P. J. ŠAFÁRIK UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
Jesenná 5, 041 54 Košice, Slovakia

**I. Mojsej and A. Tartaľová**

# On bounded nonoscillatory solutions of third-order nonlinear differential equations

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# On bounded nonoscillatory solutions of third-order nonlinear differential equations

Ivan Mojsej[1] and Alena Tartaľová[2]

[1] *Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovak Republic*
*E-mail address*: ivan.mojsej@upjs.sk

[2] *Department of Applied Mathematics and Business Informatics, Faculty of Economics, Technical University, Nemcovej 32, 040 01 Košice, Slovak Republic*
*E-mail address*: alena.tartalova@tuke.sk

**Abstract**

This paper is concerned with the asymptotic behavior of solutions of nonlinear differential equations of the third-order with quasiderivatives. We give the necessary and sufficient conditions guaranteeing the existence of bounded nonoscillatory solutions. Sufficient conditions are proved via a topological approach based on the Banach fixed point theorem.

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## 1 Introduction

Consider the third-order nonlinear differential equations with quasiderivatives of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(t)) = 0, \quad t \geq a. \tag{N}$$

Throughout the paper, we always assume that

$$r, p, q \in C([a,\infty),\mathbb{R}),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [a,\infty),$$

$$f \in C(\mathbb{R},\mathbb{R}), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0.$$

For the sake of brevity, we put

$$x^{[0]} = x,\ x^{[1]} = \frac{1}{r}x',\ x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)',\ x^{[3]} = \left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \left(x^{[2]}\right)'.$$

The functions $x^{[i]}$, i=0, 1, 2, 3 we call the *quasiderivatives* of $x$.

By a *solution* of an equation of the form $(N)$ we mean a function $w : [a, \infty) \to \mathbb{R}$ such that quasiderivatives $w^{[i]}(t)$, $0 \leq i \leq 3$ exist and are continuous on the interval $[a, \infty)$ and it satisfies the equation $(N)$ for all $t \geq a$. A solution $w$ of equation $(N)$ is said to be *proper* if it is nontrivial in any neighbourhood of infinity, it means that satisfies the condition

$$\sup \{|w(s)| : t \leq s < \infty\} > 0 \quad \text{for any } t \geq a.$$

A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to $\infty$; otherwise it is said to be *nonoscillatory.* Furthermore, equation $(N)$ is called *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if all its nontrivial solutions are nonoscillatory.

Let $\mathcal{N}(N)$ denote the set of all proper nonoscillatory solutions of equation $(N)$. The set $\mathcal{N}(N)$ can be divided into the following four classes in the same way as in [3, 5, 14]:

$$\mathcal{N}_0 = \{x \in \mathcal{N}(N), \exists\, t_x \ : \ x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \geq t_x\}$$

$$\mathcal{N}_1 = \{x \in \mathcal{N}(N), \exists\, t_x \ : \ x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) < 0 \ \text{ for } t \geq t_x\}$$

$$\mathcal{N}_2 = \{x \in \mathcal{N}(N), \exists\, t_x \ : \ x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \geq t_x\}$$

$$\mathcal{N}_3 = \{x \in \mathcal{N}(N), \exists\, t_x \ : \ x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) < 0 \ \text{ for } t \geq t_x\}$$

The object of our interest are bounded nonoscillatory solutions of equation $(N)$ in the classes $\mathcal{N}_1$ and $\mathcal{N}_2$, i.e. the solutions that belong to the following two classes

$$\mathcal{N}_1^B = \left\{x \in \mathcal{N}_1 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\},$$

$$\mathcal{N}_2^B = \left\{x \in \mathcal{N}_2 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\}.$$

Various types of differential equations of the third-order has been subject of intensive studying in the literature. There are numerous results (see, e.g., [2, 3, 4, 5, 14]) devoted to the oscillatory and asymptotic behavior of equation $(N)$. Many other authors deal with the qualitative properties of solutions of differential equations of the third-order with deviating argument. Among the extensive literature on this topic, we mention here [7, 8, 9, 12, 13, 15, 16, 17].

