Toeplitz localization operators: spectral functions density

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Abstract

We consider two classes of localization operators based on the Calderón and Gabor reproducing formulas and represent them in a uniform way as Toeplitz operators. We restrict our attention to the generating symbols depending on the first coordinate in the phase space. In this case, the Toeplitz localization operators (TLOs) exhibit an explicit diagonalization, i.e., there exists an isometric isomorphism that transforms all TLOs to the multiplication operators by some specific functions – we call them *spectral functions*. We show that these spectral functions can be written in the form of a convolution of the generating symbol of TLO with a kernel function incorporating an admissible wavelet/window. Using the Wiener's deconvolution technique on the real line, we prove that the set of spectral functions is dense in the C*-algebra of bounded uniformly continuous functions on the real line under the assumption that the Fourier transform of the kernel function does not vanish on the real line. This provides an explicit and independent description of the C*-algebra generated by the set of spectral functions. The result is then applied to the case of a parametric family of wavelets related to Laguerre functions. Thereby we also provide an explicit description of the C*-algebra generated by vertical Toeplitz operators on true poly-analytic Bergman spaces over the upper half-plane.

1 Introduction

Motivation According to [10], there are three non-trivial model classes of symbols such that the corresponding Toeplitz operators generate commutative algebras of operators on the Bergman spaces over the unit disc, or over the upper half-plane: *elliptic* (realized by radial symbols, i.e., functions depending only on $|z|$ on the unit disc), *parabolic* (realized by vertical symbols, i.e., functions depending only on $\mathrm{Im}(z)$ on the upper half-plane) and *hyperbolic* (realized by angular symbols, i.e., functions depending only on $\arg z$ on the upper half-plane). In each one of these three cases there exists an isometric isomorphism that transforms all Toeplitz operators of the corresponding C*-algebra to the multiplication operators by some specific functions. An isometric characterization of the elliptic, parabolic and hyperbolic case commutative algebra of Toeplitz operators reads as follows:

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(i) the C*-algebra generated by the set of Toeplitz operators with bounded *radial* symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{N}$) of slowly oscillating sequences, see [31, 9, 2];

(ii) the C*-algebra generated by the set of Toeplitz operators with bounded *vertical* symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}_+$) of very slowly oscillating functions on $\mathbb{R}_+$, see [12, 11];

(iii) the C*-algebra generated by the set of Toeplitz operators with bounded *angular* symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}$), see [5].

Recall that VSO($\mathbb{R}_+$) consists of all such bounded functions $\mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous with respect to the metric $\rho(x, y) = |\ln x - \ln y|$ on $\mathbb{R}_+$. Note that the C*-algebras VSO($\mathbb{R}_+$) and VSO($\mathbb{R}$) corresponding to the vertical and angular cases, respectively, are isometrically isomorphic to the C*-algebra $C_u(\mathbb{R})$ of bounded uniformly continuous functions on the real line. Moreover, the sequences from VSO($\mathbb{N}$) are nothing but the restrictions of functions from VSO($\mathbb{R}_+$) to natural numbers $\mathbb{N} = \{1, 2, \dots\}$.

This paper focuses on studying the C*-algebras generated by Toeplitz localization operators. Operators of that kind have been extensively studied by many authors, see, for example, [3, 4, 6, 25, 27, 30]. We aim to give a time-frequency analogy of the above mentioned description of commutative C*-algebras of Toeplitz operators on Bergman spaces. For that reason we shall briefly recall some necessary notions and notation.

Toeplitz localization operators A starting point for the construction of time-frequency localization (filter) operators are the reproducing formulas of Calderón in wavelet analysis and of Gabor in time-frequency analysis. They can be written in a unified form as follows (see Section 2 for more details): for every $f \in L_2(\mathbb{R})$,

$$f = \int_{\mathbb{X}} \langle f, \Psi_\zeta \rangle \Psi_\zeta \, \mathrm{d}\zeta,$$

where $\mathbb{X}$ is either the affine group and Ψ is an admissible wavelet (in wavelet case), or $\mathbb{X}$ is the whole plane and Ψ is an admissible window (in time-frequency case). The transform $W_\Psi : L_2(\mathbb{R}) \to L_2(\mathbb{X}, \mathrm{d}\zeta)$ defined by $(W_\Psi f)(\zeta) = \langle f, \Psi_\zeta \rangle$ is known as the continuous wavelet transform (in wavelet case), or short-time Fourier transform (in time-frequency case). The image $W_\Psi(L_2(\mathbb{R}))$ is a closed subspace of the Hilbert space $L_2(\mathbb{X}, \mathrm{d}\zeta)$. It is called the space of wavelet transforms, or short-time Fourier transforms.

For a given bounded function a on $\mathbb{X}$, the *Toeplitz localization operator* (TLO) T_a^Ψ with defining symbol a acts on $W_\Psi(L_2(\mathbb{R}))$ by the rule $T_a^\Psi f = P_\Psi(af)$, where P_Ψ stands for the orthogonal projection from $L_2(\mathbb{X}, \mathrm{d}\zeta)$ onto $W_\Psi(L_2(\mathbb{R}))$. In the wavelet case, the Toeplitz operator T_a^Ψ is usually called the Calderón-Toeplitz operator, whereas in the time-frequency case the operator T_a^Ψ is usually called the Gabor-Toeplitz operator. In this paper, we restrict our attention to TLOs with generating symbols a depending only on the first coordinate in the phase space, and we denote this set of operators by $\mathfrak{T}_\Psi$. As shown in [17], for each fixed admissible Ψ there exists a unitary operator R_Ψ that diagonalizes each TLO T_a^Ψ, i.e., $R_\Psi T_a^\Psi R_\Psi^* = \gamma_a^\Psi I$ with a certain explicitly given function γ_a^Ψ called the *spectral function*. This simultaneous diagonalization of the operators from the class $\mathfrak{T}_\Psi$ ensures that the C*-algebra $\mathscr{T}_\Psi$ generated by $\mathfrak{T}_\Psi$ is commutative and provides direct access to their properties: norms, boundedness, spectra, invariant subspaces, etc. Thus, the operators from the set $\mathfrak{T}_\Psi$ can be considered as "domestic", i.e., simple to study and understand. This knowledge extends naturally to other operators that can be diagonalized in the same way and approximated by $\mathfrak{T}_\Psi$ in the uniform operator topology, or even to the C*-algebra $\mathscr{T}_\Psi$ generated by $\mathfrak{T}_\Psi$. Under certain additional hypotheses, the class of such domestic operators can be explicitly described by providing a simple description of the C*-algebra $\mathscr{G}_\Psi$ generated by the corresponding spectral functions, as it is stated in what follows.

