails $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_{i-1}^2)$, $i = 2,3$, and $\bar{T}_i \in \varphi_n^1(\bar{T}_i^1) + \varphi_n^2(\bar{T}_i^2)$, $i = 1,2$. Then $(\varphi_n^1(T_4^1), \varphi_n^1(T_5^1), \varphi_n^1(T_1^1), T_2, T_3)\ \varphi_n^1(\mathcal{T}^1)\ \varphi_n^2(\mathcal{T}^2)$ and $(\varphi_n^1(\bar{T}_3^1), \varphi_n^1(\bar{T}_4^1), \varphi_n^1(\bar{T}_5^1), \bar{T}_1, \bar{T}_2)\varphi_n^1(\bar{\mathcal{T}}^1)\varphi_n^2(\bar{\mathcal{T}}^2)$ are H_n-good realisations of $\mathrm{nd}(L)$; the former satisfies "(8,2)-conditions" and the latter one "(7,2)-conditions" of Table 2.

(3232) If $\varepsilon \in [1,2]$, then $l_r = \sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1} l_i \ge e_n + \varepsilon - (e_n - 3) = 3 + \varepsilon$. With $l := \min(l_i : i \in [1,r], l_i \ge 3 + \varepsilon)$ we have $3 + \varepsilon \le l \le l_r$.

(32321) If $l_p \ge l + 3 - \varepsilon$, let $j \in [1,r]$ be such that $l_j = l$. Then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i = e_n + \varepsilon - l \le e_n + \varepsilon - (3 + \varepsilon)$ and $l_j + \sum_{i=r+1}^{p-1} l_i = l + (e_n - \varepsilon) - l_p \le e_n - 3 \to 2.1$.

(32322) $\forall i \in [1,p]\ l_i \in [3, 2+\varepsilon] \cup [l, l+2-\varepsilon]$

(323221) $\varepsilon = 1$

(3232211) If $l_r = l + 1$, then $e_n - 1 = (p - r)(l + 1)$ in contradiction with (G1)–(G4).

(3232212) $l_r = l$

(32322121) If $l_{p-1} = l$, then $e_n - 1 = (p - r)l + m$ for some $m \in [0,1]$ and $e_n \equiv 1 + m \pmod l$ in contradiction with (G1)–(G4).

(32322122) $l_{p-1} = l + 1$

(323221221) If $f_3 = 0$, then $e_n + 1 = rl$, hence from $l \le 6$ and (G1)–(G3) it follows that $l = 4$ and $n = 0$.

(3232212211) If $f_5 \ge 3$, then $\sum_{i=5}^{r} l_i + \sum_{i=p-2}^{p} l_i = e_n \to 1.1$.

(3232212212) If $f_5 = 2$, then $\mathrm{nd}(L) = (4)^{35}(5)^2$; for a good H_0-realisation of $\mathrm{nd}(L)$ see Graph H_0.

(323221222) $f_3 \geq 1$

(3232212221) $l = 4$

(32322122211) If $f_3 = 1$, then $e_n + 1 = 3 + 4(r-1)$ in contradiction with (G2) and (G3).

(32322122212) If $f_3 \geq 2$, then $\sum_{i=3}^{r} l_i + l_p = e_n \rightarrow 1.1$.

(3232212222) $l \in [5, 6]$

(32322122221) If $l_{r-1} = 3$, then $e_n + 1 = 3(r-1) + l$ in contradiction with (G1).

(32322122222) If $l_{r-1} = l$, then $\sum_{i=1}^{r-2} l_i + l_{p-1} = (e_n + 1 - 2l) + (l+1) \leq e_n - 3$ and $\sum_{i=r-1}^{p-2} l_i = e_n - 3 \rightarrow 2.1$.

(323222) If $\varepsilon = 2$, then $e_n - 2 = (p-r)l$ in contradiction with (G1)–(G4).

To conclude the proof of the implication $S(G_n) \Rightarrow S(H_n)$ we have to find good H_n-realisations of sequences satisfying "$(t, 2)$-conditions" of Table 2, $t \in [1, 6]$.

$t = 1$: If $7 | 2e_n - 12$, then $7 | e_n - 6$ and $7 | e_n - 13$. By $S(G_n)$ (the row 2 of Table 1) and Proposition 6 there is a good G_n-realisation $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of the sequence $(3)^3(4)(7)^{\frac{e_n-13}{7}}$ such that $v_{1,1} \in V(T_4^1)$ and $v_{i,2} \in V(T_i^1)$, $i = 1, 2, 3$. By $S(G_n)$ and Proposition 6 there is a good G_n-realisation $(T_1^2, T_2^2)\mathcal{T}^2$ of the sequence $(3)^2(7)^{\frac{e_n-6}{7}}$ with $v_{1,1} \in V(T_1^2)$. As $V(\varphi_n^1(T_4^1)) \cap V(\varphi_n^2(T_1^2)) \supseteq \{v_{1,1}^1\}$, there is a trail $T_4 \in \varphi_n^1(T_4^1) + \varphi_n^2(T_1^2)$ and $(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), \varphi_n^1(T_3^1), \varphi_n^2(T_2^2), T_4)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is an appropriate good H_n-realisation of the sequence $(3)^4(7)^{\frac{2e_n-12}{7}}$.

$t = 2$: By $S(G_n)$, Propositions 6 and 16.2, there is a good G_n-realisation $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of the sequence $(3)^4(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1} \in V(T_i^1)$, $v_{i,2} \in V(T_{i+2}^1)$, $i = 1, 2$. By $S(G_n)$ and Proposition 16.1 there exists a good G_n-realisation $(T_1^2, T_2^2)\mathcal{T}^2$ of the sequence $(4)^3(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1} \in V(T_i^2)$, $i = 1, 2$. As $V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_i^2)) \supseteq \{v_{i,1}^1\}$, there is a trail $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_i^2)$, $i = 1, 2$. Then $(\varphi_n^1(T_3^1), \varphi_n^1(T_4^1))\varphi_n^2(\mathcal{T}^2)(T_1, T_2)\varphi_n^1(\mathcal{T}^1)$ is a good H_n-realisation of the sequence $(3)^2(4)(7)^{\frac{2e_n-10}{7}}$ having required properties.

$t = 3$: By $S(G_n)$ there are good G_n-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(7)^{\frac{e_n-6}{7}}$ and $\mathcal{T}^2$ of $(6)(7)^{\frac{e_n-6}{7}}$. By Proposition 16.1 we may suppose without loss of generality that $v_{i,2} \in V(T_i^1)$, $i = 1, 2$. Then $(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1))\, \varphi_n^2(\mathcal{T}^2)\, \varphi_n^1(\mathcal{T}^1)$ is a necessary good H_n-realisation of $(3)^2(6)(7)^{\frac{2e_n-12}{7}}$.

$t = 4$: By $S(G_n)$ and Propositions 6 and 16.1 there exist good G_n-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)^2(4)(7)^{\frac{e_n-10}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{e_n-3}{7}}$ such that $v_{i,2} \in V(T_i^1)$, $i = 1, 2$, and $v_{1,1} \in V(T_3^1) \cap V(T_1^2)$. There is a trail $T_3 \in \varphi_n^1(T_3^1) + \varphi_n^2(T_1^2)$ and $(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), T_3)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good H_n-

realisation of $(3)^2(7)^{\frac{2e_n-6}{7}}$ we are seeking for.

