Pareto optimality in the kidney exchange game*

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Abstract

We study Pareto optimality and weak Pareto optimality in the kidney exchange game. We show that the problems to decide whether a given permutation is not Pareto optimal or whether it is not weakly Pareto optimal are NP-complete in the cases with strict as well as dichotomous preferences.

1 Introduction

The most effective currently known treatment for endstage renal failure is a kidney transplantation. Lack of cadaveric kidneys together with better survival rates of live-donor kidneys lead to an increase of living donations. Usually, a donor is a genetic or an emotional relative of a patient. Yet not rarely the kidney cannot be transplanted to the intended recipient for immunological reasons and the willing donor is lost.

Already in 1986, Rapaport [15] suggested a creation of an international living-related donor and recipient exchange and nowadays, organized systematic kidney exchange programs exist in Romania [12], the Netherlands [11], the USA [10, 13, 14], United Kingdom [8], Middle East and South Korea [6]. Moreover, there is a discussion whether there is a need to create such a program in Belgium [1].

In this paper we consider only direct exchanges, i. e. a donor donates only when his intended recipient receives a kidney from another living donor. In this case, the pool of patient-donor pairs is partitioned into disjoint cycles that represent the donations, e. g. cycle $A-B-C-A$ means that patient A receives the kidney of donor B, patient B the kidney of donor C and patient C of donor A.

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In addition to various medical, ethical or legal problems, kidney exchange posed a lot of questions connected with the formulation of a suitable model, choice of optimality criteria etc, which gave rise to a handful of works studying for various possible algorithms their theoretical efficiency as well as their practical performance on simulated or real data [2, 3, 5, 16, 17, 18, 19, 20]. Here we follow the approach suggested in [3], where the kidney exchange problem was modelled as a cooperative game between patient-donor pairs. As each patient wishes to receive for him the medically most suitable kidney and to be on a transplantation cycle that is as short as possible (since all the transplantations on a cycle should be executed simultaneously [6, 8]), patients' preferences in [3] combine these two criteria.

The resulting allocation should fulfill some welfare criteria. In [16] Pareto optimality was suggested. On the other hand, [5] studies a stronger concept, the core of the game. However, various questions about core turned out to be computationally difficult [2, 5]. In this paper we show that this is also the case with Pareto optimality in the kidney exchange game.

In Section 2 the kidney exchange game is formulated together with the studied solution concepts, in Section 3 we prove NP-completeness of Pareto optimality related problems and Section 4 concludes.

2 Definitions

The kidney exchange problem can be represented by a finite simple digraph $G = (V, A)$ where each vertex represents a patient - donor pair (in general, a patient can have several donors, but this assumption can easily be dealt with). Loops are not allowed in G as they correspond to patients who have their own compatible donors and these usually do not take part in a kidney exchange program. An arc $(v, u) \in A$ if patient v can accept the kidney from donor u. Moreover, we suppose that each vertex v has a linear ordering $\preceq_v$ of the vertices adjacent from v, meaning that patient v orders compatible kidneys from the medically most suitable one to the worst one.

If $u \preceq_v w$, we say that v prefers u to w. If $u \preceq_v w$ and $w \preceq_v u$, then v is *indifferent* between u and w, written $u \sim_v w$. If $u \preceq_v w$ but not $w \preceq_v u$, then v *strictly prefers* u to w, written $u \prec_v w$.

There are two extreme cases – the case with *strict preferences* where no indifferences in the preferences of vertices are allowed, and *dichotomous preferences* where each vertex is indifferent between all endvertices adjacent from it. Under strict preferences, $f(v)$ and $s(v)$ denote the first and the second best vertex for $v \in V$ (if they exist).

Definition 1 *An instance of* the kidney exchange game *(KE for short) is a triple* $\Gamma = (V, G, \mathcal{O})$*, where* V *is the set of patient-donor pairs (players),* $G = (V, A)$ *is a digraph and* $\mathcal{O} = \{\preceq_v, v \in V\}$.

Definition 2 *A* solution *of a KE game* $\Gamma = (V, G, \mathcal{O})$ *is a permutation* π *of* V *such that* $v \neq \pi(v)$ *implies* $(v, \pi(v)) \in A$ *for each* $v \in V$.

We say, that v is *uncovered* by π iff $\pi(v) = v$. Otherwise, v is *covered* by permutation π. In what follows, $C^\pi(v)$ denotes the cycle of π containing v and we represent a permutation by its cycles.

