The smallest semicopula-based universal integrals I: properties and characterizations

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Abstract

In this article we provide a detailed study of basic properties of the smallest universal integral $\mathbf{I}_S$ with S being the underlying semicopula and review some of the recent developments in this direction. The class of integrals under study is also known under the name seminormed integrals and includes the well-known Sugeno as well as Shilkret integral as special cases. We present some representations and equivalent formulations for $\mathbf{I}_S$. Further we discuss basic properties such as translatability, linearity, homogeneity, and maxitivity. In particular, we provide the characterizations of the two prominent integrals of Sugeno and Shilkret. We also raise and discuss some open questions.

1 Introduction

This is the first part of the two-paper set devoted to a detailed study of the weakest semicopula-based universal integrals. Both are motivated by the recent ideas of universal integrals in the sense of [14] which can be defined for arbitrary measurable spaces, arbitrary monotone measures and arbitrary measurable functions. As far as we know, the concept of the smallest semicopula-based integral was already known in work of Suárez-García and Álvarez-Gil [23] under the name seminormed integral, followed by other papers dealing with this integral in general, or at least in some remarkable cases (such as the Shilkret integral [22], also called the (N)fuzzy integral in Zhao's work [27], or the Sugeno-Weber family of integrals studied in [26]). In fact, this class of integrals is only a special case of the so-called *universal integrals*, which are briefly described below.

Universal integrals in general Let us consider the measurable spaces $(X, \mathscr{A})$, where $\mathscr{A}$ is a σ-algebra of the subsets of a non-empty set X, and let $\mathscr{S}$ denote the family of all measurable spaces. A class of all $\mathscr{A}$-measurable functions $f : X \to [0, +\infty]$ will be denoted by $\mathscr{F}_{(X,\mathscr{A})}$. For each $a \in]0, +\infty]$ we denote by $\mathscr{M}^a_{(X,\mathscr{A})}$ the set of all monotone set functions such that $m(X) = a$. Put

$$\mathscr{M}_{(X,\mathscr{A})} = \bigcup_{a\in]0,+\infty]} \mathscr{M}^a_{(X,\mathscr{A})}.$$

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Finally, put

$$\mathscr{D}_{[0,+\infty]} = \bigcup_{(X,\mathscr{A})\in\mathscr{S}} \mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}.$$

The two pairs $(m_1, f_1) \in \mathscr{M}_{(X_1,\mathscr{A}_1)} \times \mathscr{F}_{(X_1,\mathscr{A}_1)}$ and $(m_2, f_2) \in \mathscr{M}_{(X_2,\mathscr{A}_2)} \times \mathscr{F}_{(X_2,\mathscr{A}_2)}$ are said to be *integrally equivalent*, we write $(m_1, f_1) \sim (m_2, f_2)$, if for each $t \in]0, +\infty]$ it holds

$$m_1(\{x \in X_1;\ f_1(x) \geq t\}) = m_2(\{x \in X_2;\ f_2(x) \geq t\}).$$

The philosophy of universal integral is then formulated in the following three natural axioms.

Definition 1.1 (cf. [14]) A functional $\mathbf{I} : \mathscr{D}_{[0,+\infty]} \to [0, +\infty]$ is said to be a *universal integral*, if the following conditions are satisfied:

I1) for an arbitrary measurable space $(X, \mathscr{A})$ a restriction of the function $\mathbf{I}$ to $\mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}$ is non-decreasing in each component;

I2) for all pairs $(m, c \cdot \mathbf{1}_A) \in \mathscr{D}_{[0,+\infty]}$, where $c \in [0, +\infty]$ and $A \in \mathscr{A}$, it holds

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A)$$

for some fixed pseudo-multiplication $\otimes : [0, +\infty]^2 \to [0, +\infty]$;

I3) for all integrally equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathscr{D}_{[0,+\infty]}$ it holds

$$\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2).$$

Due to property **I3)** we may "aggregate" the information which is included in the pair $(m, f) \in \mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}$ into a single function $h_{m,f} : [0, +\infty] \to [0, +\infty]$ given by

$$h_{m,f}(t) := m(\{x \in X;\ f(x) \geq t\}).$$

Easily, this function is always non-increasing on $[0, +\infty]$, but need not be always continuous.