T_a^Ψ	γ_a^Ψ
$\mathfrak{T}_\Psi := \left\{T_a^\Psi;\ a \in L_\infty(\mathbb{R})\right\}$	$\mathfrak{G}_\Psi := \left\{\gamma_a^\Psi;\ a \in L_\infty(\mathbb{R})\right\}$
$\mathscr{T}_\Psi :=$ C*-algebra$(\mathfrak{T}_\Psi)$	$\mathscr{G}_\Psi :=$ C*-algebra$(\mathfrak{G}_\Psi)$

Table 1: Passing from operators to their spectral functions

Main results We first show that each spectral function γ_a^Ψ corresponding to TLO T_a^Ψ with a bounded generating symbol a on $\mathbb{R}$ is a convolution on the real line with an explicitly given kernel K_Ψ. Then using Wiener's deconvolution on the real line we get the following result:

Theorem 1. *If the Fourier transform $\widehat{K}_\Psi$ of a convolution kernel $K_\Psi \in L_1(\mathbb{R})$ does not vanish on $\mathbb{R}$, then the set*

$$\mathfrak{G}_\Psi = \{K_\Psi * a;\ a \in L_\infty(\mathbb{R})\}$$

is a dense subset of $C_u(\mathbb{R})$. Consequently, the C-algebra $\mathscr{T}_\Psi$ generated by the set $\mathfrak{T}_\Psi$ is isometrically isomorphic to $C_u(\mathbb{R})$.*

It follows that the uniform closure of the set $\mathfrak{T}_\Psi$ coincides with the C*-algebra $\mathscr{T}_\Psi$ generated by the set $\mathfrak{T}_\Psi$. Furthermore, there is no qualitative difference between the wavelet/window cases when studying the structure of commutative algebra $\mathscr{T}_\Psi$. Indeed, under the assumption that $\widehat{K}_\Psi$ does not vanish on $\mathbb{R}$, each C*-algebra $\mathscr{T}_\Psi$ generated by TLOs with symbols $a \in L_\infty(\mathbb{R})$ coincides with the closure of the set of its initial generators. Furthermore, in that case Theorem 1 gives an isometric characterization of the "time-frequency" case commutative algebra of TLOs as an analogy of the description of commutative C*-algebras of Toeplitz operators with radial, vertical and angular symbols acting on Bergman spaces. The proof of Theorem 1 is more general, but not so explicit in comparison with the constructions used in the proofs of density in [12] and [11]. Note that the method used in the proof of Theorem 1 yields an elegant and short proof of the main result of [11].

There is another connection between [12] and Theorem 1. Surprisingly, it is described in [18] that the case of Toeplitz operators acting on true poly-analytic Bergman spaces can be viewed as TLOs related to a special case of wavelets, namely, to the wavelets constructed from Laguerre functions on the half-line. It is also known from [18] that the C*-algebra generated by Toeplitz operators with bounded vertical symbols acting on true poly-analytic Bergman space of order $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ over the upper half-plane is commutative, and is isometrically isomorphic to the C*-algebra generated by the set

$$\mathfrak{G}_k := \{\gamma_{a,k};\ a \in L_\infty(\mathbb{R}_+)\}$$

where the corresponding spectral functions are given by

$$\gamma_{a,k}(u) := \int_{\mathbb{R}_+} a\left(\frac{v}{2u}\right) \mathrm{e}^{-v} L_k^2(v)\,\mathrm{d}v,$$

with $L_k(x)$ being the Laguerre polynomial of order k. Thus, as a particular case of Theorem 1, we extend the characterization of the C*-algebra generated by vertical Toeplitz operators acting on the non-weighted Bergman space of analytic functions from [12] to the true poly-analytic case proving the following density result:

Theorem 2. *For each $k \in \mathbb{Z}_+$, the set $\mathfrak{G}_k$ is dense in the C*-algebra* $\mathrm{VSO}(\mathbb{R}_+)$.

Theorem 2 with $k = 0$ is the main result of [12]. As a consequence of Theorem 2 we have that the C*-algebra $\mathscr{T}_k$ generated by Toeplitz operators acting on true poly-analytic Bergman space of order k with bounded vertical symbols is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$. Moreover, Theorem 2 shows that the set of initial generators of $\mathscr{T}_k$ is dense in $\mathscr{T}_k$, i.e., the C*-algebraic closure of the set of initial generators coincides with its topological closure. These results do not depend on the value of the weight parameter k.

Note that the C*-algebra generated by all Toeplitz operators with bounded symbols, acting on the Bergman space over the unit ball in $\mathbb{C}^n$, also coincides with the topological closure of the set of initial generators. This result was recently proved by Xia [34].

Organization of paper Section 2 summarizes the basic facts on time-frequency analysis tools used in the paper, Toeplitz localization operators and their diagonalization. In Section 3 we prove Theorem 1 using Wiener's division lemma. The special case of Toeplitz operators related to wavelets constructed from Laguerre functions on the half-line is investigated in Section 4, where a detailed proof of Theorem 2 is given. The main concept of the proof of Theorem 2 is to move from the half-line to the real line and to use the approximate deconvolution technique developed in Section 3. Verification that the Fourier transform of the corresponding convolution kernel does not vanish on the real line (i.e., the assumption of Theorem 1) is a fine part of the proof and leads to investigation of zeros of a family of certain complex polynomials related to symmetric Meixner-Pollaczek polynomials. We solve this task using techniques from the theory of orthogonal polynomials.

2 Preliminaries

In this section we recall some basic facts we work with in this paper. As a standard overview on the rich subject of time-frequency analysis methods, concepts and basic results we recommend the books [8, 13, 33].