$t = 5$: By $S(G_n)$ (the row 5 of Table 1) and Proposition 6 there are good G_n-realisations $(T_1^j, T_2^j, T_3^j)\mathcal{T}^j$ of the sequence $(4)^3(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1+j} \in V(T_i^j)$, $i = 1,2,3$, $j = 1,2$. Then $(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), \varphi_n^1(T_3^1), \varphi_n^2(T_1^2), \varphi_n^2(T_2^2), \varphi_n^2(T_3^2))\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good H_n-realisation of $(4)^6(7)^{\frac{2e_n-24}{7}}$ having required properties.

$t = 6$: By $S(G_n)$ (the row 3 of Table 1) and Proposition 6 there exists a good G_n-realisation $[\prod_{i=1}^{6}(T_i^1)]\mathcal{T}^1$ of the sequence $(4)^6(7)^{\frac{e_n-24}{7}}$ with $v_{i,2} \in V(T_i^1)$, $i = 1,2,3$, $v_{i,3} \in V(T_{3+i}^1)$, $i = 1,2$, and $v_{1,1} \in V(T_6^1)$. By $S(G_n)$ and Proposition 6 there exists also a good G_n-realisation $(T_1^2)\mathcal{T}^2$ of the sequence $(3)(7)^{\frac{e_n-3}{7}}$ with $v_{1,1} \in V(T_1^2)$. There is a trail $T_6 \in \varphi_n^1(T_6^1) + \varphi_n^2(T_1^2)$ and $[\prod_{i=1}^{5}(\varphi_n^1(T_i^1))](T_6)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is an appropriate good H_n-realisation of $(4)^5(7)^{\frac{2e_n-20}{7}}$.

(b) $(S(G_n) \wedge S(H_n)) \Rightarrow S(G_{n+1})$

Claim 3 *A sequence $(l_1, \dots, l_p) \in \mathrm{Sct}(G_{n+1})$ has a good G_{n+1}-realisation whenever one of the following conditions is fulfilled:*

1. *There is $I^1 \subseteq [1,p]$ such that $\sum_{i\in I^1} l_i = 2e_n$.*
2. *There is $m \in [1,p]$ and a decomposition $\{\{i_1, \dots, i_m\}, I^1, I^2\}$ of $[1,p]$ such that, for both $j = 1,2$, there is a sequence $L^j = (l_{i_1}^j, \dots, l_{i_m}^j)$ such that $L^j L\langle I^j\rangle \in \mathrm{Sct}(H_n)$ and a good H_n-realisation $(T_{i_1}^j, \dots, T_{i_m}^j)\mathcal{T}^j$ of $L^j L\langle I^j\rangle$ satisfying $l_{i_k} = l_{i_k}^1 + l_{i_k}^2$ and $V(T_{i_k}^1) \cap V(T_{i_k}^2) \cap V_{n,2}^1 \neq \emptyset$ for any $k \in [1,m]$.*
3. *There is a decomposition $\{\{i_1, i_2\}, I^1, I^2\}$ of $[1,p]$ and $l_{i_k}^1 \in [3, l_{i_k} - 3]$, $k = 1,2$, such that there is a good $(l_{i_1}^1, l_{i_2}^1)$-global H_n-realisation of the sequence $(l_{i_1}^1, l_{i_2}^1)L\langle I^1\rangle$.*

Proof. 1. With $I^2 := [1,p] - I^1$ we have $\sum_{i\in I^2} l_i = e_{n+1} - \sum_{i\in I^1} l_i = 2e_n$. By $S(H_n)$ there exists a good H_n-realisation $\mathcal{T}^j$ of $L\langle I^j\rangle$, $j = 1,2$. Let T be a trail of length $\neq 4$ of $\mathcal{T}^j$, $j \in [1,2]$, and let $i \in [1,3]$. Since $\psi_n^j(W_{n,i}) \subseteq V_{n+1,i}$, from $V(T) \cap W_{n,i} \neq \emptyset$ it follows that $V(\psi_n^j(T)) \cap V_{n+1,i} \neq \emptyset$. Therefore, $\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $L\langle I^1\rangle L\langle I^2\rangle \sim L$.

2. As $V(\psi_n^1(T_{i_k}^1)) \cap V(\psi_n^2(T_{i_k}^2)) \supseteq \psi_n^1(V(T_{i_k}^1) \cap V(T_{i_k}^2) \cap V_{n,2}^1) \neq \emptyset$, there is a trail $T_{i_k} \in \psi_n^1(T_{i_k}^1) + \psi_n^2(T_{i_k}^2)$ for any $k \in [1,m]$. If $(l_{i_k})_4 \neq 0$, there is $j \in [1,2]$ such that $(l_{i_k}^j)_4 \neq 0$. Clearly, the trail $\psi_n^j(T_{i_k}^j)$ is G_{n+1}-good (as above), hence, because of $V(T_{i_k}) \supseteq V(\psi_n^j(T_{i_k}^j))$, so is T_{i_k}. Thus, $(T_{i_1}, \dots, T_{i_m})\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(l_{i_1}, \dots, l_{i_m})L\langle I^1\rangle\, L\langle I^2\rangle \sim L$.

3. Let $(T_{i_1}^1, T_{i_2}^1)\mathcal{T}^1$ be a good $(l_{i_1}^1, l_{i_2}^1)$-global H_n-realisation of $(l_{i_1}^1, l_{i_2}^1)L\langle I^1\rangle$ such that $|E(T_{i_k}^1)| = l_{i_k}^1$, $k = 1,2$; there is $j \in [1,2]$ with $V(T_{i_1}^1) \cap (V_{n,2}^j \cup V_{n,3}^j) \neq \emptyset$ and $V(T_{i_2}^1) \cap (V_{n,2}^{3-j} \cup V_{n,3}^{3-j}) \neq \emptyset$. If $l_{i_k}^2 := l_{i_k} - l_{i_k}^1$, $k = 1,2$, from $l_{i_1} + l_{i_2} + \sum_{i\in I^1\cup I^2} l_i = 4e_n$ and $l_{i_1}^1 + l_{i_2}^1 + \sum_{i\in I^1} l_i = 2e_n$ it follows that $L^2 := (l_{i_1}^2, l_{i_2}^2)L\langle I^2\rangle \in \mathrm{Sct}(H_n)$. Hence, by $S(H_n)$ there is a good H_n-realisation $(T_{i_2}^1, T_{i_2}^2)\mathcal{T}^2$ of L^2.

If there is $k \in [1,2]$ such that $V(T^2_{i_1}) \cap (V^k_{n,2} \cup V^k_{n,3}) \neq \emptyset$ and $V(T^2_{i_2}) \cap (V^{3-k}_{n,2} \cup V^{3-k}_{n,3}) \neq \emptyset$, by Proposition 19.3 we may suppose without loss of generality that $v^m_{1,2} \in V(T^1_{i_m}) \cap V(T^2_{i_m})$, $m = 1,2$. Since $V(\psi^1_n(T^1_{i_m})) \cap V(\psi^2_n(T^2_{i_m})) \supseteq \{v^{m,1}_{1,2}\}$, there is a trail $T_m \in \psi^1_n(T^1_{i_m}) + \psi^2_n(T^2_{i_m})$, $m = 1,2$, and $(T_1, T_2)\psi^1_n(\mathcal{T}^1)\psi^2_n(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(l_{i_1}, l_{i_2})L\langle I^1\rangle L\langle I^2\rangle \sim L$.