Universal integrals based on semicopulas Considering monotone set functions $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ (i.e., m is a *capacity* on $\mathscr{A}$), and functions $f \in \mathscr{F}_{(X,\mathscr{A})}$ satisfying $\mathrm{Ran}(f) \subseteq [0, 1]$ (in this case we will write $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$), the universal integrals are related to pseudo-multiplication $\otimes$ with the neutral element 1. We also put

$$\mathscr{D}_{[0,1]} = \bigcup_{(X,\mathscr{A})\in\mathscr{S}} \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}.$$

The restriction of pseudo-multiplication $\otimes$ to $[0, 1]^2$ is usually called a binary semicopula, or a t-seminorm. Indeed, a binary *semicopula* is an operation $\otimes : [0, 1]^2 \to [0, 1]$, which is non-decreasing in both coordinates, its neutral element is 1, and it satisfies the inequality $x \otimes y \leq M(x, y)$ for all $(x, y) \in [0, 1]^2$, where $M(x, y) = \min\{x, y\}$ is the minimum t-norm. A set of all binary semicopulas will be denoted by $\mathfrak{S}$.

The idea of construction of universal integrals on $[0,+\infty]$ may be applied to the interval $[0,1]$ as well. In such a case the smallest, resp. the greatest $[0,1]$-valued universal integral $\mathbf{I}_{\circledast}$, resp. $\mathbf{I}^{\circledast}$, is given by

$$\mathbf{I}_{\circledast}(m,f) := \max\left\{m(\{f=1\}), \sup_{t\in]0,1]}\{h_{m,f}(t)=1\}\right\}$$
$$\mathbf{I}^{\circledast}(m,f) := \min\left\{m(\{f>0\}), \inf_{t\in]0,1]}\{h_{m,f}(t)>0\}\right\}.$$

We may observe that $\mathbf{I}_{\circledast}$ is closely related to the semicopula D (the so-called drastic product), because of the formula

$$\mathbf{I}_{\circledast}(m,f) = \sup_{t\in[0,1]} D(t, h_{m,f}(t)).$$

On the other hand, the integral $\mathbf{I}^{\circledast}$ is related to the strongest semicopula $M = \min$ (known also as the copula of co-monotone dependence). More precisely, a *copula* C is a semicopula which is 2-increasing, i.e., for all $0 \le x \le y \le 1$ and $0 \le u \le v \le 1$ the inequality

$$C(x,u) + C(y,v) - C(y,u) - C(x,v) \ge 0$$

holds. The latter property implies the Lipschitz property of C, i.e., for each $x, x', y, y' \in [0,1]$ it holds

$$|C(x,y) - C(x',y')| \le |x-x'| + |y-y'|.$$

From it follows that each copula is uniformly continuous on $[0,1]^2$. The fact that copulas are special semicopulas enables one to construct universal integrals in different ways. In what follows we will be interested in the extremal (the smallest) case of a class of integrals $\mathbf{I}_S$, where S is a (semi)copula modelling a relationship between the values of functions and measures under consideration. For further reading on hierarchical families of (semi)copula-based universal integrals we recommend the recent papers [15, 16].

Organization and description of results The main object of our interest in this paper is a class of universal integrals given by

$$\mathbf{I}_S(m,f) := \sup_{t\in[0,1]} S(t, h_{m,f}(t)),$$

where $(X,\mathscr{A}) \in \mathscr{S}$, $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and S is a (binary) semicopula. In Section 2.1 we provide an equivalent representation of the integral in the form

$$\mathbf{I}_S(m,f) = \sup_{A\in\mathscr{A}} S\left(\inf_{x\in A} f(x), m(A)\right).$$