Reproducing formulas of time-frequency analysis in a unified setting In this paper we work with both the reproducing formulas of Calderón in wavelet analysis and of Gabor in time-frequency analysis, using the following notation: for every $f \in L_2(\mathbb{R})$,

$$f = \int_{\mathbb{X}} \langle f, \Psi_\zeta \rangle \Psi_\zeta \, \mathrm{d}\zeta. \tag{1}$$

In the wavelet case, $\mathbb{X}$ is the affine group $\mathbb{G} = \{\zeta = (u, v);\ u > 0,\ v \in \mathbb{R}\}$ with hyperbolic metric and corresponding hyperbolic measure $\mathrm{d}\zeta = u^{-2}\,\mathrm{d}u\,\mathrm{d}v$; and $\Psi = \psi$ is a wavelet, i.e., a function from $L_2(\mathbb{R})$ satisfying the admissibility condition

$$\int_{\mathbb{R}_+} |\widehat{\psi}(t\xi)|^2 \frac{\mathrm{d}t}{t} = 1 \text{ for a.e. } \xi \in \mathbb{R},$$

where $\widehat{f}$ is the Fourier transform $\mathscr{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, given by

$$\widehat{f}(\xi) = (\mathscr{F} f)(\xi) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, \mathrm{d}x.$$

The group $\mathbb{G}$ acts on $L_2(\mathbb{R})$ by

$$\psi_\zeta(x) = \frac{1}{\sqrt{u}}\, \psi\left(\frac{x - v}{u}\right).$$

On the other hand, in the time-frequency case, $\mathbb{X}$ is the plane $\mathbb{R}^2 = \{\zeta = (q, p);\ q, p \in \mathbb{R}\}$ with Lebesgue measure $\mathrm{d}\zeta = \mathrm{d}q\,\mathrm{d}p$; $\Psi = \phi$ is an admissible window such that $\|\phi\|_{L_2(\mathbb{R})} = 1$. The action of $\mathbb{R}^2$ on $L_2(\mathbb{R})$ is defined by

$$\phi_\zeta(x) = \mathrm{e}^{2\pi \mathrm{i} p x} \phi(x - q).$$

In what follows, Ψ always means an admissible wavelet ψ, or an admissible window ϕ.

Toeplitz localization operators For a fixed Ψ, the transform $W_\Psi : L_2(\mathbb{R}) \to L_2(\mathbb{X}, \mathrm{d}\zeta)$ defined by $(W_\Psi f)(\zeta) = \langle f, \Psi_\zeta \rangle$ is a linear isometry. Its image $W_\Psi(L_2(\mathbb{R}))$ is a closed subspace of the Hilbert space $L_2(\mathbb{X}, \mathrm{d}\zeta)$, and is called the space of wavelet transforms, or short-time Fourier transforms, respectively. Function $(\eta, \zeta) \mapsto \langle \Psi_\eta, \Psi_\zeta \rangle$ is the reproducing kernel of the space $W_\Psi(L_2(\mathbb{R}))$, and the orthogonal projection from $L_2(\mathbb{X}, \mathrm{d}\zeta)$ onto $W_\Psi(L_2(\mathbb{R}))$ can be expressed as the integral operator

$$(P_\Psi F)(\eta) = \int_{\mathbb{X}} F(\zeta) \langle \Psi_\eta, \Psi_\zeta \rangle \, \mathrm{d}\zeta.$$

For a given bounded function a on $\mathbb{X}$, we define the Toeplitz localization operator (TLO) $T_a^\Psi : W_\Psi(L_2(\mathbb{R})) \to W_\Psi(L_2(\mathbb{R}))$ with generating symbol a by $T_a^\Psi = P_\Psi M_a$, where M_a is the operator of pointwise multiplication by a on $L_2(\mathbb{X}, \mathrm{d}\zeta)$. Alternatively, TLO T_a^Ψ may be viewed as acting on $L_2(\mathbb{R})$, given by

$$T_a^\Psi f = \int_{\mathbb{X}} a(\zeta)(W_\Psi f)(\zeta)\Psi_\zeta \, \mathrm{d}\zeta,$$

where the integral is interpreted in a weak sense. In this form it resembles the reproducing formula (1), i.e., $T_1^\Psi f = f$.

Spectral functions as convolutions In this paper we restrict our attention to the generating symbols depending on the first coordinate in the phase space. As shown in [17], for each admissible Ψ, there exists a unitary operator R_Ψ that diagonalizes each TLO T_a^Ψ acting on $W_\Psi(L_2(\mathbb{R}))$ with a bounded symbol a on $\mathbb{R}$. That is,

$$R_\Psi T_a^\Psi R_\Psi^* = \gamma_a^\Psi I$$

with an explicitly stated spectral function $\gamma_a^\Psi : \mathbb{R} \to \mathbb{C}$. On the one hand, for the time-frequency case with the generating symbol $a \in L_\infty(\mathbb{R})$, the spectral function takes the form

$$\gamma_a^\phi(\xi) = \int_{\mathbb{R}} a(q) \, |\phi(\xi - q)|^2 \, \mathrm{d}q = (K_\phi * a)(\xi), \quad \xi \in \mathbb{R},$$

where $*$ is the usual (additive) convolution on $\mathbb{R}$, and $K_\phi(z) = |\phi(z)|^2$ is the (time-frequency case) kernel function. On the other hand, for the wavelet case with the generating symbol $b \in L_\infty(\mathbb{R}_+)$, the spectral function was obtained in the following form:

$$\breve{\gamma}_b^\psi(x) = \int_{\mathbb{R}_+} b(u) \, |\widehat{\psi}(ux)|^2 \, \frac{\mathrm{d}u}{u} = \left(\breve{K}_\psi \star \widetilde{b}\right)(x), \quad x \in \mathbb{R},$$

where $\star$ is the Mellin (multiplicative) convolution on $\mathbb{R}_+$ given by

$$(f \star g)(x) := \int_{\mathbb{R}_+} f(u) \, g\left(\frac{x}{u}\right) \frac{\mathrm{d}u}{u}$$

whenever $f, g \in L_1\left(\mathbb{R}_+, \frac{\mathrm{d}u}{u}\right)$. Moreover, $\widetilde{b}(u) := b(1/u)$ for $b : \mathbb{R}_+ \to \mathbb{C}$ is an involution, and $\breve{K}_\psi(z) = |\widehat{\psi}(z)|^2$ is the (wavelet case) kernel function.