Fixed point theorems are important tool in the oscillation and nonoscillation theory of ordinary differential equations. In particular, when one proves the existence of nonoscillatory solutions. Various interesting results on this subject and fairly comprehensive bibliography of the earlier work can be found in the books [1, 11]. In the sequel, we will need the following fixed point theorem.

**Theorem 1.1** *(Banach fixed point theorem) Any contraction mapping of a complete non-empty metric space $\mathcal{M}$ into $\mathcal{M}$ has a unique fixed point in $\mathcal{M}$.*

The aim of this paper is to investigate the existence and asymptotic behavior of some nonoscillatory solutions of equation $(N)$. We give the necessary and sufficient conditions for the existence of bounded nonoscillatory solutions in the classes $\mathcal{N}_1$ and $\mathcal{N}_2$. Our research is based on a study of the asymptotic properties of nonoscillatory solutions as well as on a topological approach via the Banach fixed point theorem. Presented results are interesting in themselves by virtue of their necessary and sufficient character. Moreover, they complement and extend some other results that have been stated in the papers [6] and [10], respectively. Several illustrative examples are also provided.

Finally, we introduce the following notation:

$$I(u_i) = \int_a^\infty u_i(t)\,dt, \qquad I(u_i, u_j) = \int_a^\infty u_i(t) \int_a^t u_j(s)\,ds\,dt, \quad i, j = 1, 2,$$

where $u_i$, $i = 1, 2$ are continuous positive functions on the interval $[a, \infty)$. For simplicity, we will sometimes write $u(\infty)$ instead of $\lim_{t\to\infty} u(t)$.

## 2 Main results

We begin our consideration with several results concerning the asymptotic behavior of solutions of equation $(N)$ in the class $\mathcal{N}_1^B$. The following theorem provides the sufficient conditions for the existence of those solutions.

**Theorem 2.1** *Assume that function $f$ satisfies Lipschitz condition on some interval $[c, d]$ where $c$, $d$ are constants such that $0 < c < d$. Let one of the following conditions be satisfied:*

*(a)* $\quad I(p, q) < \infty$ *and* $I(r) < \infty$,

*(b)* $\quad I(p, r) < \infty$ *and* $I(q) < \infty$.

*Then equation $(N)$ has a bounded solution $x$ in the class $\mathcal{N}_1$, i.e. $\mathcal{N}_1^B \neq \emptyset$.*

**Proof.** We prove the existence of a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_1$.

Suppose *(a)*. Let L denote Lipschitz constant of function $f$ on the interval $[c, d]$, let $K = \max\{f(u) : u \in [c, d]\}$ and $t_0 \geq a$ be such that

$$\left(\int_{t_0}^\infty p(k) \int_{t_0}^k q(l)\,dl\,dk\right)\left(\int_{t_0}^\infty r(s)\,ds\right) \leq \min\left\{\frac{d-c}{K}, \frac{1}{L+1}\right\}. \qquad (1)$$

Let us define the set

$$\Delta = \{u \in C([t_0, \infty), \mathbb{R}) : c \leq u(t) \leq d\},$$

where $C([t_0,\infty),\mathbb{R})$ will denote the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|\,, t \geq t_0\}$. Clearly, $\Delta$ is a non-empty closed subset of $C([t_0,\infty),\mathbb{R})$ and so $\Delta$ is a non-empty complete metric space. For every $u \in \Delta$ we consider a mapping $T_1 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_1 u)(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k) f(u(k))\, dk\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply to the mapping $T_1$ the Banach fixed point theorem (Theorem 1.1), it is sufficient to prove that $T_1$ maps $\Delta$ into itself and $T_1$ is a contraction mapping in $\Delta$.