Moreover, we describe a natural Lebesgue-type construction of the integral using the simple functions approach. We also show that for symmetric (commutative) componentwise concave semicopulas, i.e., they are concave in each argument when the other is held fixed, we may transform the integral $\mathbf{I}_S$ defined on a measurable space $(X,\mathscr{A},m)$ to another integral $\mathbf{I}_S$ defined on Borel measurable subsets of the unit interval with respect to Lebesgue measure. Basic properties of the integral $\mathbf{I}_S$ (such as the necessary and sufficient condition for null-value of the integral, necessary and sufficient condition for equality of two integrals of almost everywhere equal functions) are investigated in Section 2.2. In Section 2.3 we present a characterization of the well-known Sugeno integral with respect to the equivalence $\mathbf{I}_S(m, f) \geq t \Leftrightarrow h_{m,f}(t) \geq t$ with $t \in [0, 1]$. Translation invariance of the studied integral with respect to constants under the condition of concave horizontal sections of the underlying semicopula S is described in Section 2.4. Further properties of the integral $\mathbf{I}_S$, such as linearity, homogeneity and maxitivity are studied in Section 2.5. In connection with these results there are formulated some open problems. One (more or less academic) application of the integral is discussed in Section 3.

2 Semicopula-based universal integral $\mathbf{I}_S$: basic properties and characterizations

It is well-known, see for instance [14], that the set of all $[0, 1]$-valued universal integrals with S as the underlying semicopula, is a convex set. A class of the smallest semicopula-based integrals has the form

$$\mathbf{I}_S(m, f) := \sup_{t\in[0,1]} S(t, h_{m,f}(t)),$$

where $(X, \mathscr{A}) \in \mathscr{S}$ and $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Obviously, if $\mathbf{I}_S(m, f) = a$, then for each $t \in [0, 1]$ it holds $S(t, h_{m,f}(t)) \leq a$, as well as for each $\varepsilon \in [0, 1]$ there exists $t_\varepsilon \in [0, 1]$ such that $S(t_\varepsilon, h_{m,f}(t_\varepsilon)) \geq a-\varepsilon$. Since for each semicopula S we have $D \leq S \leq M$ (with D being the drastic product and M the minimum), then obviously $\mathbf{I}_D \leq \mathbf{I}_S \leq \mathbf{I}_M$. Indeed, the integral $\mathbf{I}_D$ is really the weakest one in the class of integrals $\mathbf{I}_S$ (as well as in the class of all $[0, 1]$-valued universal integral related to a semicopula S).

Some special cases of the integral $\mathbf{I}_S$ are already known in the literature. If we take $S = M$, we recover the original definition of the Sugeno integral [24]. The case $S = \Pi$ yields the Shilkret integral, whose combination with maxitive set functions was first studied by Shilkret in [22]. Also, interesting special cases of the integral $\mathbf{I}_S$ for operators S with more specific properties have been studied by Weber [26] for strict triangular norms T, and by Mesiar [19] with respect to possibility measures. In this paper as well as in its continuation [5] we will refer to the integral $\mathbf{I}_S$ as (S)-*universal integral*. Let us note that this integral is called a *seminormed fuzzy integral* in [23], the term also used elsewhere e.g. in [3, 1, 6, 12, 13, 21]. For a fairly general definition of seminormed and semi(co)normed fuzzy integrals see [7], where the monotone set functions taking values in a complete lattice are considered. Futhermore, it is shown that these integrals have very nice properties when associated with possibility measures.

Easily, if $\mathbf{I}_S(m, f) = S(t_0, h_{m,f}(t_0))$ for some $t_0 \in [0, 1]$, then

$$\mathbf{I}_S(m, f) = S(t_0, h_{m,f}(t_0)) \leq S(t_0, 1) = t_0$$

for each semicopula S. A sufficient condition for the equality $\mathbf{I}_S(m, f) = S(t_0, h_{m,f}(t_0))$ is provided in [28].