The relationship between the additive and multiplicative convolution may be described via the uniform homeomorphisms $\eta(u) = -\ln u$ and $\theta(x) = \ln x$. Thus, given a function $a \in L_\infty(\mathbb{R})$, we transform it to $b = a \circ \eta \in L_\infty(\mathbb{R}_+)$, calculate the wavelet case spectral function $\breve{\gamma}_b^\psi$ and compose it with θ^{-1}. The resulting function $\gamma_a^\psi : \mathbb{R} \to \mathbb{C}$ has the form

$$\gamma_a^\psi(\xi) = \breve{\gamma}_{a\circ\eta}^\psi\left(\theta^{-1}(\xi)\right).$$

In other words, after the changes of variables we can consider the generating symbols from $L_\infty(\mathbb{R})$ and the spectral functions defined on $\mathbb{R}$. Then $a \mapsto \gamma_a^\psi$ is a mapping connecting the original a with its transformed spectral function,

$$\gamma_a^\psi(\xi) = \big(K_\psi * a\big)(\xi) = \int_{\mathbb{R}} a(x)\, |\widehat{\psi}(\mathrm{e}^{\xi - x})|^2 \,\mathrm{d}x, \quad \xi \in \mathbb{R},$$

where the convolution kernel K_ψ is defined on $\mathbb{R}$ by $K_\psi = \breve{K}_\psi \circ \theta^{-1}$. In this paper, when working with $a \in L_\infty(\mathbb{R})$, we always mean the transformed generating symbol in the case of TLOs related to wavelets.

In the following proposition we summarize the results of this section for both cases (time-frequency as well as wavelet) of spectral functions.

Proposition 3. *Let $a \in L_\infty(\mathbb{R})$. Then γ_a^Ψ can be expressed as $\gamma_a^\Psi = K_\Psi * a$. Consequently, each γ_a^Ψ belongs to the C^*-algebra $C_u(\mathbb{R})$ of uniformly continuous functions on $\mathbb{R}$.*

3 Proof of Theorem 1: approximate deconvolution on the real line

In this section we describe a technique that provides a sufficient condition for density of the set $\mathfrak{G}_\Psi$ in terms of a convolution kernel K_Ψ (expressed in terms of admissible wavelets/windows Ψ). In order to prove Theorem 1, we use the following concept of a Dirac sequence, see [23, XI, § 1].

Definition 4. A sequence $(f_n)_{n=1}^\infty$ of functions $f_n : \mathbb{R} \to [0, +\infty)$ is called a *Dirac sequence*, if it satisfies the following conditions:

(a) $\int_{\mathbb{R}} f_n(x)\,\mathrm{d}x = 1$ for each $n \in \mathbb{N}$;

(b) for each $\delta > 0$ it holds
$$\lim_{n\to\infty} \int_{\mathbb{R}\setminus[-\delta,\delta]} f_n(x)\,\mathrm{d}x = 0.$$

The Dirac sequences form a subclass of approximate identities in the non-unital Banach algebra $L_1(\mathbb{R})$, and the following proposition implies that the Wiener's condition ($\widehat{K}$ does not vanish) is sufficient for the approximate invertibility of K in $L_1(\mathbb{R})$.

Proposition 5 (approximate deconvolution on the real line)**.** *Suppose that $K \in L_1(\mathbb{R})$, such that $\widehat{K}$ does not vanish. Then there exists a sequence $(\phi_n)_{n=1}^\infty$ in $L_1(\mathbb{R})$, such that $(K * \phi_n)_{n=1}^\infty$ is a Dirac sequence.*

Proof. Let $(\psi_n)_{n=1}^\infty$ be the Fejér kernel on the real line:

$$\psi_n(x) = \frac{\sin(n\pi x)^2}{n\pi^2 x^2}.$$

It is well known (see, for example, [26, subsection 2.3.7]) that $(\psi_n)_{n=1}^{\infty}$ is a Dirac sequence in $L_1(\mathbb{R})$. Additionally, the Fourier transforms of the functions ψ_n have compact supports:

$$\widehat{\psi}_n(t) = \begin{cases} 1 - \frac{|t|}{n}, & |t| \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, from Wiener's Division Lemma (see, for example, [29, Lemma 1.4.2]), for every $n \in \mathbb{N}$ there exists a function $\phi_n \in L_1(\mathbb{R})$, such that $K * \phi_n = \psi_n$. $\square$

The following lemma is a well-known result for uniformly continuous functions, see, for example, [26, Theorem 2.2.21].

Lemma 6. *Let $f \in C_u(\mathbb{R})$. If $(h_n)_{n=1}^{\infty}$ is a Dirac sequence, then* $\lim_{n\to\infty} \|f * h_n - f\|_{\infty} = 0$.

Proposition 7. *Suppose that $K \in L_1(\mathbb{R})$, such that $\widehat{K}$ does not vanish on $\mathbb{R}$. Then the set $\mathfrak{S} = \{K * a;\ a \in L_{\infty}(\mathbb{R})\}$ is a dense subset of $C_u(\mathbb{R})$.*

Proof. It is well-known that the convolution of a bounded function with an integrable function is bounded and uniformly continuous. Therefore, $\mathfrak{S}$ is contained in $C_u(\mathbb{R})$.

Let $\sigma \in C_u(\mathbb{R})$, and choose sequences $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ as in Proposition 5. We may define a sequence of functions $(f_n)_{n=1}^{\infty}$ by $f_n = \phi_n * \sigma$, and then

$$K * f_n = K * \phi_n * \sigma = \psi_n * \sigma.$$

Since $(\psi_n)_{n=1}^{\infty}$ is a Dirac sequence, $\psi_n * \sigma$ converges uniformly to σ by Lemma 6. $\square$

Proof of Theorem 1. Let $a \in L_{\infty}(\mathbb{R})$. By Proposition 3 we have $\gamma_a^{\Psi} = K_{\Psi} * a$. If Ψ is such that $K_{\Psi} \in L_1(\mathbb{R})$ and the Fourier transform of K_{Ψ} does not vanish on $\mathbb{R}$, then by Proposition 7 the set $\mathfrak{G}_{\Psi} = \{K_{\Psi} * a;\ a \in L_{\infty}(\mathbb{R})\}$ is a dense subset of $C_u(\mathbb{R})$. It follows that the C*-algebra $\mathscr{T}_{\Psi}$ generated by TLOs with symbols $a \in L_{\infty}(\mathbb{R})$ is isomorphic to the C*-algebra $C_u(\mathbb{R})$ and coincides with the closure of the set of its initial generators. $\square$

The following simple result states that, for a particular case, functions of the form $K * a$ are not only uniformly continuous, but also Lipschitz continuous. Let $\mathrm{AC}(\mathbb{R})$ be the algebra of absolutely continuous functions on $\mathbb{R}$. Then $f \in \mathrm{AC}(\mathbb{R})$ iff f has a derivative almost everywhere and $f' \in L_1(\mathbb{R})$. In particular, the condition of Proposition 8 holds if K belongs to the Schwartz class.