If there is $k \in [1,2]$ such that $V(T^2_{i_1}) \cup V(T^2_{i_2}) \subseteq V(G^k_n)$, by Proposition 19.2,3 we may suppose without loss of generality that $v^1_{1,2} \in V(T^1_{i_1}) \cap V(T^2_{i_1})$, $v^2_{1,3} \in V(T^1_{i_2})$ and $v^1_{1,3} \in V(T^2_{i_2})$. As above, there is a trail $T_1 \in \psi^1_n(T^1_{i_1}) + \psi^2_n(T^2_{i_1})$. Further, since $V(\psi^1_n(T^1_{i_2})) \cap V(\psi^2_n(T^2_{i_2})) \supseteq \{v^{2,1}_{1,3}\} = \{v^{1,2}_{1,3}\}$, there exists a trail $T_2 \in \psi^1_n(T^1_{i_2}) + \psi^2_n(T^2_{i_2})$ and $(T_1, T_2)\psi^1_n(\mathcal{T}^1)\psi^2_n(\mathcal{T}^2)$ is a good G_{n+1}-realisation of a sequence changeable to L. ■

Claim 4 *A sequence $(l_1, \dots, l_p) \in \mathrm{Sct}(G_{n+1})$ has a good G_{n+1}-realisation whenever one of the following conditions is fulfilled:*

1. *There is a decomposition $\{\{i_1\}, I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^j} l_i \leq 2e_n - 3$, $j = 1,2$.*
2. *There is a decomposition $\{\{i_1, i_2\}, I^1, I^2\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$, such that $((l^1_{i_1})_4, (l^1_{i_2})_4), ((l_{i_1} - l^1_{i_1})_4, (l_{i_2} - l^1_{i_2})_4) \notin P$ and $l^1_{i_1} + l^1_{i_2} + \sum_{i\in I^1} l_i = 2e_n$.*
3. *There is a decomposition $\{\{i_1, i_2, i_3\}, I^1, I^2, I^3\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2,3$, such that $l^1_{i_1} + l^1_{i_3} + \sum_{i\in I^1} l_i = e_n = l^1_{i_2} + (l_{i_3} - l^1_{i_3}) + \sum_{i\in I^2} l_i$.*
4. *There is a decomposition $\{\{i_1, i_2\}, I^1, I^2, I^3\}$ of $[1,p]$ and $l^1_{i_j} \in [3, l_{i_j} - 3]$, $j = 1,2$, such that $l^1_{i_j} + \sum_{i\in I^j} l_i = e_n$, $j = 1,2$.*

Proof. 1. If $l^j_{i_1} := 2e_n - \sum_{i\in I^j} l_i$, then $l^j_{i_1} \in [3, 2e_n]$, $j = 1,2$, and $l_{i_1} = l^1_{i_1} + l^2_{i_1}$. By $S(H_n)$ there exists a good H_n-realisation $(T^j_{i_1})\mathcal{T}^j$ of $(l^j_{i_1})L\langle I^j\rangle$, $j = 1,2$. By Proposition 19.1 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_1}) \cap V^1_{n,2} \neq \emptyset$ (notice that a closed trail in H_n necessarily contains a vertex of an eccentric part of H_n). Thus, we are done by Claim 3.2.

2. If $l^2_{i_k} := l_{i_k} - l^1_{i_k}$, then $l^2_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$. Since $L^j := (l^j_{i_1}, l^j_{i_2})L\langle I^2\rangle \in \mathrm{Sct}(H_n)$, by $S(H_n)$ there is a good H_n-realisation $\mathcal{T}^j = (T^j_{i_1}, T^j_{i_2})\ \mathcal{T}^j$ of L^j, $j = 1,2$. If there is $j \in [1,2]$ such that $\mathcal{T}^j$ is $(l^j_{i_1}, l^j_{i_2})$-global, we are done by Claim 3.3. So, let $j_1, j_2 \in [1,2]$ be such that $V(T^k_{i_1}) \cup V(T^k_{i_2}) \subseteq V(G^{j_k}_n)$, $k = 1,2$. By Proposition 16.2 there is $m_k \in [2,3]$ such that $\{V(T^k_{i_1}), V(T^k_{i_2})\}$ has a system of distinct representatives in $V^{j_k}_{n,m_k}$, $k = 1,2$. By Proposition 19.1 we may suppose without loss of generality that $v_{k,2} \in V(T^1_{i_k}) \cup V(T^2_{i_k})$, $k = 1,2$. Now, it suffices to use Claim 3.2.

3. Put $l^2_{i_3} := l_{i_3} - l^1_{i_3} \in [3, l_{i_3} - 3]$. By $S(G_n)$ there is a good G_n-realisation $(T^1_{i_j}, T^j_{i_3})\mathcal{T}^j$ of $(l^1_{i_j}, l^j_{i_3})L\langle I^j\rangle$, $j = 1,2$; we may suppose without loss of generality that $v_{1,1} \in V(T^1_{i_3}) \cap V(T^2_{i_3})$. As $V(\varphi^1_n(T^1_{i_3})) \cap V(\varphi^2_n(T^2_{i_3})) \supseteq \{v^1_{1,1}\}$, there

is a trail $T_3 \in \varphi_n^1(T_{i_3}^1) + \varphi_n^2(T_{i_3}^2)$ and $(\varphi_n^1(T_{i_1}^1), \varphi_n^2(T_{i_2}^1), T_3)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $(l_{i_1}^1, l_{i_2}^1)$-global H_n-realisation of the sequence $(l_{i_1}^1, l_{i_2}^1, l_{i_3}^1)L\langle I^1\rangle L\langle I^2\rangle \sim (l_{i_1}^1, l_{i_2}^1)L\langle \{i_3\} \cup I^1 \cup I^2\rangle$. Now, we are done by Claim 3.3 with the decomposition $\{\{i_1, i_2\}, \{i_3\} \cup I^1 \cup I^2, I^3\}$ of $[1,p]$ (and by Lemma 1).

4. By $S(G_n)$ there is a good G_n-realisation $(T_{i_j}^1)\mathcal{T}^j$ of $(l_{i_j}^1)L\langle I^j\rangle$, $j = 1, 2$. Since $(\varphi_n^1(T_{i_1}^1), \varphi_n^2(T_{i_2}^1))\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $(l_{i_1}^1, l_{i_2}^1)$-global H_n-realisation of $(l_{i_1}^1, l_{i_2}^1)L\langle I^1\rangle L\langle I^2\rangle \sim (l_{i_1}^1, l_{i_2}^1)L\langle I^1 \cup I^2\rangle$, Claim 3.3 with the decomposition $\{\{i_1, i_2\}, I^1 \cup I^2, I^3\}$ of $[1,p]$ can be used. ■

Consider $L \in \mathrm{Sct}(G_{n+1})$, assume that $\mathrm{nd}(L) = (l_1, \ldots, l_p)$ and let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \le 2e_n - 3$ and $\sum_{i=1}^{q-1} l_i + l_p > 2e_n - 3$.

Instead of (G1)–(G4) we are going to use the following assertions:

$$\forall m \in \{3,5,6\}\ 2e_n \equiv 0 \pmod m, \tag{H1}$$
$$n = 0 \Rightarrow 2e_n \equiv 2 \pmod 4, \tag{H2}$$
$$n \in [1,\infty) \Rightarrow 2e_n \equiv 0 \pmod 4, \tag{H3}$$
$$\exists m \in \{3,5,6\}\ 2e_n \equiv m \pmod 7, \tag{H4}$$
$$n = 0 \Rightarrow 2e_n \equiv 6 \pmod 8, \tag{H5}$$
$$n \in [1,\infty) \Rightarrow 2e_n \equiv 0 \pmod 8. \tag{H6}$$

(1) $\sum_{i=1}^{q-1} l_i + l_p = 2e_n \to 3.1$.