Proposition 2.1 *Let $S \in \mathfrak{S}$ be continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous. Then for each $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ there exists $t_0 \in [0,1]$ such that $\mathbf{I}_S(m,f) = S(t_0, h_{m,f}(t_0))$.*

Note that the continuity of m is not a necessary condition in the proposition. Indeed, for $X = \mathbb{N}$, $\mathscr{A} = 2^X$ and

$$m(A) = \begin{cases} \frac{1}{2n}, & A = \{n\}, \\ \frac{1}{\min A}, & |A| > 1, \\ 0, & A = \emptyset, \end{cases}$$

we have that for the function $f(x) = \mathbf{1}_{\{1\}}(x)$, $x \in X$, and an arbitrary continuous semicopula S it holds $\mathbf{I}_S(m,f) = S(1, h_{m,f}(1)) = \frac{1}{2}$, however the monotone set function m is not continuous.

2.1 Representation theorems for $\mathbf{I}_S$

Inspired by the well-known result for the Sugeno integral we present the following representation of integral $\mathbf{I}_S$. Its proof is analogous (with a slight modification) to the proof of the corresponding result for the Sugeno integral, but for the sake of completeness we give it here, see also [23, Lemma 3.8].

Theorem 2.2 *Let $S \in \mathfrak{S}$. Then for all $(X,\mathscr{A}) \in \mathscr{S}$ and $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ we have*

$$\mathbf{I}_S(m,f) = \sup_{A \in \mathscr{A}} S\left(\inf_{x \in A} f(x), m(A)\right).$$

Proof. For $t \in [0,1]$ put $A_t = \{x \in X; f(x) \geq t\} \in \mathscr{A}$. Then $h_{m,f}(t) = m(A_t)$ and $\inf\limits_{x \in A_t} f(x) \geq t$, i.e.,

$$S(t, h_{m,f}(t)) \leq S\left(\inf_{x \in A_t} f(x), m(A_t)\right) \leq \sup_{A \in \mathscr{A}} S\left(\inf_{x \in A} f(x), m(A)\right),$$

and therefore,

$$\mathbf{I}_S(m,f) = \sup_{t \in [0,1]} S(t, h_{m,f}(t)) \leq \sup_{A \in \mathscr{A}} S\left(\inf_{x \in A} f(x), m(A)\right). \tag{1}$$

On the other hand, take an arbitrary set $A \in \mathscr{A}$ and put $t_0 = \inf\limits_{x \in A} f(x)$. Then $A \subset A_{t_0}$ and $m(A) \leq m(A_{t_0}) = h_{m,f}(t_0)$. Furthermore,

$$S(t_0, m(A)) \leq S(t_0, h_{m,f}(t_0)) \leq \sup_{t \in [0,1]} S(t, h_{m,f}(t)) = \mathbf{I}_S(m,f).$$

Taking supremum over all $A \in \mathscr{A}$ we get

$$\sup_{A \in \mathscr{A}} S\left(\inf_{x \in A} f(x), m(A)\right) \leq \mathbf{I}_S(m,f). \tag{2}$$

The inequalities (1) and (2) imply our result. $\square$

The next representation connects the construction of the so-called max-seminormed fuzzy integral and integral $\mathbf{I}_S$, and in some sense it resembles the well-known Lebesgue construction, see [23]. Let $S \in \mathfrak{S}$. For a simple non-negative function from $\mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ of the form

$$\mathfrak{s} = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \mathbf{1}_{A_i},$$

where $\alpha_i \in [0,1]$ with $0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, $A_i = \{x \in X; \mathfrak{s}(x) \geq \alpha_i\}$, put

$$Q_S(\mathfrak{s}) := \max\{S(\alpha_i, m(A_i));\ i = 1, 2, \ldots, n\}.$$

We denote by **Sim** the set of all simple functions from $\mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$.