By easy computation, we obtain

$$\int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\, d\tau \leq \left(\int_{t_0}^{\infty} r(\tau)\, d\tau\right)\left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\right). \tag{2}$$

$T_1$ maps $\Delta$ into $\Delta$. In fact, $x_u(t) \geq c$ and in view of (1) and (2), we get

$$\begin{aligned} x_u(t) &= c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k) f(u(k))\, dk\, ds\, d\tau \\ &\leq c + K \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\, d\tau \\ &\leq c + K \left(\int_{t_0}^{\infty} r(\tau)\, d\tau\right)\left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\right) \leq d. \end{aligned}$$

Now, let $u_1, u_2 \in \Delta$ and $t \geq t_0$. Then, taking into account the inequalities (1) and (2) and the fact that function $f$ satisfies Lipschitz condition on the interval $[c,d]$, we have the following

$$\begin{aligned} |(T_1u_1)(t) - (T_1u_2)(t)| &\leq \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k) |f(u_1(k)) - f(u_2(k))|\, dk\, ds\, d\tau \\ &\leq L \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k) |u_1(k) - u_2(k)|\, dk\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \int_{t}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \left(\int_{t_0}^{\infty} r(\tau)\, d\tau\right)\left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(k)\, dk\, ds\right) \\ &\leq \frac{L}{L+1}\|u_1 - u_2\| = Q\|u_1 - u_2\|. \end{aligned}$$

This immediately implies that for every $u_1, u_2 \in \Delta$

$$\|T_1u_1 - T_1u_2\| \leq Q\|u_1 - u_2\| \qquad \text{where } 0 < Q < 1.$$

Hence, we proved that $T_1$ is a contraction mapping in $\Delta$. Now, the Banach fixed point theorem yields the existence of the unique fixed point $x \in \Delta$ such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(k) f(x(k))\, dk\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = \int_{t}^{\infty} p(s) \int_{t_0}^{s} q(k) f(x(k))\, dk\, ds > 0$$

and

$$x^{[2]}(t) = -\int_{t_0}^{t} q(k) f(x(k))\, dk < 0,$$

it is clear that $x$ is a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_1$, i.e. $\mathcal{N}_1^B \neq \emptyset$.

Suppose *(b)*. Using similar arguments as in the case *(a)*, we are led to the conclusion that $\mathcal{N}_1^B \neq \emptyset$. Therefore, we omit it. This completes the proof. □

Theorem 2.1 is illustrated by the following example.

**Example 1** Let us consider the differential equation

$$\left( \left(t^2+1\right) \left( \left(t^2+1\right) x'(t) \right)' \right)' + \frac{8t}{\left(2t^2+1\right)^2} x^2(t) \operatorname{sgn} x(t) = 0, \quad t \geq 2. \tag{3}$$

This equation has the form $(N)$ where $r(t) = p(t) = \dfrac{1}{t^2+1}$, $q(t) = \dfrac{8t}{\left(2t^2+1\right)^2}$ and $f(u) = u^2 \operatorname{sgn} u$. Since function $f$ is Lipschitz on the interval $[1, 2]$ with the Lipschitz constant $L = 4$ and the integrals $I(p, r)$, $I(q)$ are convergent, Theorem 2.1 secures that equation (3) has a solution in the class $\mathcal{N}_1^B$. One such solution is the function $x(t) = \dfrac{2t^2+1}{t^2+1}$.

We also have the following result for the solutions in the class $\mathcal{N}_1^B$.

**Theorem 2.2** *If $I(p, q) = \infty$, then $\mathcal{N}_1^B = \emptyset$.*

**Proof.** Assume that $x \in \mathcal{N}_1^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. Let $x(\infty) = M_x < \infty$. As $x$ is a positive increasing function and $f$ is a continuous function on the interval $[x(T), M_x]$, there exists a positive constant m such that

$$m = \min \left\{ f(u) : u \in [x(T), M_x] \right\}. \tag{4}$$

Integrating equation $(N)$ twice in $[T, t]$, we obtain

$$x^{[1]}(t) = x^{[1]}(T) + x^{[2]}(T) \int_{T}^{t} p(s)\, ds - \int_{T}^{t} p(s) \int_{T}^{s} q(k) f(x(k))\, dk\, ds$$

and therefore

$$x^{[1]}(t) < x^{[1]}(T) - \int_T^t p(s) \int_T^s q(k) f(x(k))\, dk\, ds.$$

Using this inequality with (4), we have

$$x^{[1]}(t) < x^{[1]}(T) - m \int_T^t p(s) \int_T^s q(k)\, dk\, ds,$$

which gives a contradiction as $t \to \infty$, because function $x^{[1]}(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way.