(2) If $\sum_{i=1}^{q-1} l_i + l_p \ge 2e_n + 3$, then $\sum_{i=q}^{p-1} l_i \le 2e_n - 3 \to 4.1$.

(3) $\exists \delta \in \{-2,-1,1,2\}\ \sum_{i=1}^{q-1} l_i + l_p = 2e_n + \delta$

(31) $l_{p-1} \ge 9$

(311) $\delta = -2$

(3111) If $l_q \le 4$, then $\sum_{i=2}^{q-1} l_i + l_p \le 2e_n - 2 - 3$ and $l_1 + \sum_{i=q}^{p-2} l_i = l_1 + e_{n+1} - (2e_n - 2) - l_{p-1} \le 4 + 2e_n + 2 - 9 \to 4.1$.

(3112) If $l_q \in [5, l_p - 1]$, then $\sum_{i=1}^{q} l_i \le 2e_n - 3$ and $\sum_{i=q+1}^{p-1} l_i = 2e_n + 2 - l_q \le 2e_n - 3 \to 4.1$.

(3113) If $l := l_p$ and $l_i = l$ for any $i \in [q,p]$, we obtain $2e_n + 2 = (p-q)l$ and $2e_n - 2 \ge l | 2e_n + 2$, so that $l \le e_n + 1$ and $p - q \ge 2$. Using (H1)–(H3) we see that $3 \nmid l$, $4 \nmid l$ (because of (H3), $4|l$ implies $n = 0$, $l|152$ and $l \le 8$, while we have $l \ge 9$) and $5 \nmid l$, hence $l \ge 11$.

(31131) If $l_1 \le l - 5$, then $\sum_{i=2}^{q-1} l_i + l_p \le 2e_n - 2 - 3$ and $l_1 + \sum_{i=q}^{p-2} l_i = 2e_n + 2 - (l - l_1) \le 2e_n - 3 \to 4.1$.

(31132) If $l_j = l - 2$ for some $j \in [1, q-1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{q+1} l_i = 2e_n \to 3.1$.

(31133) $\forall i \in [1, q-1]\ l_i \in \{l-4, l-3, l-1, l\}$

(311331) If $f_{l-1} \ge 2$ and $l_j = l_{j+1} = l - 1$ for some $j \in [1, q-1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{q+2} l_i = 2e_n \to 3.1$.

(311332) If $f_{l-4} + f_{l-3} \ge 2$, then $l - 4 \le l_1 \le l_2 \le l - 3$, $\sum_{i=3}^{q+1} l_i = 2e_n - 2 + l - (l_1 + l_2) \le 2e_n + 6 - l \le 2e_n - 3$ and $l_1 + l_2 + \sum_{i=q+2}^{p-1} l_i = 2e_n + 2 - (l - l_1) - (l - l_2) \le 2e_n - 4 \to 4.1$.

(311333) If $f_{l-4}+f_{l-3}+f_{l-1}\le 2$, then $2e_n-2\equiv -4f_{l-4}-3f_{l-3}-f_{l-1}\pmod l$, $2e_n+2\equiv 0\pmod l$, $-4f_{l-4}-3f_{l-3}-f_{l-1}\equiv -4\pmod l$, $\mathrm{nd}(L)=(l_1,l_2)(l)^{p-2}$, $p=2q\ge 4$ and $(l_1,l_2)\in\{(l-3,l-1),(l-4,l)\}$. Since $pl-4=4e_n$, we have $pl\equiv 0\pmod 4$.

(3113331) If q is even, $q=2r$, then $3\le l_1^1:=\lfloor\frac{l-2}{2}\rfloor\le l-7\le l_1-3$, $3\le l_2^1:=\lceil\frac{l-2}{2}\rceil\le l-4\le l_2-3$, $3\le l_3^1:=\lceil\frac{l}{2}\rceil\le l-3=l_3-3$ and $\lfloor\frac{l-2}{2}\rfloor+\lceil\frac{l}{2}\rceil+(r-1)l=e_n=\lceil\frac{l-2}{2}\rceil+(l-\lceil\frac{l}{2}\rceil)+(r-1)l\to 4.3$ (with $i_1:=1$, $i_2:=2$, $i_3:=3$, $I^1:=[4,r+2]$, $I^2:=[r+3,2r+1]$ and $I^3=[2r+2,4r]$).

(3113332) If q is odd, $q=2r+1$, then $pl=2(2r+1)l\equiv 0\pmod 4$ implies that l is even, $l=2m\ge 12$, $3\le l_1^1:=m-1\le 2m-7\le l_1-3$, $3\le l_2^1:=m-1\le 2m-7\le l_2-3$ and $m-1+r\cdot 2m=e_n\to 4.4$ (with $i_1:=1$, $i_2:=2$, $I^1:=[3,r+2]$, $I^2:=[r+3,2r+2]$ and $I^3:=[2r+3,4r+2]$).

(312) If $\delta=-1$, then $3\le l_{p-1}^1:=6\le l_{p-1}-3$, $3\le l_p^1:=l_p-5\le l_p-3$, $((6)_4,(l_p-5)_4),((l_{p-1}-6)_4,(5)_4)\notin P$ and $6+(l_p-5)+\sum_{i=1}^{q-1}l_i=2e_n\to 4.2$.

(313) If $\delta=1$, then $3\le l_{p-1}^1:=5\le l_{p-1}-3$, $3\le l_p^1:=l_p-6\le l_p-3$, $((5)_4,(l_p-6)_4),((l_{p-1}-5)_4,(6)_4)\notin P$ and $5+(l_p-6)+\sum_{i=1}^{q-1}l_i=2e_n\to 4.2$.

(314) $\delta=2$

(3141) If there is $j\in[1,q-1]$ such that $l_j\in[5,l_{p-1}-1]$, then $\sum_{i=1}^{j-1}l_i+\sum_{i=j+1}^{q-1}l_i+l_p\le 2e_n+2-5$ and $l_j+\sum_{i=q}^{p-2}l_i=2e_n-2-(l_{p-1}-l_j)\le 2e_n-3\to 4.1$.

(3142) $\forall i\in[1,q-1]\ l_i\in\{3,4,l_{p-1}\}$

(31421) If $f_3+f_4\ge 2$, then $\sum_{i=3}^{q-1}l_i+l_p\le(2e_n+2)-2\cdot 3$ and $l_1+l_2+\sum_{i=q}^{p-2}l_i\le 2\cdot 4+(2e_n-2)-9\to 4.1$.

(31422) If $f_3+f_4\le 1$, then $l_i=l_{p-1}=:l$ for any $i\in[2,p-1]$, $2e_n+2=\sum_{i=1}^{q-1}l_i+l_p\equiv j+l_p\pmod l$ for some $j\in\{0,3,4\}$ and $2e_n-2\equiv 0\pmod l$, so that $j+l_p\equiv 4\pmod l$ and, using (H1)–(H3) $3\nmid l$, $4\nmid l$ (as $l\ge 9$), $5\nmid l$ and $l\ge 11$.

(314221) If $l_p=l+2$, then $\sum_{i=1}^{q}l_i=2e_n\to 3.1$.

(314222) If $l_p\ge l+5$, then $\sum_{i=1}^{q}l_i=(2e_n+2)-(l_p-l)\le 2e_n-3$ and $\sum_{i=q+1}^{p-1}l_i=(2e_n-2)-l\le 2e_n-11\to 4.1$.

(314223) If $l_p=l+k$ for some $k\in\{0,1,3,4\}$, then $j+k\equiv 4\pmod l$. Consequently, from $l\ge 11$ it follows that $j+k=4$ and $(j,k)\in\{(0,4),(3,1),(4,0)\}$.