Theorem 2.3 *Let $S \in \mathfrak{S}$. Then for all $(X,\mathscr{A}) \in \mathscr{S}$ and $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ it holds*

$$\mathbf{I}_S(m,f) = \sup\{Q_S(\mathfrak{s});\ \mathfrak{s} \in \mathbf{Sim},\ \mathfrak{s} \leq f\}.$$

Proof. For $t \in [0,1]$ put $A_t = \{x \in X; f(x) \geq t\}$. Since for all $t \in [0,1]$ it holds $S(t, h_{m,f}(t)) = \mathbf{I}_S(m, t \cdot \mathbf{1}_{A_t})$, then we get

$$\begin{aligned}\{S(t, h_{m,f}(t)); t \in [0,1]\} &= \{\mathbf{I}_S(m, t \cdot \mathbf{1}_{A_t}); t \in [0,1]\} \\ &\subseteq \{\mathbf{I}_S(m, \mathfrak{s});\ \mathfrak{s} \in \mathbf{Sim},\ \mathfrak{s} \leq f\}.\end{aligned}$$

From it follows that

$$\mathbf{I}_S(m,f) = \sup\{S(t, h_{m,f}(t)); t \in [0,1]\} \leq \sup\{\mathbf{I}_S(m,\mathfrak{s});\ \mathfrak{s} \in \mathbf{Sim},\ \mathfrak{s} \leq f\}.$$

On the other hand, from the monotonicity of integral $\mathbf{I}_S$, for each $\mathfrak{s} \in \mathbf{Sim}$, $\mathfrak{s} \leq f$ we have $\mathbf{I}_S(m,\mathfrak{s}) \leq \mathbf{I}_S(m,f)$, and therefore

$$\sup\{\mathbf{I}_S(m,\mathfrak{s});\ \mathfrak{s} \in \mathbf{Sim},\ \mathfrak{s} \leq f\} \leq \mathbf{I}_S(m,f).$$

This completes the proof. $\square$

Let $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Since the function $h_{m,f} : [0,1] \to [0,1]$ is non-increasing, it is Borel measurable on $[0,1]$. Therefore, we discuss how to transform the integral $\mathbf{I}_S(m,f)$, which is defined on a monotone measure space $(X, \mathscr{A}, m)$, into another (S)-universal integral $\mathbf{I}_S(\lambda, g)$ defined on the Lebesgue measure space $([0,1], \mathscr{B}([0,1]), \lambda)$, where $\mathscr{B}([0,1])$ is the class of all Borel sets in $[0,1]$ and λ is the Lebesgue measure.

Example 2.4 Consider the non-additive set function m defined for each $A \subseteq X = [0,1]$ by

$$m(A) = \begin{cases} 1, & \lambda(A) \geq \frac{3}{4}, \\ \frac{1}{4} + \lambda(A), & \lambda(A) \in [\frac{1}{8}, \frac{3}{4}[, \\ 0, & \lambda(A) < \frac{1}{8}, \end{cases}$$

where λ is the Lebesgue measure on all Borel subsets $\mathscr{B}(X)$ of X. For the identity function $f(x) = x$ on X we get

$$h_{m,f}(t) = \begin{cases} 1, & t \in [0, \frac{1}{4}], \\ \frac{5}{4} - t, & t \in [\frac{1}{4}, \frac{7}{8}], \\ 0, & t \in]\frac{7}{8}, 1] \end{cases} \quad \text{and} \quad h_{\lambda, h_{m,f}}(t) = \begin{cases} 1, & t = 0, \\ \frac{7}{8}, & t \in]0, \frac{3}{8}], \\ \frac{5}{4} - t, & t \in]\frac{3}{8}, 1]. \end{cases}$$

Thus, $\mathbf{I}_\Pi(m, f) = \frac{25}{64} = \mathbf{I}_\Pi(\lambda, h_{m,f})$, as it is graphically demonstrated in Fig.1.