Further we will define an extension of preferences from vertices to preferences over permutations. These preferences incorporate both the suitability of kidneys as well as cycle lengths.

Definition 3 *Let* $\Gamma = (V, G, \mathcal{O})$ *be a KE game,* $v \in V$ *a player,* π *and* σ *permutations of* V. *We say that player* v prefers *permutation* π *to permutation* σ, *written* $\pi \leq_v \sigma$, *if:*

(i) $\pi(v) \prec_v \sigma(v)$ *or*

(ii) $\pi(v) \sim_v \sigma(v)$ *and* $|C^\pi(v)| \leq |C^\sigma(v)|$.

Player v strictly prefers *permutation* π *to permutation* σ, *written* $\pi <_v \sigma$, *if* $\pi \leq_v \sigma$ *but not* $\sigma \leq_v \pi$.

Note that in the dichotomous case, preferences of players over permutations depend only on the lengths of cycles. More precisely, player v prefers permutation π to permutation σ, if either v is covered in π and uncovered in σ, or if v is covered both in π and σ, but $C^\pi(v)$ is shorter than $C^\sigma(v)$.

With the players' preferences over permutations, we can define Pareto optimal and core permutations.

Definition 4 *A coalition* $S \subseteq V$ weakly blocks *a solution* π *if there exists a permutation* σ *of* V *such that each player in* S *prefers* σ *to* π *and at least one player in* S *strictly prefers* σ *to* π. *Coalition* $S \subseteq V$ strongly blocks *a solution* π *if there exists a permutation* σ *of* V *such that each player in* S *strictly prefers* σ *to* π.

Definition 5 *A permutation* π *is* Pareto optimal *for game* Γ *(*$\pi \in PO(\Gamma)$ *for short) if the grand coalition* V *does not weakly block* π. *A permutation* π *is* weakly Pareto optimal *for game* Γ *(*$\pi \in WPO(\Gamma)$ *for short) if the grand coalition* V *does not strongly block* π.

Definition 6 *A permutation* π *is in the* core $C(\Gamma)$ *of game* Γ *if no coalition weakly blocks* π, *and it is in the* weak core $WC(\Gamma)$ *of game* Γ *if no coalition strongly blocks it.*

As each strongly blocking coalition is also weakly blocking, we have

$$C(\Gamma) \subseteq WC(\Gamma) \subseteq WPO(\Gamma),$$

$$C(\Gamma) \subseteq PO(\Gamma) \subseteq WPO(\Gamma).$$

and the above inclusions can be proper [3]. So core concepts are stronger notions, more persistent against possible disruption by participating players or groups of players, but (weakly) Pareto optimal permutations may cover more vertices than any (weak) core permutation does [5] and/or contains shorter cycles.

In the case with strict preferences, the *Top Trading Cycles algorithm* (TTC for short) [3] finds for a KE game Γ a permutation $\pi \in C(\Gamma)$. Disadvantages of the TTC permutation are that it can contain very long cycles or cover too few vertices (see examples in [5]). However, questions whether there exist some other permutations in the weak core with given properties lead to NP-complete, even to inapproximable problems [5, 2].

In the case with indifferences, the TTC algorithm does not work. In [4] it was shown that it is even NP-hard to decide whether $WC(\Gamma) \neq \emptyset$ and also whether $C(\Gamma) \neq \emptyset$.

With dichotomous preferences, it is possible in polynomial time to find a permutation $\pi \in WC(\Gamma)$ [3]. However, the problem of deciding whether $C(\Gamma) \neq \emptyset$ is NP-complete. In [3] it was also argued that $PO(\Gamma) \neq \emptyset$ for each KE game Γ, but it is NP-hard to find a permutation $\pi \in PO(\Gamma)$.

To our surprise, Pareto optimality in the KE game turned out to be computationally very difficult. We considered the following decision problems: problems KE–NONPO–TEST and KE–NONWPO–TEST ask whether a given permutation π is not Pareto optimal and not weakly Pareto optimal, respectively, for a given KE game Γ. Notice that both problems belong to the class NP, as when another permutation σ is given, it can be polynomially verified that each player of the grand coalition (strictly) improves compared to π. In the next section we show that they are are NP-complete, even with two extreme types of preferences – dichotomous and strict.