(3142231) If $(j,k)=(0,4)$, then $p=2q$, $\mathrm{nd}(L)=(l)^{2q-1}(l+4)$ and $2ql+4=4e_n$, so that $2ql\equiv 0\pmod 4$.

(31422311) If q is even, $q=2r$, then $3\le l_p^1:=\lfloor\frac{l+2}{2}\rfloor\le l+1=l_p-3$, $3\le l_1^1:=\lceil\frac{l+2}{2}\rceil\le l-3=l_1-3$, $3\le l_2^1:=\lceil\frac{l}{2}\rceil\le l-3=l_2-3$ and $\lfloor\frac{l+2}{2}\rfloor+\lceil\frac{l}{2}\rceil+(r-1)l=e_n=\lceil\frac{l+2}{2}\rceil+(l-\lceil\frac{l}{2}\rceil)+(r-1)l\to 4.3$.

(31422312) If q is odd, $q=2r+1$, then l must be even, $l=2m$, $3\le l_p^1:=m+1\le 2m+1=l_p-3$, $3\le l_1^1:=m+1\le 2m-3=l_1-3$ and

$m+1+r\cdot 2m = e_n \to 4.4$.

(3142232) If $(j,k)\in\{(3,1),(4,0)\}$, then $p=2q+1$, $\mathrm{nd}(L)=(j)(l)^{2q-1}(l+4-j)$ and $2ql\equiv 0 \pmod 4$.

(31422321) If q is even, $q=2r$, then $3\le l_p^1 := \lfloor\frac{l+1-j}{2}\rfloor \le l-3\le l_p-3$, $3\le l_2^1 := \lceil\frac{l+3-j}{2}\rceil \le l-3 = l_2-3$, $3\le l_3^1 := \lceil\frac{l+1-j}{2}\rceil \le l-3=l_3-3$ and $\lfloor\frac{l+1-j}{2}\rfloor + \lceil\frac{l+1-j}{2}\rceil + j + (r-1)l = e_n = \lceil\frac{l+3-j}{2}\rceil + (l-\lceil\frac{l+1-j}{2}\rceil) + (r-1)l \to 4.3$.

(31422322) If q is odd, $q=2r+1$, then l is even, $l=2m\ge 12$, $3\le l_p^1 := m+1-j \le 2m+4-j = l_p$, $3\le l_2^1 := m+1\le 2m-3 = l_2-3$ and $(m+1-j)+j+r\cdot 2m = e_n = m+1+r\cdot 2m \to 4.4$.

(32) If $l_{p-1}\le 8$, let $r\in[q,p]$ be defined by the inequalities $\sum_{i=1}^{r-1} l_i \le 2e_n-3$ and $\sum_{i=1}^{r} l_i > 2e_n-3$.

(321) $\sum_{i=1}^{r} l_i = 2e_n \to 3.1$.

(322) If $\sum_{i=1}^{r} l_i \ge 2e_n+3$, then $\sum_{i=r+1}^{p} l_i \le 2e_n-3 \to 4.1$.

(323) $\exists\varepsilon\in\{-2,-1,1,2\}$ $\sum_{i=1}^{r} l_i = 2e_n+\varepsilon$

(3231) $\varepsilon\in[-2,-1]$

(32311) If $l_p\ge l_1+3-\varepsilon$, then $\sum_{i=2}^{r} l_i = 2e_n+\varepsilon-l_1\le 2e_n-4$ and $l_1+\sum_{i=r+1}^{p-1} l_i = l_1+(2e_n-\varepsilon)-l_p\le 2e_n-3\to 4.1$.

(32312) $l_p\le l_1+2-\varepsilon$

(323121) If there are $j\in[1,r]$ and $k\in[r+1,p]$ such that $l_k=l_j-\varepsilon$, then $\sum_{i=1}^{j-1} l_i+\sum_{i=j+1}^{r} l_i + l_k = 2e_n\to 3.1$.

(323122) $\forall j\in[1,r]\ \forall k\in[r+1,p]\ l_k\ne l_j-\varepsilon$

(3231221) $\varepsilon=-2$

(32312211) If $l_r=l_1$, then $2e_n-2\equiv 0 \pmod{l_1}$ and from (H1)–(H6) it follows that $n=0$, $l_1=4$ and $r=37$.

(323122111) If $f_5\ge 2$, then $35\cdot 4+2\cdot 5 = 2e_0\to 3.1$.

(323122112) If $f_7\ge 2$, then $34\cdot 4+2\cdot 7=2e_0\to 3.1$.

(323122113) If $f_5+f_7\le 2$, then $2e_0+2=\sum_{i=38}^{p} l_i\equiv 5f_5+7f_7 \pmod 2$ (recall that $l_p\le 8$), and so $f_5=f_7\le 1$.

(3231221131) If $l_{p-1}\ge 7$, then $3=:l_{p-1}^1\le l_{p-1}-3$, $3=:l_p^1\le l_p-3$, $((3)_4,(3)_4)$, $((l_{p-1}-3)_4,(l_p-3)_4)\notin P$ (here we use the inequality $f_7\le 1$) and $3+3+36\cdot 4=2e_0\to 4.2$.

(3231221132) If $l_{p-1}\le 6$, then $(l_{p-1},l_p)\in\{(4,4),(4,8),(5,7)\}$.

(32312211321) If $\mathrm{nd}(L)\in\{(4)^{75},(4)^{73}(8)\}$, we can use the fact that the graph $G_1=K_{10,10,10}$ is an edge-disjoint union of graphs $K^i\cong K_{10,10}$, $i=1,2,3$. By [7], the graph $K_{10,10}$ is ADTC, there is a K^i-realisation $\mathcal{T}^i$ of $(4)^{25}$, $i=1,2,3$, and a K^3-realisation $\bar{\mathcal{T}}^3$ of $(4)^{23}(8)$. Then $\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3$ is a good H_1-realisation of $(4)^{75}$ and $\mathcal{T}^1\mathcal{T}^2\bar{\mathcal{T}}^3$ is a good H_1-realisation of $(4)^{73}(8)$.

(32312211322) If $\mathrm{nd}(L)=(4)^{72}(5,7)$, consider again the above H_1-realisation of $(4)^{75}$. By Lemma 1 and Proposition 6 we may suppose without loss of generality that $\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3=(T_1,T_2,T_3)\mathcal{T}$, where $V(T_i)=\{v_{1,i},\ v_{2,i},\ v_{1,i+1},$

$v_{2,i+1}\}$, $i=1,2$, and $V(T_3)=\{v_{1,1}, v_{2,1}, v_{1,3}, v_{2,3}\}$. Now $\mathcal{T}((v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,3}, v_{1,1}), (v_{1,1}, v_{2,2}, v_{2,3}, v_{2,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1}))$ is a good G_1-realisation of $\mathrm{nd}(L)$.

(32312212) If $l_r = l_1+1$, then $f_{l_1+2} = f_{l_1+3} = 0$.

(323122121) If $l_{p-1} = l_1+1$, then $2e_n+2 \le (p-r)(l_1+1)+3$, $p-r \ge \frac{2e_n-1}{l_1+1} \ge \lfloor \frac{149}{8} \rfloor = 18$ and $l_{r+1} = l_{r+2} = l_1+1$.

(3231221211) If $l_2 = l_1$, then $\sum_{i=3}^{r+2} l_i = 2e_n \to 3.1$.

(3231221212) If $l_2 = l_1+1$, then $2e_n - 2 = r(l_1+1) - 1$ and $2e_n \equiv 1 \pmod{l_1+1}$ in contradiction with (H1)–(H6).