Proposition 8. *Let $K \in L_1(\mathbb{R}) \cap \mathrm{AC}(\mathbb{R})$. Then, for every $a \in L_{\infty}(\mathbb{R})$, the function $K * a$ is Lipschitz continuous, i.e.,*

$$|(K * a)(x_1) - (K * a)(x_2)| \leq \|K'\|_1 \, \|a\|_{\infty} \, |x_1 - x_2| \qquad (x_1, x_2 \in \mathbb{R}).$$

Proof. We bound a by $\|a\|_{\infty}$,

$$|(K * a)(x_1) - (K * a)(x_2)| \leq \|a\|_{\infty} \int_{\mathbb{R}} |K(x_1 - y) - K(x_2 - y)| \, \mathrm{d}y.$$

Without loss of generality, suppose that $x_1 < x_2$. Applying the second fundamental theorem of calculus and the Tonelli–Fubini theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}} |K(x_1-y)-K(x_2-y)|\,\mathrm{d}y &= \int_{\mathbb{R}} \left| \int_{x_1-y}^{x_2-y} K'(u)\,\mathrm{d}u \right| \mathrm{d}y \le \int_{\mathbb{R}} \int_{x_1-y}^{x_2-y} |K'(u)|\,\mathrm{d}u\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \int_{x_1-u}^{x_2-u} |K'(u)|\,\mathrm{d}y\,\mathrm{d}u = (x_2-x_1)\|K'\|_1,\end{aligned}$$

which completes the proof. □

Note that if the convolution kernel K_Ψ belongs to $L_1(\mathbb{R})\cap\mathrm{AC}(\mathbb{R})$, then by Proposition 8 the set $\mathfrak{G}_\Psi$ is a *proper subset* of $C_u(\mathbb{R})$, because there are bounded uniformly continuous functions on $\mathbb{R}$ that are not Lipschitz continuous.

4 Proof of Theorem 2: a vertical true poly-analytic case study

In this section we apply the general density result from previous section to an important special case of TLOs related to wavelets from Laguerre functions. In what follows we recall necessary notions and constructions from [18].

Let $\mathbb{G} = \{\zeta = (u,v);\ u > 0, v \in \mathbb{R}\}$ be the affine group with left-invariant Haar measure $\mathrm{d}\nu_L(\zeta) = u^{-2}\,\mathrm{d}u\mathrm{d}v$. We identify $\mathbb{G}$ with the upper half-plane $\Pi = \{\zeta = v+\mathrm{i}u;\ v \in \mathbb{R}, u > 0\}$ in the complex plane $\mathbb{C}$, with hyperbolic metric and corresponding (hyperbolic) measure $\mathrm{d}\nu_L$. Then $L_2(\mathbb{G}, \mathrm{d}\nu_L)$ is the Hilbert space of all square-integrable complex-valued functions on $\mathbb{G}$ with respect to the measure $\mathrm{d}\nu_L$.

For $k \in \mathbb{Z}_+$, consider the parameterized family of admissible wavelets $\psi^{(k)}$ on $\mathbb{R}$ defined on the Fourier transform side as

$$\widehat{\psi}^{(k)}(\xi) = \chi_+(\xi)\sqrt{2\xi}\,\ell_k(2\xi),$$

where $\ell_n(x) := \mathrm{e}^{-x/2}L_n(x)$ are the (simple) Laguerre functions, where $L_n(x)$ is the Laguerre polynomial of order $n \in \mathbb{Z}_+$; and χ_+ is the characteristic function of the positive half-line. Then, for a fixed $k \in \mathbb{Z}_+$, the space $W_{\psi^{(k)}}(L_2(\mathbb{R}))$ is isometrically isomorphic to

$$A^{(k)} := \{W_k f(u,v);\ f \in H_2(\mathbb{R})\} \subset L_2(\mathbb{G}, \mathrm{d}\nu_L),$$

where $W_k f$ is the continuous wavelet transforms of Hardy-space functions $f \in H_2(\mathbb{R})$ with respect to wavelets $\psi^{(k)}$. Importantly, for $k \in \mathbb{N}$, the space $A^{(k)}$ coincides with the *true poly-analytic Bergman spaces* of order $k+1$ on the upper half-plane. For further details we refer the interested reader to [16] and [28].

Given a generating symbol $a = a(\zeta)$, $\zeta \in \mathbb{G}$, the Toeplitz operator $T_a^{\psi^{(k)}}$ acting on $W_{\psi^{(k)}}(L_2(\mathbb{R}))$ is usually called the Calderón-Toeplitz operator (CTO), see [30, 25]. From the diagonalization described in Section 2, each CTO $T_a^{\psi^{(k)}}$ with a vertical symbol $a = a(u)$

on $\mathbb{G}$ is unitarily equivalent to the multiplication operator acting on $L_2(\mathbb{R}_+)$ with the spectral function $\gamma_a^{\psi^{(k)}}$. For simplicity, we write $T_a^{(k)}$ instead of $T_a^{\psi^{(k)}}$, and $\gamma_{a,k}$ instead of $\gamma_a^{\psi^{(k)}}$. The explicit form of spectral function $\gamma_{a,k} : \mathbb{R}_+ \to \mathbb{C}$ is as follows:

$$\gamma_{a,k}(u) = \int_{\mathbb{R}_+} a\left(\frac{v}{2u}\right) \ell_k^2(v)\, \mathrm{d}v,$$