□

As a consequence of Theorems 2.1 and 2.2, we get the following result.

**Corollary 2.1** *Let function $f$ satisfy Lipschitz condition on some interval $[c, d]$ where $c$, $d$ are constants such that $0 < c < d$ and $I(r) < \infty$. Then a necessary and sufficient condition for equation $(N)$ to have a solution $x$ in the class $\mathcal{N}_1^B$ is that $I(p, q) < \infty$.*

Now, we are interested in the existence and asymptotic properties of solutions of equation $(N)$ in the class $\mathcal{N}_2^B$. We state here the sufficient and necessary conditions that guarantee the existence of those solutions. The following theorems hold.

**Theorem 2.3** *Assume that function $f$ satisfies Lipschitz condition on some interval $[c, d]$ where $c$, $d$ are constants such that $0 < c < d$. Let one of the following conditions be satisfied:*

*(a)* $I(r, p) < \infty$ *and* $I(q) < \infty$,

*(b)* $I(q, p) < \infty$ *and* $I(r) < \infty$.

*Then equation $(N)$ has a bounded solution $x$ in the class $\mathcal{N}_2$, i.e $\mathcal{N}_2^B \neq \emptyset$.*

**Proof.** We prove the existence of a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_2$.

Suppose *(a)*. Let L denote Lipschitz constant of function $f$ on the interval $[c, d]$, let $K = \max\{f(u) : u \in [c, d]\}$ and $t_0 \geq a$ be such that

$$\left( \int_{t_0}^{\infty} r(k) \int_{t_0}^{k} p(l)\, dl\, dk \right) \left( \int_{t_0}^{\infty} q(s)\, ds \right) \leq \min\left\{ \frac{1}{L+1}, \frac{d-c}{K} \right\}. \tag{5}$$

Let us define the set

$$\Delta = \{u \in C([t_0, \infty), \mathbb{R}) : c \leq u(t) \leq d\},$$

where $C([t_0,\infty),\mathbb{R})$ will denote the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|, t \geq t_0\}$. Clearly, $\Delta$ is a non-empty closed subset of $C([t_0,\infty),\mathbb{R})$ and so $\Delta$ is a non-empty complete metric space. For every $u \in \Delta$ we consider a mapping $T_2 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_2u)(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k) f(u(k))\, dk\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply to the mapping $T_2$ the Banach fixed point theorem (Theorem 1.1), it is sufficient to prove that $T_2$ maps $\Delta$ into itself and $T_2$ is a contraction mapping in $\Delta$.

It is easy to verify that the following inequality holds:

$$\int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k)\, dk\, ds\, d\tau \leq \left(\int_{t_0}^{\infty} q(k)\, dk\right) \left(\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau\right) \tag{6}$$

$T_2$ maps $\Delta$ into $\Delta$. Really, $x_u(t) \geq c$ and in view of (5) and (6), we have the following

$$\begin{aligned} x_u(t) &= c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k) f(u(k))\, dk\, ds\, d\tau \\ &\leq c + K \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k)\, dk\, ds\, d\tau \\ &\leq c + K \left(\int_{t_0}^{\infty} q(k)\, dk\right) \left(\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau\right) \leq d. \end{aligned}$$

Now, let $u_1, u_2 \in \Delta$ and $t \geq t_0$. Then, the inequalities (5) and (6) and the fact that function $f$ satisfies Lipschitz condition on the interval $[c,d]$ yield

$$\begin{aligned} |(T_2u_1)(t) - (T_2u_2)(t)| &\leq \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k) |f(u_1(k)) - f(u_2(k))|\, dk\, ds\, d\tau \\ &\leq L \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k) |u_1(k) - u_2(k)|\, dk\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k)\, dk\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \left(\int_{t_0}^{\infty} q(k)\, dk\right) \left(\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau\right) \\ &\leq \frac{L}{L+1}\|u_1 - u_2\| = Q_1\|u_1 - u_2\|. \end{aligned}$$