3 NP-completeness

Theorem 1 *Problem* KE-NONPO-TEST *is NP-complete already in the case with dichotomous preferences.*

Proof. To prove the NP-hardness, we will use a polynomial transformation from the problem EXACT 3-COVER, shown to be NP-complete in [7]. In EXACT 3-COVER a finite set X, $|X| = 3q$ and a family $\mathcal{F}$ of three-element subsets of X are given. The question is whether a subfamily $\mathcal{F}'$ of $\mathcal{F}$ exists such that each element of X belongs to exactly one set from $\mathcal{F}'$.

For each instance $(X, \mathcal{F})$ of EXACT 3-COVER, we construct a KE game $\Gamma = (V, G, \mathcal{O})$ with dichotomous preferences and a permutation π.

Suppose that the elements of X are ordered $x_1, x_2, \ldots, x_n,\ n = 3q > 3$, that $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ and $F_i = \{x_i^1, x_i^2, x_i^3\}$. For each set $F_i \in \mathcal{F}$, there will be 9 vertices $a_i^k, b_i^k, c_i^k,\ k = 1, 2, 3$ in V and for each element $x_j \in X$ there will be one vertex x_j. The arcs of G are defined in Figure 1 in the form of incidence lists, where A_j is the set of those a_i^k that correspond to the occurence of x_j as x_i^k in F_i.

$$\begin{array}{lll} a_i^k: & b_i^k & i = 1, \dots, m;\ k = 1, 2, 3 \\ b_i^k: & x_i^k,\ c_i^k & i = 1, \dots, m;\ k = 1, 2, 3 \\ c_i^k: & c_i^{k+1},\ a_i^k & i = 1, \dots, m;\ k = 1, 2, 3\ (\textit{modulo}\ 3) \\ x_j: & A_j,\ x_{j+1} & j = 1, \dots, n\ (\textit{modulo}\ n) \end{array}$$

Figure 1: Arcs of G

In the obtained KE game, construct permutation

$$\pi = \{(x_1, \dots, x_n)(a_i^k, b_i^k, c_i^k),\ i = 1, \dots, m,\ k = 1, 2, 3\} \tag{1}$$

and for brevity, call $(x_1, \dots, x_n)$ the *long cycle.*

We will show, that $(X, \mathcal{F})$ admits an exact 3-cover if and only if permutation $\pi \notin PO(\Gamma)$.

Suppose that $(X, \mathcal{F})$ admits an exact 3-cover $\mathcal{F}' = \{F_i, i \in I\}$. Let us define permutation σ of V consisting of the following cycles:

$$(c_i^1, c_i^2, c_i^3),\quad (a_i^k, b_i^k, x_i^k)\quad k = 1, 2, 3,\ i \in I \tag{2}$$

$$(a_i^k, b_i^k, c_i^k)\quad k = 1, 2, 3,\ i \notin I, \tag{3}$$

As $\mathcal{F}'$ is an exact 3-cover, permutation σ is well defined and $|C^\sigma(v)| = 3$ for each player $v \in V$. Hence each player x_j strictly prefers σ to π and other players are indifferent between σ and π. So permutation π is weakly blocked via σ and therefore it is not Pareto optimal.

For the other direction, suppose that $\pi \notin PO(\Gamma)$ and σ weakly blocks it. We will show, that $\mathcal{F}$ admits an exact 3-cover.

As π covers each vertex, so does σ. We will moreover show, that σ consists only of 3-cycles. As G does not contain cycles of length 2, we must have $|C^\sigma(v)| = |C^\pi(v)| = 3$ for each player $v \in V' = \{a_i^k, b_i^k, c_i^k;\ i = 1, \dots, m;\ k = 1, 2, 3\}$. So vertices in V' cannot improve and therefore we must have $\sigma <_{x_j} \pi$ for at least one vertex x_j. Hence x_j cannot be on the long cycle. Then necessarily $\sigma(x_j) = a_i^k$ for some $a_i^k \in A_j$, which implies $C^\sigma(x_j) = C^\sigma(a_i^k) = (x_j, a_i^k, b_i^k)$.

If $\sigma(x_j) \neq x_{j+1}$ for some j, then also $C^\sigma(x_{j+1}) = (x_{j+1}, a_r^s, b_r^s)$, where $x_{j+1} \in F_r$ as its s^{th} element. By induction, we get that σ contains only cycles of length 3.