see [16, Theorem 3.2]. We denoted the set $\{\gamma_{a,k};\, a \in L_\infty(\mathbb{R}_+)\}$ by $\mathfrak{G}_k$. Recall that in the case $k = 0$, corresponding to Toeplitz operators with bounded vertical symbols acting on the non-weighted Bergman space of analytic functions over the upper half-plane, the diagonalization (i.e., the reduction of $T_a^{(0)}$ to $\gamma_{a,0}$) was found already by Vasilevski [32]. Immediatelly, it follows that the C*-algebra generated by CTOs $T_a^{(0)}$ with bounded vertical symbols a is commutative and is isometrically isomorphic to the C*-algebra generated by the set $\mathfrak{G}_0$. It was proved in [12] that the set $\mathfrak{G}_0$ is dense in the C*-algebra $\mathrm{VSO}(\mathbb{R}_+)$. In the case $k = 0$ the corresponding convolution kernel is somewhat simpler, and an approximately inverse sequence $(\phi_n)_{n=1}^\infty$ was constructed explicitly in [12]. For a general $k \in \mathbb{Z}_+$ we were not able to construct an approximately inverse sequence $(\phi_n)_{n=1}^\infty$ in such an explicit manner. Instead of that, we apply Theorem 1, which means that we use a much less explicit construction of ϕ_n explained in the proof of Proposition 5. Thus, the present proof of Theorem 2 is divided into four steps:

I. We transform the density problem to the real line via an appropriate change of variables to allow Wiener's deconvolution technique to be applied.

II. We express the Fourier transform of the convolution kernel as a product of the gamma function with a complex polynomial P_k.

III. To prove that P_k has no real zero, we express P_k as a linear combination of symmetric Meixner-Pollaczek polynomials and move the zeros to the real line using certain linear transformations.

IV. We show that the transformed polynomials satisfy a three-term recurrence relation, thus being orthogonal. This verifies Wiener's sufficiency condition $\widehat{K_k}(t) \neq 0$ for the set $\mathfrak{G}_k$ being a dense set in $\mathrm{VSO}(\mathbb{R}_+)$ for each $k \in \mathbb{Z}_+$.

Step I: Transforming the problem to the real line

After the change of variables $u = \mathrm{e}^x/2$, $v = \mathrm{e}^y$, with

$$b(x) = a(\mathrm{e}^{-x}) \qquad \text{and} \qquad \Gamma_{b,k}(x) = \gamma_{a,k}(\mathrm{e}^x/2),$$

we obtain

$$\Gamma_{b,k}(x) = \int_{\mathbb{R}} b(x-y) K_k(y)\, \mathrm{d}y,$$

where

$$K_k(x) = \mathrm{e}^x \ell_k^2(\mathrm{e}^x) = \frac{\mathrm{e}^x}{\mathrm{e}^{\mathrm{e}^x}} (L_k(\mathrm{e}^x))^2.$$

Note that $K_k \in L_1(\mathbb{R})$ for each k, and

$$\|K_k\|_1 = \int_{\mathbb{R}} K_k(x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^x \ell_k^2(\mathrm{e}^x)\,\mathrm{d}x = \int_{\mathbb{R}_+} \ell_k^2(t)\,\mathrm{d}t = 1.$$

We aim to prove that, for each $k \in \mathbb{Z}_+$, the set

$$\mathfrak{G}_k = \{\Gamma_{b,k};\ b \in L_\infty(\mathbb{R})\}$$

is dense in the C*-algebra $C_u(\mathbb{R})$. Since $b \in L_\infty(\mathbb{R})$ and $K_k \in L_1(\mathbb{R})$, each function $\Gamma_{b,k}$ belongs to $C_u(\mathbb{R})$. From Theorem 1, it is sufficient to prove that the Fourier transform of K_k does not vanish on $\mathbb{R}$, i.e.,

$$(\forall t \in \mathbb{R}) \quad \widehat{K}_k(t) \neq 0.$$

Step II: Fourier transform of the convolution kernel

Let us start with Howell's formula [14], and also [7, formula 8.976(3^6)],

$$\big(L_n^{(\lambda)}(x)\big)^2 = \frac{\Gamma(1+\lambda+n)}{2^{2n} n!} \sum_{r=0}^{n} \binom{2n-2r}{n-r} \frac{(2r)!}{r!} \frac{1}{\Gamma(1+\lambda+r)} L_{2r}^{(2\lambda)}(2x).$$

In particular, for $\lambda = 0$,

$$L_k^2(x) = \frac{1}{2^{2k}} \sum_{r=0}^{k} \alpha_r(k) L_{2r}(2x), \quad \text{where } \alpha_r(k) := \binom{2k-2r}{k-r}\binom{2r}{r}.$$

Then the convolution kernel K_k can be written as

$$K_k(x) = \frac{1}{2^{2k}} \frac{\mathrm{e}^x}{\mathrm{e}^{\mathrm{e}^x}} \sum_{r=0}^{k} \alpha_r(k)\, L_{2r}(2\mathrm{e}^x),$$

and the Fourier transform of each summand is

$$\int_{\mathbb{R}} L_{2r}(2\mathrm{e}^x)\,\mathrm{e}^{-2\pi\,\mathrm{i}\,xt - \mathrm{e}^x}\,\mathrm{e}^x \mathrm{d}x = 2^{2\pi\,\mathrm{i}\,t-1} \int_{\mathbb{R}_+} L_{2r}(z)\,\mathrm{e}^{-z/2} z^{-2\pi\,\mathrm{i}\,t}\,\mathrm{d}z.$$

Applying [7, formula 7.414(7)] for the last integral,

$$\begin{aligned} 2^{2\pi \mathrm{i} t-1} \int_{\mathbb{R}_+} L_{2r}(z)\,\mathrm{e}^{-z/2} z^{-2\pi\,\mathrm{i}\,t}\,\mathrm{d}z &= \Gamma(1-2\pi\,\mathrm{i}\,t)\ {}_2F_1(-2r, 1-2\pi\,\mathrm{i}\,t; 1; 2) \\ &= \Gamma(1-2\pi\,\mathrm{i}\,t)\ {}_2F_1(-2r, 2\pi\,\mathrm{i}\,t; 1; 2), \end{aligned}$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function. Thus, the Fourier transform of K_k is

$$\widehat{K}_k(t) = \frac{(-1)^k}{(k!)^2}\,\Gamma(1-2\pi\,\mathrm{i}\,t) P_k(2\pi t),$$

where

$$P_k(t) = \frac{(-1)^k\,(k!)^2}{2^{2k}}\ \sum_{r=0}^{k} \alpha_r(k)\ {}_2F_1(-2r, \mathrm{i}\,t; 1; 2).$$