(323122122) If $l_{p-1} = l_1+4$, then $l_1 \le 4$.

(3231221221) If $l_1 = 3$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r]-I} l_i + l_{p-1} + l_p = 2e_n \to 3.1$.

(3231221222) If $l_1 = 4$, then $4f_4 + 5f_5 = 2e_n - 2 \equiv 0 \pmod 2$, hence $f_5 \ge 2$, $\sum_{i=1}^{r-2} l_i + l_p = 2e_n - 4$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 4 \to 4.1$.

(32312213) If $l_r = l_1+2$, then $l_j = l_1+3$ for any $j \in [r+1,p]$, $l_i \in \{l_1, l_1+2\}$ for any $i \in [1,r]$, $2e_n+2 \equiv 0 \pmod{l_1+3}$ and, since $l_1 \le 5$, from (H1) it follows that $l_1 \in [4,5]$. We have also $2e_n+2 = (p-r)(l_1+3)$, hence $l_{r+1} = l_{r+2} = l_1+3$.

(323122131) If $l_1 = 4$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$, hence $\sum_{i\in[1,r+2]-I} l_i = 2e_n \to 3.1$.

(323122132) If $l_1 = 5$, then $n = 0$ and $5f_5 + 7f_7 = 2e_0 - 2 = 148$.

(3231221321) If $f_5 \ge 2$, then $\sum_{i=3}^{r+1} l_i = 2e_0 - 4 = l_1 + l_2 + \sum_{i=r+2}^{p-1} l_i \to 4.1$.

(3231221322) If $f_7 \ge 2$, then $\sum_{i=1}^{r-2} l_i + l_{r+1} + l_{r+2} = 2e_0 \to 3.1$.

(32312214) $l_r = l_1+3$

(323122141) If $l_{p-1} = l_1+3$, then $l_i \in \{l_1, l_1+2, l_1+3\}$ for any $i \in [1,r]$, there is $m \in [0,1]$ such that $2e_n+2 \equiv m \pmod{l_1+3}$, $l_1 \in [4,5]$ and $l_{r+1} = l_{r+2} = l_1+3$.

(3231221411) $l_1 = 4$

(32312214111) If $f_4 \ge 3$ or $f_6 \ge 2$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r+2]-I} l_i = 2e_n \to 3.1$.

(32312214112) $f_4 \le 2 \wedge f_6 \le 1$

(323122141121) If $l_p = 7$, then $\mathrm{nd}(L) = (4,6)(7)^{2r-2}$, $14r-4 = 4e_n$, hence r is even, $r = 2s$ and $7(s-1)+6 = e_n$. By $S(G_n)$ (see the row 2 of Table 1) and Proposition 6 there are good G_n-realisations $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of $(3)^3(4)(7)^{s-2}$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(3,4,6)(7)^{s-2}$ such that $v_{i,2} \in V(T_i^1)$, $i = 1,2,3$, and $v_{1,1} \in V(T_4^1) \cap V(T_1^2)$. As $V(\varphi_n^1(T_4^1)) \cap V(\varphi_n^2(T_1^2)) \supseteq \{v_{1,1}^1\}$, there is $T \in \varphi_n^1(T_4^1) + \varphi_n^2(T_1^2)$ and $[\prod_{i=1}^3 (\varphi_n^1(T_i^1))](\varphi_n^2(T_2^2), \varphi_n^2(T_3^2), T)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good H_n-realisation of $(3)^3(4,6)(7)^{2s-3}$ with $v_{i,2}^1 \in V(\varphi_n^1(T_i^1))$, $i = 1,2,3$. By $S(H_n)$ (the row 7 of Table 2) there is a good H_n-realisation $(T_1^3, T_2^3, T_3^3)\mathcal{T}^3$ of $(4)^3(7)^{2s-2}$ such that $v_{i,2}^1 \in V(T_i^3)$, $i = 1,2,3$. Since $V(\psi_n^1(\varphi_n^1(T_i^1))) \cap$

$V(\psi_n^2(T_i^3)) \supseteq \{v_{i,2}^{1,1}\}$, there is a trail $T_i \in \psi_n^1(\varphi_n^1(T_i^1)) + \psi_n^2(T_i^3)$, $i = 1, 2, 3$, and $(\psi_n^1(\varphi_n^2(T_2^2)), \psi_n^1(\varphi_n^2(T_3^2)), T_1, T_2, T_3, \psi_n^1(T))\psi_n^1(\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2))\psi_n^2(\mathcal{T}^3)$ is a good G_{n+1}-realisation of $\mathrm{nd}(L)$.

(323122141122) If $l_p = 8$, then $\mathrm{nd}(L) = (4)(7)^{2r-2}(8)$, $14r-2 = 4e_n$, hence r is odd, $r = 2s+1$, $3 =: l_p^1 \leq l_p - 3$, $3 =: l_2^1 \leq l_2 - 3$, $((3)_4, (3)_4), ((5)_4, (4)_4) \notin P$ and $3 + 3 + 2s \cdot 7 = 2e_n \to 4.2$.

(3231221412) If $l_1 = 5$, then $n = 0$, $l_p = 8$, $2e_0 - 2 = 148 \equiv 5f_5 + 7f_7 \pmod 2$ and $f_5 + f_7 \geq 2$.

(32312214121) If $f_5 \geq 2$, then $\sum_{i=3}^{r+1} l_i = 2e_0 - 4 = l_1 + l_2 + \sum_{i=r+2}^{p-1} l_i \to 4.1$.

(32312214122) If $f_7 \geq 2$, and $l_j = l_{j+1} = 7$ for some $j \in [2, r-2]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+2} l_i = 2e_0 \to 3.1$.

(32312214123) If $f_5 = f_7 = 1$, then $\mathrm{nd}(L) = (5, 7)(8)^{36}$. By $S(G_0)$ and Propositions 6 and 16.1 there are good G_0-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)(4)^2(8)^8$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(4)^2(3)(8)^8$ such that $v_{i,1} \in V(T_i^1) \cap V(T_i^2)$, $i = 1, 2$. As $V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_i^2)) \supseteq \{v_{i,1}^1\}$, there is a trail $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_i^2)$, $i = 1, 2$, $(\varphi_n^2(T_3^2), \varphi_n^1(T_3^1), T_1, T_2)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $(3, 4)$-global H_0-realisation of $(3, 4, 7)(8)^{17}$. Since $3 =: l_3^1 \leq l_3 - 3$ and $3 \leq l_4^1 := 4 \leq l_4 - 3$, we are done by Claim 3.3.

(323122142) If $l_{p-1} = l_1 + 4$, then $l_1 \leq 4$ and $f_{l_1+2} = 0$.

(3231221421) If $l_1 = 3$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1,r]-I} l_i + l_{p-1} + l_p = 2e_n \to 3.1$.

(3231221422) If $l_1 = 4$, then $\sum_{i=2}^{r-1} l_i + l_p = 2e_n - 5$ and $l_1 + \sum_{i=r}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(32312215) If $l_r = l_1 + 4$, then $2e_n + 2 \equiv 0 \pmod{l_1 + 4}$, $2e_n - 2 \equiv l_1 \pmod{l_1 + 4}$, $f_{l_1+2} = 0$ and $l_1 \leq 4$.

(323122151) $l_1 = 3$

(3231221511) If $f_3 \geq 4$, $f_4 \geq 3$ or $f_6 \geq 2$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1,r+2]-I} l_i = 2e_n \to 3.1$.