Further, if $(a_i^k, b_i^k, x_i^k) \in \sigma$ for some i and k, then necessarily $C^\sigma(c_i^k) = (c_i^1, c_i^2, c_i^3)$ and therefore also $(a_i^{k+1}, b_i^{k+1}, x_i^{k+1}) \in \sigma$ and $(a_i^{k+2}, b_i^{k+2}, x_i^{k+2}) \in \sigma$

(k modulo 3). So for each i, either $(a_i^k, b_i^k, x_i^k) \in \sigma$ for all k or for none. Hence if we set

$$I = \{i : \ (a_i^k, b_i^k, x_i^k) \in \sigma \text{ for some } k\},$$

then it is immediate that $\mathcal{F}' = \{F_i, i \in I\}$ is an exact 3-cover of $\mathcal{F}$.

As the above transformation is polynomial, we conclude that problem KE–NONPO–TEST is NP-hard under dichotomous preferences. ■

Theorem 2 *Problem* KE–NONPO–TEST *is NP-complete already under strict preferences.*

Proof. To prove the NP-hardness, we use exactly the same polynomial transformation from the problem EXACT 3-COVER as in the Theorem 1, but now the orders of vertices in the incidence lists of Figure 1 define strict preferences of vertices, with the entries in A_j ordered strictly, but arbitrarily.

The proof is also very similar, we just add some remarks.

When σ is defined by (2)–(3), $\sigma <_v \pi$ not only for players x_j, but also for all players $v \in \{c_i^k, b_i^k; i \in I\}$, as for them $\sigma(v) \prec_v \pi(v)$.

For the converse implication, if the grand coalition weakly blocks π via a permutation σ, then necessarily $\sigma(a_i^k) = \pi(a_i^k) = b_i^k$, as $\pi(a_i^k) = f(a_i^k)$ and $C^\pi(a_i^k)$ is shortest possible. Hence these players cannot improve. Further, as $C^\sigma(a_i^k) = C^\sigma(b_i^k)$, we have $|C^\sigma(b_i^k)| = 3$ for each b_i^k. To have at least one player who strictly improves, σ must contain at least one cycle of the form (a_i^k, b_i^k, x_i^k) and the rest of the proof follows. ■

Theorem 3 KE–NONWPO–TEST *is NP-complete already under dichotomous preferences.*

Proof.

We will use a polynomial transformation from the problem RESTRICTED SAT shown to be NP-complete in [9]. In this problem one asks whether a Boolean formula B in CNF containing n Boolean variables $x_1, x_2, \ldots, x_n$ and m clauses $K_1, K_2, \ldots, K_m$, such that each variable appears exactly twice nonnegated and exactly twice negated in B is satisfiable.

For each instance B of RESTRICTED SAT we construct a KE game $\Gamma = (V, G, \mathcal{O})$ with dichotomous preferences and a permutation π.

For each variable x_j, $j = 1, 2, \ldots, n$ there will be a *variable cell* $\Gamma(x_j)$ of 8 *variable players* $x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2, w_j^1, w_j^2$ where x_j^1 (y_j^1) corresponds to the first and x_j^2 (y_j^2) to the second occurence of literal x_j ($\overline{x_j}$). Players $x_j^1, x_j^2, y_j^1, y_j^2$ will be called *proper* variable players.

For each clause $K_k = \{p_k^1, p_k^2, \ldots, p_k^{i_k}\}$, $k = 1, 2, \ldots, m$ (without loss of generality we suppose $i_k > 1$ for each k) there is a *clause cell* $\Gamma(K_k)$ consisting of $4i_k$

clause players $p_k^i, t_k^i, q_k^i, r_k^i,\ i = 1, \ldots, i_k$. Player p_k^i corresponds to the ith entry of K_k and will be called a *proper clause* player, $i = 1, \ldots, i_k$.

Hence game Γ consists of $24n$ players; there are 8 variable players for each variable and 4 clause players for each of $4n$ literals.

We will use the following notation: for each proper variable player v, $c(v)$ denotes the proper clause player corresponding to the position of the corresponding literal in B; and for each proper clause player c, the corresponding proper variable player will be denoted by $v(c)$.