The first of these polynomials are

$$\begin{aligned}
P_0(t) &= 1,\\
P_1(t) &= -1 + \mathrm{i}\,t + t^2,\\
P_2(t) &= 4 - 6\,\mathrm{i}\,t - 7t^2 + 2\,\mathrm{i}\,t^3 + t^4, \ \text{ and}\\
P_3(t) &= -36 + 66\,\mathrm{i}\,t + 85t^2 - 39\,\mathrm{i}\,t^3 - 22t^4 + 3\,\mathrm{i}\,t^5 + t^6.
\end{aligned}$$

Since Γ has no zeros, it remains to show that each polynomial P_k has no real root, i.e., $P_k(t) \neq 0$ for every $t \in \mathbb{R}$.

Step III: P_k as a linear combination of orthogonal polynomials

Meixner polynomials of the second kind $m_n^{(\lambda)}(x, \phi)$ (also known as *Meixner-Pollaczek polynomials*) are [21, formula (9.7.1)]

$$m_n^{(\lambda)}(x, \phi) := \frac{(2\lambda)_n}{n!}\,\mathrm{e}^{\mathrm{i}\,n\phi}\ {}_2F_1\left(-n, \lambda + \mathrm{i}\,x; 2\lambda; 1 - \mathrm{e}^{-2\,\mathrm{i}\,\phi}\right).$$

Since

$${}_2F_1(-2r, \mathrm{i}\,t; 1; 2) = (-1)^r\, m_{2r}^{(1/2)}\left(t + \frac{\mathrm{i}}{2}, \frac{\pi}{2}\right),$$

each P_k can be written as

$$P_k(t) = \frac{(-1)^k (k!)^2}{2^{2k}} \sum_{r=0}^{k} (-1)^r\, \alpha_r(k)\, m_{2r}^{(1/2)}\left(t + \frac{\mathrm{i}}{2}, \frac{\pi}{2}\right).$$

Some numerical experiments in Wolfram Mathematica show that the zeros of P_k lie on the horizontal line $\operatorname{Im} t = -1/2$. This observation motivates the following change of variables:

$$Q_k(x) = 4^k\, P_k\left(\frac{x - \mathrm{i}}{2}\right).$$

Substituting the expression for P_k yields the following form:

$$Q_k(x) = (-1)^k (k!)^2 \sum_{r=0}^{k} (-1)^r \alpha_r(k)\, m_{2r}^{(1/2)}\left(\frac{x}{2}, \frac{\pi}{2}\right).$$

The first of these polynomials are

$$\begin{aligned}
Q_0(x) &= 1,\\
Q_1(x) &= -3 + x^2,\\
Q_2(x) &= 41 - 22x^2 + x^4, \ \text{ and}\\
Q_3(x) &= -1323 + 907x^2 - 73x^4 + x^6.
\end{aligned}$$

The polynomials $m_n^{(1/2)}(\frac{x}{2}, \frac{\pi}{2})$ are of special interest because they possess many interesting properties, see [1]. They are known as the *symmetric Meixner-Pollaczek* polynomials, because they satisfy the symmetric relation

$$m_n^{(1/2)}\left(\frac{x}{2}, \frac{\pi}{2}\right) = (-1)^n\, m_n^{(1/2)}\left(-\frac{x}{2}, \frac{\pi}{2}\right),$$

which means that each polynomial $m_{2r}^{(1/2)}(\frac{x}{2}, \frac{\pi}{2})$ is an even function. Thus, each polynomial Q_k is also an even function.

Let M_n be the (transformed) monic form of $m_n^{(1/2)}(\frac{x}{2}, \frac{\pi}{2})$, i.e.,

$$M_n(x) := (2n)!\, m_{2n}^{(1/2)}\left(\frac{\sqrt{x}}{2}, \frac{\pi}{2}\right).$$

These monic forms are completely described by the recurrence formula [1]

$$xM_n(x) = M_{n+1}(x) + a_n M_n(x) + b_n M_{n-1}(x), \quad n \in \mathbb{Z}_+, \tag{2}$$

where $M_{-1}(x) = 0$, $M_0(x) = 1$, and $a_n = 4n(2n+1)+1$, $b_n = 4n^2(2n-1)^2$. Then we write $Q_k(x)$ as $R_k(x^2)$, where

$$R_k(x) = \sum_{r=0}^{k} \beta_r(k) M_r(x) \quad \text{with} \quad \beta_r(k) := (-1)^{k-r}\binom{k}{r}^2 (2k-2r)!$$

The first of these polynomials are

$$\begin{aligned}
&R_0(x) = 1,\\
&R_1(x) = -3 + x,\\
&R_2(x) = 41 - 22x + x^2, \text{ and}\\
&R_3(x) = -1323 + 907x - 73x^2 + x^3.
\end{aligned}$$

Thus, each R_k is a monic polynomial of degree k and R_k alternates its sign (because M_k alternates sign).

Remark 9. The appearance of the symmetric Meixner-Pollaczek polynomials on the Fourier transform side of the convolution kernel related to Laguerre functions is quite natural, because the Laguerre functions and Meixner-Pollaczek polynomials are connected via the Mellin transform, see [20, 22].

Step IV: $(R_k)_{k=0}^{\infty}$ is a family of orthogonal polynomials

Proposition 10. *The polynomials R_k satisfy the recurrence relation*

$$xR_k(x) = R_{k+1}(x) + A_k R_k(x) + B_k R_{k-1}(x) \tag{3}$$

with $A_k = 8k(k+1)+3$, $B_k = 16k^4$, and $R_{-1}(x) = 0$, $R_0(x) = 1$.