(3231221512) $f_3 \leq 3 \wedge f_4 \leq 2 \wedge f_6 \leq 1$

(32312215121) If $f_4 = 2$ and $l_j = l_{j+1} = 4$ for some $j \in [2, r-2]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+1} l_i = 2e_n - 3$ and $l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = 2e_n - 4 \to 4.1$.

(32312215122) $f_4 = 1$

(323122151221) If $\mathrm{nd}(L) = (3)^2(4)(7)^{\frac{4e_n-10}{7}}$, by $S(H_n)$ (the rows 1 and 7 of Table 2) there are good H_n-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)^4(7)^{\frac{2e_n-12}{7}}$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$ such that $V(T_i^1) \cap V(T_i^2) \cap V_{n,2}^1 \supseteq \{v_{i,2}^1\}$, $i = 1, 2, 3 \to 3.2$ (with $i_1 := 4$, $i_2 := 5$, $I^1 := [1, 2] \cup [6, r+2]$, $I^2 := \{3\} \cup [r+3, 2r-1]$ and $l_{i_k}^1 := 3$, $l_{i_k}^2 := 4$, $k = 1, 2$).

(323122151222) If $\mathrm{nd}(L) = (4, 6)(7)^{\frac{4e_n-10}{7}}$, by $S(H_n)$ (the rows 4 and 7 of Table 2) there are good H_n-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(6)(7)^{\frac{2e_n-12}{7}}$

and $(T_1^2,T_2^2)\mathcal{T}^2$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$ such that $V(T_i^1)\cap V(T_i^2)\cap V_{n,2}^1 \supseteq \{v_{i,2}^1\}$, $i=1,2 \to 3.2$.

(32312215123) If $f_4=0$, then $\mathrm{nd}(L)=(3)(7)^{\frac{4e_n-3}{7}}$ and we can proceed analogously as in (323122151221) (with $i_1:=2$, $i_2:=3$, $i_3:=4$, $I^1:=\{1\}\cup[5,r+2]$, $I^2:=[r+3,2r]$ and $l_{i_k}^1:=3$, $l_{i_k}^2:=4$, $k=1,2,3$).

(3231222) If $\varepsilon=-1$, then $l_p\le l_1+3$.

(32312221) If $l_r=l_1$, then $2e_n-1\equiv 0 \pmod{l_1}$ in contradiction with (H1)–(H6).

(32312222) If $l_r=l_1+1$, then $l_j=l_1+3$ for any $j\in[r+1,p]$, $2e_n+1\equiv 0 \pmod{l_1+3}$, $l_1\le 5$ and, because of (H1)–(H6), $l_1=4$. Since $4f_4+5f_5=2e_n-1\ge 149$, there is $I\subseteq[1,r]$ such that $\sum_{i\in I} l_i=20$ and $\sum_{i\in[1,r+3]-I} l_i=2e_n \to 3.1$.

(32312223) If $l_r=l_1+2$, then $l_j=l_1+2$ for any $j\in[r+1,p]$, hence $2e_n+1\equiv 0 \pmod{l_1+2}$, $l_1\le 6$ and, because of (H1)–(H6), $l_1=5$ and $2e_n-1\equiv 5 \pmod 7$. Moreover, $l_i\in\{5,7\}$ for any $i\in[1,r]$.

(323122231) If $f_5\ge 4$, then $\sum_{i=5}^{r+3} l_i=2e_n\to 3.1$.

(323122232) If $f_5\le 3$, then $\mathrm{nd}(L)=(5)(7)^{\frac{4e_n-5}{7}}$, $l_i^1:=3\le l_i-3$, $i=2,3$, by $S(G_n)$ there is a good G_n-realisation $\mathcal{T}^1$ of $(3)(7)^{\frac{e_n-3}{7}}$ and $\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^1)$ is a good $(3,3)$-global H_n-realisation of $(3)^2(7)^{\frac{2e_n-6}{7}}\to 3.3$.

(32312224) If $l_r=l_1+3$, then $2e_n+1\equiv 0 \pmod{l_1+3}$, $l_1\le 5$, from (H1)–(H6) it follows that $l_1=4$, hence $f_6=0$ and $2e_n-1\equiv 5 \pmod 7$.

(323122241) $\exists j\in[2,r-1]\ l_j=5$

(3231222411) If $f_4\ge 2$, then $\sum_{i=3}^{j-1} l_i+\sum_{i=j+1}^{r+2} l_i=2e_n\to 3.1$.

(3231222412) $f_4=1$

(32312224121) If $f_5\ge 4$, then $l_1+\sum_{i=6}^{r+3} l_i=2e_n\to 3.1$.

(32312224122) If $f_5\le 3$, then $\mathrm{nd}(L)=(4)(5)^3(7)^{\frac{4e_n-19}{7}}$, $l_1+l_2+\sum_{i=5}^{r+1} l_i=2e_n-4$ and $l_3+l_4+\sum_{i=r+2}^{p-1} l_i=2e_n-3\to 4.1$.

(323122242) $f_5=0$

(3231222421) If $f_4\ge 5$, then $\sum_{i=6}^{r+3} l_i=2e_n\to 3.1$.

(3231222422) If $f_4\le 4$, then $\mathrm{nd}(L)=(4)^3(7)^{\frac{4e_n-12}{7}}$. By $S(H_n)$ (the rows 4 and 6 of Table 2) and Proposition 19.1 there are good H_n-realisations $(T_1^1,T_2^1)\mathcal{T}^1$ of $(3)^2(7)^{\frac{2e_n-6}{7}}$ and $[\prod_{i=1}^5(T_i)]\mathcal{T}^2$ of $(4)^5(7)^{\frac{2e_n-20}{7}}$ such that $v_{i,3}^2\in V(T_i^1)$, $i=1,2$, $v_{i,2}^1\in V(T_i^2)$, $i=1,2,3$, and $v_{i,3}^1\in V(T_{3+i}^2)$, $i=1,2$. As $V(\psi_n^1(T_i^1))\cap V(\psi_n^2(T_{3+i}^2))\supseteq\{v_{i,3}^{2,1}\}=\{v_{i,3}^{1,2}\}$, there is a trail $T_i\in\psi_n^1(T_i^1)+\psi_n^2(T_{3+i}^2)$, $i=1,2$, and $\psi_n^2((T_1^2,T_2^2,T_3^2)\mathcal{T}^2)(T_1,T_2)\psi_n^1(\mathcal{T}^1)$ is a good G_{n+1}-realisation of $\mathrm{nd}(L)$ with $v_{i,2}^{1,2}\in\psi_n^2(T_i^2)$, $i=1,2,3$. Since $p_{(5\cdot 2^{n+1})3}(v_{i,2}^{1,2})=2$, $i=1,2,3$, by Proposition 6 there is a good G_{n+1}-realisation $(\bar{T}_1,\bar{T}_2,\bar{T}_3)\bar{\mathcal{T}}$ of $\mathrm{nd}(L)$ such that $v_{i,1}\in V(\bar{T}_i)$, $i=1,2,3$ (satisfying the conditions of the row 5 of Table 1).

(3232) If $\varepsilon \in [1,2]$, then $l_r = \sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1} l_i \geq 2e_n + \varepsilon - (2e_n - 3) = 3 + \varepsilon$. With $l := \min(l_i : i \in [1,r], l_i \geq 3 + \varepsilon)$ we have $3 + \varepsilon \leq l \leq l_r$.