The arc set of G is defined in Figure 2 and the construction is illustrated in Figures 3 and 4.

$$
\begin{array}{lllll}
x_j^i: & z_j^i & i = 1, 2; & j = 1, \ldots, n & \\
z_j^i: & c(y_j^{3-i}), y_j^i & i = 1, 2; & j = 1, \ldots, n & \\
y_j^i: & w_j^i & i = 1, 2; & j = 1, \ldots, n & \\
w_j^i: & c(x_j^i), x_j^{3-i} & i = 1, 2; & j = 1, \ldots, n & \\
p_k^i: & v(p_k^i), r_k^i & i = 1, \ldots, i_k; & k = 1, \ldots, m & \\
r_k^i: & q_k^i & i = 1, \ldots, i_k; & k = 1, \ldots, m & \\
q_k^i: & t_k^i & i = 1, \ldots, i_k; & k = 1, \ldots, m & \\
t_k^i: & r_k^{i+1}, p_k^{i+1} & i = 1, \ldots, i_k; & k = 1, \ldots, m; & (i + 1 \ mod \ i_k)
\end{array}
$$

Figure 2: Arcs of G

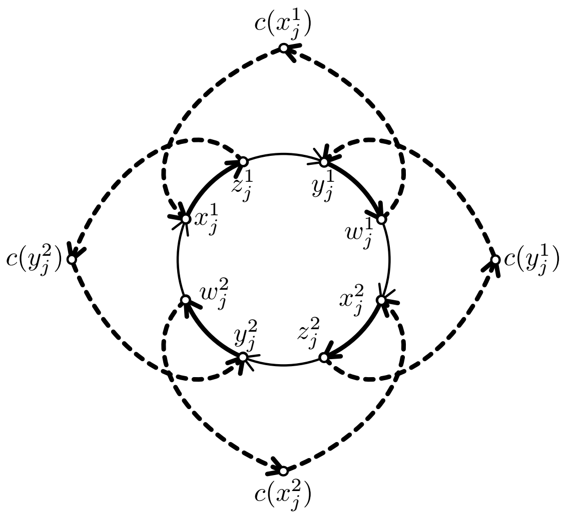

Figure 3: Variable cell with adjacent inter-cell cycles

The construction will be completed by the definition of permutation π. For each variable x_j, $j = 1, \ldots, n$ we put

$$(x_j^1, z_j^1, y_j^1, w_j^1, x_j^2, z_j^2, y_j^2, w_j^2) \in \pi,$$

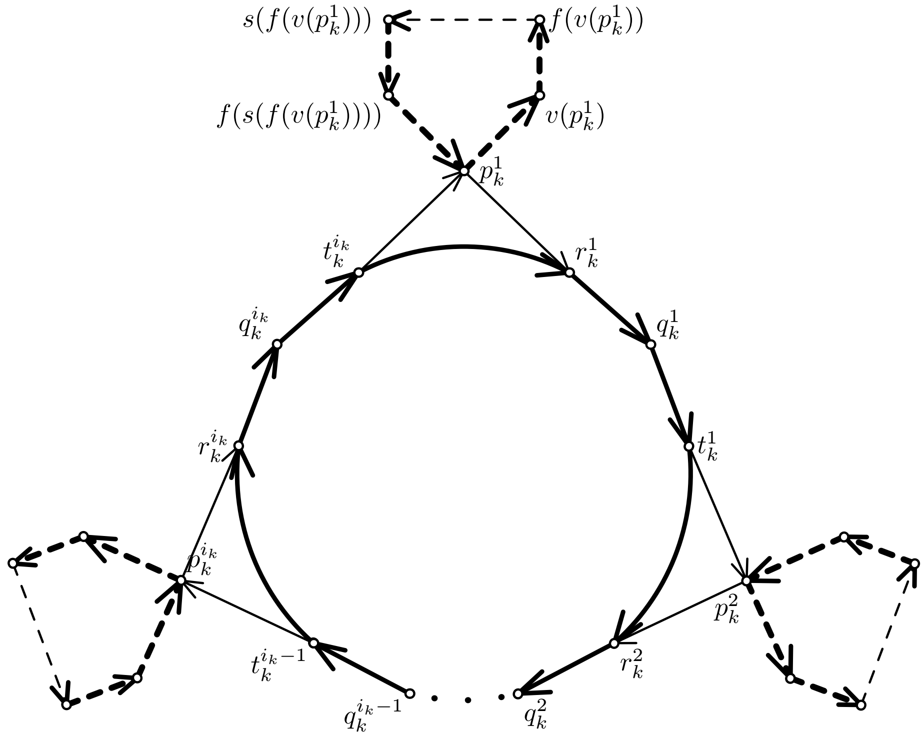

Figure 4: Clause cell with adjacent inter-cell cycles

and for each clause K_k, , $k = 1, \ldots, m$ we put

$$(p_k^1, r_k^1, q_k^1, t_k^1, p_k^2, r_k^2, q_k^2, t_k^2, \ldots, p_k^{i_k}, r_k^{i_k}, q_k^{i_k}, t_k^{i_k}) \in \pi.$$