Proof. For $k = 0$ the result is trivial. Consider $k \geq 1$. From

$$xR_k(x) = xM_k(x) + \sum_{r=0}^{k-1} \beta_r(k) x M_r(x)$$

and the recurrence relation (2) for polynomials M_r, it follows that

$$\begin{aligned} xR_k(x) &= R_{k+1}(x) + A_k R_k(x) + B_k R_{k-1}(x) \\ &\quad + [a_k + \beta_{k-1}(k) - \beta_k(k+1) - A_k] M_k(x) \\ &\quad + \sum_{r=0}^{k-1} [b_{r+1}\beta_{r+1}(k) + (a_r - A_k)\beta_r(k) + \beta_{r-1}(k) - \beta_r(k+1) - B_k \beta_r(k-1)] M_r(x), \end{aligned}$$

using the convention $\beta_{-1}(k) := 0$. Then R_k satisfies the three-term recurrence relation if and only if

$$\begin{gathered} a_k + \beta_{k-1}(k) - \beta_k(k+1) = A_k, \\ b_{r+1}\beta_{r+1}(k) + (a_r - A_k)\beta_r(k) - B_k \beta_r(k-1) = \beta_r(k+1) - \beta_{r-1}(k) \end{gathered}$$

for $r = 0, 1, \ldots, k-1$. The first identity holds since $\beta_n(n+1) = -2(n+1)^2$ for each n, and so

$$a_k + \beta_{k-1}(k) - \beta_k(k+1) = 4k(2k+1) + 1 - 2k^2 + 2(k+1)^2 = 8k^2 + 8k + 3 = A_k.$$

For the second identities we apply some binomial identities. For each $k \geq 1$ and $r = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} \beta_r(k+1) - \beta_{r-1}(k) &= (-1)^{k+r+1}(2k-2r+2)! \left(\binom{k+1}{r}^2 - \binom{k}{r-1}^2 \right) \\ &= (-1)^{k+r+1}(2k-2r+2)! \binom{k}{r}\binom{k+1}{r} \frac{k+1+r}{k+1} \\ &= \lambda_r(k) \cdot (2k-2r+1)(2k-2r-1)(k+r+1), \end{aligned}$$

where Pascal's identity,

$$\binom{n+1}{m} - \binom{n}{m-1} = \binom{n}{m},$$

as well as the binomial coefficient identities,

$$\begin{gathered} \binom{n+1}{m} + \binom{n}{m-1} = \binom{n+1}{m} \frac{n+1+m}{n+1}, \\ (m+1)\binom{n}{m+1} = n\binom{n-1}{m}, \end{gathered}$$

have been used, and

$$\lambda_r(k) := 4\,(-1)^{r+k+1}\binom{k}{r}^2(k-r)\cdot(2k-2r+2)!.$$

On the other hand, the same manipulation with binomial coefficient identities yields

$$\begin{aligned} b_{r+1}\beta_{r+1}(k) &= \lambda_r(k)\cdot(2r+1)^2(k-r),\\ (a_r-A_k)\beta_r(k) &= \lambda_r(k)\cdot(2k-2r-1)[4k(k+1)-2r(2r+1)+1],\\ B_k\beta_r(k-1) &= \lambda_r(k)\cdot 4k^2(k-r). \end{aligned}$$

Thus,

$$b_{r+1}\beta_{r+1}(k)+(a_r-A_k)\beta_r(k)-B_k\beta_r(k-1)=\lambda_r(k)\cdot\Theta(k,r),$$

where

$$\Theta(k,r) := (2r+1)^2(k-r)+(2k-2r-1)[4k(k+1)-2r(2r+1)+1]-4k^2(k-r).$$

Easy algebraic computations then yield

$$\Theta(k,r)=(2k-2r+1)(2k-2r-1)(k+r+1).$$

Thus, the sequence $(R_k)_{k=0}^{\infty}$ satisfies the recurrence relation (3). □

Final arguments From Favard's theorem [24], the polynomials R_k are orthogonal with respect to a positive measure on $\mathbb{R}$. Therefore, all zeros of R_k are real and simple. Thus, the zeros of Q_k belong to $\mathbb{R}\cup \mathrm{i}\,\mathbb{R}$, and the zeros of P_k belong to

$$\left(\mathbb{R}-\frac{1}{2}\,\mathrm{i}\right)\cup\left(\mathrm{i}\,\mathbb{R}-\frac{1}{2}\,\mathrm{i}\right).$$

The unique intersection of this set with the real line is the point 0, and it can be verified (using a routine manipulation with binomial coefficients) that

$$P_k(0)=(-1)^k(k!)^2\neq 0.$$

Thus, each polynomial P_k does not vanish on $\mathbb{R}$, and it follows that

$$\widehat{K}_k(t)=\frac{(-1)^k}{(k!)^2}\,\Gamma(1-2\pi\,\mathrm{i}\,t)P_k(2\pi t)$$

also does not vanish on $\mathbb{R}$. Therefore, from Theorem 1, the set $\mathfrak{S}_k=\{\Gamma_{b,k}; b\in L_\infty(\mathbb{R})\}$ is dense in $C_u(\mathbb{R})$ and this implies that the set $\mathfrak{G}_k$ is dense in $\mathrm{VSO}(\mathbb{R}_+)$ for each $k\in\mathbb{Z}_+$. The proof of Theorem 2 is complete.

Remark 11. Let $\mathfrak{T}_k$ be the set of Calderón-Toeplitz operators $T_a^{(k)}$ acting on $A^{(k)}$ with bounded vertical symbols a. From Huang [15], the von Neumann algebra $W^*(\mathfrak{T}_k)$ generated by $\mathfrak{T}_k$ is maximal, and $W^*(\mathfrak{T}_k)$ is the closure of the C*-algebra $\mathscr{T}_k$ generated by $\mathfrak{T}_k$ with respect to the strong operator topology in the algebra $\mathscr{B}\left(A^{(k)}\right)$ of all linear bounded operators on the true poly-analytic Bergman space $A^{(k)}$ over the upper half-plane. These outcomes do not depend on the value of $k \in \mathbb{Z}_+$. For a description of C*-algebras of Toeplitz operators with vertical symbols acting on each poly-analytic Bergman spaces (not only true poly-analytic ones) over the upper half-plane we refer the interested reader to recent paper [28].

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