(32321) If $l_p \geq l + 3 - \varepsilon$, let $j \in [1,r]$ be such that $l_j = l$. Then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i = 2e_n + \varepsilon - l \leq 2e_n + \varepsilon - (3 + \varepsilon)$ and $l_j + \sum_{i=r+1}^{p-1} l_i = l + (2e_n - \varepsilon) - l_p \leq 2e_n - 3 \to 4.1$.

(32322) $\forall i \in [1,p]\ l_i \in [3, 2+\varepsilon] \cup [l, l+2-\varepsilon]$

(323221) $\varepsilon = 1$

(3232211) If $l_r = l + 1$, then $2e_n - 1 \equiv 0 \pmod{l+1}$ in contradiction with $l \leq 7$ and (H1)–(H6).

(3232212) $l_r = l$

(32322121) If $l_{p-1} = l$, then $2e_n - 1 \equiv k \pmod{l}$ for some $k \in [0,1]$ and then from (H1)–(H6) it follows that $l = 4$, $n = 0$, $2e_0 + 1 \equiv 3 \pmod 4$ and $f_3 \geq 1$.

(323221211) If $f_3 \geq 3$, then $\sum_{i=4}^{r+2} l_i = 2e_0 \to 3.1$.

(323221212) If $f_3 \leq 2$, then $\mathrm{nd}(L) = (3)(4)^{73}(5)$. If $\mathcal{T}$ is the sequence of closed trails from (32312211322), then $((v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1}), (v_{1,3}, v_{2,1}, v_{2,3}, v_{2,2}, v_{1,3}))\ \mathcal{T}\ ((v_{1,1}, v_{2,2}, v_{2,1}, v_{1,2}, v_{2,3}, v_{1,1}))$ is a good G_1-realisation of $\mathrm{nd}(L)$.

(32322122) $l_{p-1} = l + 1$

(323221221) If $f_3 = 0$, then $2e_n + 1 = rl$, hence from $l \leq 7$ and (H1)–(H4) it follows that $l = 7$, $\sum_{i=1}^{r-2} l_i + l_p = 2e_n - 5$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(323221222) $f_3 \geq 1$

(3232212221) $l = 4$

(32322122211) If $f_3 = 1$, then $2e_n + 1 \equiv 3 \pmod 4$, hence from (H3) it follows that $n = 0$ and $2e_0 - 1 \equiv 1 \pmod 4$. Further, $2e_0 - 1 \equiv f_5 \pmod 4$, $f_5 \geq 5$ (as $l_{p-1} = l_p = 5$) and $\sum_{i=1}^{r-4} l_i + \sum_{i=p-2}^{p} l_i = 2e_0 \to 3.1$.

(32322122212) If $f_3 \geq 2$, then $\sum_{i=3}^{r} l_i + l_p = 2e_n \to 3.1$.

(3232212222) $l \in [5,7]$

(32322122221) If $l_{r-1} = 3$, then $2e_n + 1 \equiv l \pmod 3$, hence, by (H1), $l = 7$ and $\sum_{i=4}^{r} l_i + l_p = 2e_n \to 3.1$.

(32322122222) If $l_{r-1} = l$, then $\sum_{i=1}^{r-2} l_i + l_{p-1} = (2e_n + 1 - 2l) + (l+1) \leq 2e_n - 3$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(323222) If $\varepsilon = 2$, then $2e_n - 2 \equiv 0 \pmod{l}$ in contradiction with (H1) and (H4).

Now, it remains to be proved that there are good G_{n+1}-realisations of four sequences from $\mathrm{Sct}(G_{n+1})$ according to the row t of Table 1, $t \in [1,4]$.

$t = 1$: By $S(H_n)$ (the row 4 of Table 2) and Proposition 19.1 there are good H_n-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ and $(T_1^2)\mathcal{T}^2$ of $(3)^2(7)^{\frac{2e_n-6}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $i = 1,2$, and $v_{3,2}^1 \in V(T_1^2)$. If $\bar{T}_i := \psi_n^1(T_i^1)$, $i = 1,2$ and $\bar{T}_3 := \psi_n^2(T_1^2)$, then $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(4)^3(7)^{\frac{4e_n-12}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1,2,3$. As $p_{(5\cdot 2^{n+1})3}(v_{i,2}^{1,1}) = 2$, $i = 1,2,3$, it suffices to use Proposition 6.

$t = 2$: By $S(H_n)$ (the row 2 of Table 2) and Proposition 19.1 there are good H_n-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(4)(7)^{\frac{2e_n-10}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{2e_n-3}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $i = 1, 2$, and $v_{3,2}^1 \in V(T_1^2)$. If $\bar{T}_i := \psi_n^1(T_i^1)$, $i = 1, 2$, and $\bar{T}_3 := \psi_n^2(T_1^2)$, then $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(3)^3(4)(7)^{\frac{4e_n-13}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1, 2, 3$. As for $t = 1$, employ Proposition 6.

$t = 3$: By $S(H_n)$ (the rows 7 and 8 of Table 2) and Proposition 19.1 there are good H_n-realisations $(T_1^j, T_2^j, T_3^j)\mathcal{T}^j$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$, $j = 1, 2$, such that $v_{i,3}^1 \in V(T_i^1)$, $i = 1, 2, 3$, $v_{i,2}^1 \in V(T_i^2)$, $i = 1, 2$, and $v_{1,1}^1 \in V(T_3^2)$. If $\bar{T}_i^j := \psi_n^j(T_i^1)$, $i = 1, 2, 3$, $j = 1, 2$, then $[\prod_{i=1}^3(\bar{T}_i^1)][\prod_{i=1}^3(\bar{T}_i^2)]\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(4)^6(7)^{\frac{4e_n-24}{7}}$ with $v_{i,3}^{1,1} \in V(\bar{T}_i^1)$, $i = 1, 2, 3$, $v_{i,2}^{1,2} \in V(\bar{T}_i^2)$, $i = 1, 2$, and $v_{1,1}^{1,2} \in V(\bar{T}_3^2)$. Since $p_{(5\cdot 2^{n+1})3}(v_{i,3}^{1,1}) = 3$, $i = 1, 2, 3$, $p_{(5\cdot 2^n)3}(v_{i,2}^{1,2}) = 2$, $i = 1, 2$, and $p_{(5\cdot 2^{n+1})3}(v_{1,1}^{1,2}) = 1$, we are done again due to Proposition 6.

$t = 4$: By $S(H_n)$ (the row 5 of Table 2) and Proposition 19.1 there are good H_n-realisations $[\prod_{i=1}^6(T_i^1)]\mathcal{T}^1$ of $(4)^6(7)^{\frac{2e_n-24}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{2e_n-3}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $v_{i,3}^2 \in V(T_{3+i}^1)$, $i = 1, 2, 3$, and $v_{3,3}^1 \in V(T_1^2)$. As $V(\psi_n^1(T_6^1)) \cap V(\psi_n^2(T_1^2)) \supseteq \{v_{3,3}^{2,1}\} = \{v_{3,3}^{1,2}\}$, there is a trail $\bar{T} \in \psi_n^1(T_6^1) + \psi_n^2(T_1^2)$ and $[\prod_{i=1}^5(\psi_n^1(T_i^1))](\bar{T})\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good G_{n+1}-realisation of $(4)^5(7)^{\frac{4e_n-20}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1, 2, 3$, and $v_{i,3}^{2,1} \in V(\bar{T}_{3+i})$, $i = 1, 2$. Because of $p_{(5\cdot 2^{n+1})3}(v_{i,2}^{1,1}) = 2$, $i = 1, 2, 3$, and $p_{(5\cdot 2^n)3}(v_{i,3}^{2,1}) = 3$, $i = 1, 2$, we are done with help of Proposition 6. $\blacksquare$

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