These inequalities immediately imply that for every $u_1, u_2 \in \Delta$

$$\|T_2u_1 - T_2u_2\| \leq Q_1\|u_1 - u_2\| \qquad \text{where } 0 < Q_1 < 1.$$

Thus, we proved that $T_2$ is a contraction mapping in $\Delta$. Consequently, according to the Banach theorem there exists the unique fixed point $x \in \Delta$ such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(k) f(x(k))\, dk\, ds\, d\tau, \quad t \geq t_0.$$

It is clear that $x$ is a positive bounded solution of equation $(N)$ in the class $\mathcal{N}_2$, i.e. $\mathcal{N}_2^B \neq \emptyset$.

Suppose *(b)*. The proof is the same as in the case *(a)* except for some minor changes. Therefore, we omit it. The proof is now complete.

□

The following example shows the meaning of Theorem 2.3.
**Example 2** We consider the differential equation

$$\left(\left(t^2+1\right)\left(t^2 x'(t)\right)'\right)' + \frac{2\left(t^2-1\right)}{\left(t^2+1\right)^2 \operatorname{arctg}^3 t}\, x^3(t) = 0, \quad t \geq 2. \tag{7}$$

It is easy to verify that the assumptions of Theorem 2.3 are fulfilled and so equation (7) has a solution in the class $\mathcal{N}_2^B$. Really, one such solution is the function $x(t) = \operatorname{arctg} t$.

**Theorem 2.4** *If $I(q) = \infty$, then $\mathcal{N}_2^B = \emptyset$.*

**Proof.** Assume that $x \in \mathcal{N}_2^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Let $x(\infty) = M_x < \infty$. As $x$ is a positive increasing function and $f$ is a continuous function on the interval $[x(T), M_x]$, there exists a positive constant m such that

$$m = \min\left\{f(u) : u \in [x(T), M_x]\right\}. \tag{8}$$

By integrating equation $(N)$ in the interval $[T, t]$, we get

$$x^{[2]}(t) = x^{[2]}(T) - \int_T^t q(s) f(x(s))\, ds.$$

This equality with (8) yields that

$$x^{[2]}(t) < x^{[2]}(T) - m \int_T^t q(s)\, ds,$$

which gives a contradiction as $t \to \infty$, because function $x^{[2]}(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way.

□

From Theorems 2.3 and 2.4, one gets immediately the following result.

**Corollary 2.2** *Let function $f$ satisfy Lipschitz condition on some interval $[c, d]$ where c, d are constants such that $0 < c < d$ and $I(r, p) < \infty$. Then a necessary and sufficient condition for equation $(N)$ to have a solution $x$ in the class $\mathcal{N}_2^B$ is that $I(q) < \infty$.*

**Remark 1** Theorems 2.1, 2.3 and Corollaries 2.1, 2.2 are still valid if instead of the assumption that function $f$ satisfies Lipschitz condition on an interval $[c, d]$ where c, d are constants such that $0 < c < d$, we will require that function $f$ satisfies Lipschitz condition on an interval $[d_1, c_1]$ where $c_1$, $d_1$ are constants such that $d_1 < c_1 < 0$. Under this assumption, in Theorem 2.1 (Theorem 2.3), we can prove the existence of a negative bounded solution of equation $(N)$ in the class $\mathcal{N}_1$ ($\mathcal{N}_2$) by using similar arguments.

**Remark 2** Similar investigation of the asymptotic behavior of the second order nonlinear differential equations of the form

$$(r(t)x'(t))' + q(t)f(x(t)) = 0, \quad t \geq a$$

has been given in the paper [6]. Our results also extend some other ones published in [10]. Finally, we refer the reader to the books [1, 7, 11] and to the references contained therein for other interesting results on this topic.

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