These cycles will be called *perimeter* cycles. Notice that perimeter cycles in variable cells have length 8, while perimeter cycles in clause cells are of length $4i_k$, $k = 1, \ldots, m$.

As π covers all players, let us first analyze how all players can be covered. It is easy to see that for players of a variable cell $\Gamma(x_j)$ there are only three possibilities, all players are:

(i) either on the perimeter cycle, or

(ii) in two cycles $(x_j^i, z_j^i, y_j^i, w_j^i, c(x_j^i))$, $i = 1, 2$ (let us call them *T-cycles*), or

(iii) in two cycles $(y_j^i, w_j^i, x_j^{3-i}, z_j^{3-i}, c(y_j^i))$, $i = 1, 2$ called *F-cycles*.

As the lengths of the T-cycles and F-cycles are 5, in cases (ii) and (iii) all players from $\Gamma(x_j)$ are better off than under π. Moreover, as these cycles contain one player from a clause cell, we will call them *proper inter-cell* cycles. Notice that any other cycle involving players from several different cells has length greater than 8, so only proper inter-cell cycles could be used when we do not want to make anybody worse off.

Let us now look at a clause cell $\Gamma(K_k)$. If its non-proper clause players are to be covered without making anybody worse off, they must be on cycles that use only arcs within $\Gamma(K_k)$. Otherwise at least two variable cells have to be crossed, getting the cycle length greater than 8, which will necessarily make some proper variable player worse off. Hence the only possibility to cover all players from $\Gamma(K_k)$ is either by the perimeter cycle, or by having some proper clause players p_k^i, $i \in I$ in their corresponding proper inter-cell cycles and the remaining players of $\Gamma(K_k)$ on the common cycle which uses the perimeter arcs and the corresponding *shortcuts* (t_k^{i-1}, r_k^i), $i \in I$. In the latter case all players of $\Gamma(K_k)$ simultaneously strictly improve compared to π.

Now suppose that B is satisfied by some Boolean valuation. Create a permutation σ as follows: cover each variable cell $\Gamma(x_j)$ for which x_j is true by two T-cycles and by two F-cycles if x_j is false. As B is satisfied, each clause cell is crossed by at least one proper inter-cell cycle. Let the remaining players of each clause cell be on the common cycle with corresponding shortcuts. It is easy to see that $2 \leq |C^\sigma(v)| < |C^\pi(v)|$ for each $v \in V$, so $\sigma <_v \pi$ for each $v \in V$.

For the other direction, suppose that B is not satisfiable, but all players have strictly improved. Then all variable players are on T-cycles or F-cycles and each clause cell is crossed by at least one proper inter-cell cycle.

Let us now define a Boolean valuation as follows: for each used T-cycle assign the underlying variable true and for each used F-cycle make the corresponding variable false. Clearly such a valuation is not contradictory and it is easy to see that it satisfies B, a contradiction.

Hence B is satisfiable if and only if $\pi \notin WPO$. As the construction is polynomial, we conclude that KE–NONWPO–TEST is NP-complete. ■

Theorem 4 KE–NONWPO–TEST *problem is NP-complete even in the case with strict preferences.*

Proof.

We will use the same transformation as in Theorem 3 but now we interpret the order of entries in Figure 2 as preferences of players.

The argument is also identical, we just notice that in addition to getting shorter cycles, each player v, for whom $\sigma(v) \neq \pi(v)$, prefers $\sigma(v)$ to $\pi(v)$. ■

4 Conclusion

Efficient kidney exchange programs are nowadays in the centre of interest. In this paper we concentrated on the concept of Pareto optimality in the KE game. We showed, that already testing a given permutation for being not Pareto optimal or not weakly Pareto optimal is NP-hard even in the two extreme cases with dichotomous and with strict preferences.

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