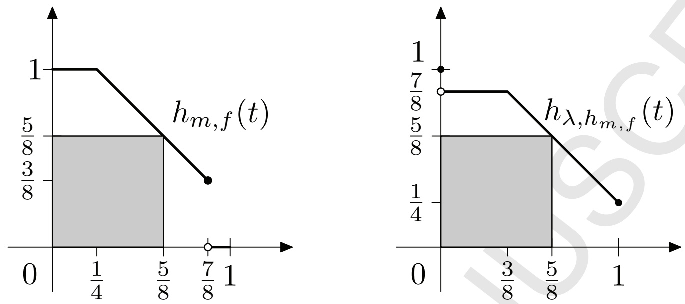

Fig. 1. Transformation of integral from Example 2.4

To be able to prove such a transformation result for a general semicopula S (but not for all!, see Remark 2.6), we fix $z \in [0, 1]$ and consider the inequality

$$S(x + y, z) \leq S(x, z) + S(y, z) \tag{3}$$

holding for each $x, y \in [0, 1]$ such that $x + y \in [0, 1]$. This inequality may be rewritten to the form

$$\frac{S(x + y, z) - S(y, z)}{x + y - y} \leq \frac{S(x, z) - S(0, z)}{x - 0}$$

for $x \in]0, 1]$ (note that for $x = 0$ the inequality (3) holds trivially). Therefore, if we fix $z \in [0, 1]$, the increments of a semicopula S do not increase, so the inequality (3) is fulfilled for each semicopula with concave horizontal sections. Recall that a horizontal section of a semicopula S at $z \in [0, 1]$ is a function $h_{S,z} : [0, 1] \to [0, 1]$ given by $h_{S,z}(t) := S(t, z)$. A vertical section of S at $z \in [0, 1]$ is a function $v_{S,z} : [0, 1] \to [0, 1]$ given by $v_{S,z}(t) := S(z, t)$. If both the vertical as well as horizontal sections of a semicopula S are concave, a semicopula S is called *componentwise concave*, see e.g. [8], viz. for all $x, y, \alpha \in [0, 1]$

$$S(\alpha x + (1 - \alpha)y, z) \geq \alpha S(x, z) + (1 - \alpha)S(y, z),$$
$$S(z, \alpha x + (1 - \alpha)y) \geq \alpha S(z, x) + (1 - \alpha)S(z, y).$$

Such kind of concavity has an important probabilistic interpretation in terms of a positive dependence property of bivariate random pairs.

Now, we are in a position to state the following result involving a transformation theorem for the Sugeno as well as the Shilkret integral.

Theorem 2.5 (transformation of integral) *Let S be a commutative semicopula such that for each $z \in [0, 1]$ its horizontal section $h_{S,z}$ is concave. Then for all $(X, \mathscr{A}) \in \mathscr{S}$ and $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ it holds*

$$\mathbf{I}_S(m, f) = \mathbf{I}_S(\lambda, h_{m,f}).$$

Proof. For any $t \in [0,1]$ put $\mathscr{B}_t = \{A \in \mathscr{B}([0,1]);\, \sup A = t\}$. Then $\{\mathscr{B}_t;\, t \in [0,1]\}$ is a partition of $\mathscr{B}([0,1])$ and $\sup\limits_{A \in \mathscr{B}_t} \lambda(A) = t$. By Theorem 2.2 we have

$$\begin{aligned}
\mathbf{I}_S(\lambda, h_{m,f}) &= \sup_{A \in \mathscr{B}([0,1])} S\left(\inf_{u \in A} h_{m,f}(u), \lambda(A)\right) \\
&= \sup_{t \in [0,1]} \sup_{A \in \mathscr{B}_t} S\left(\inf_{u \in A} h_{m,f}(u), \lambda(A)\right).
\end{aligned}$$

Further denote $h_{m,f}(t^-) = \lim\limits_{u \to t^-} h_{m,f}(u)$. Since $h_{m,f}(u)$ is non-increasing, then

$$1 \geq h_{m,f}(t^-) \geq \inf_{u \in A} h_{m,f}(u) \geq h_{m,f}(t)$$

holds for an arbitrary $A \in \mathscr{B}_t$. Moreover, since S is a commutative semicopula with concave horizontal sections, then it is componentwise concave, and thus S is continuous. Then we get

$$\begin{aligned}
\mathbf{I}_S(\lambda, h_{m,f}) &= \sup_{t \in [0,1]} \sup_{A \in \mathscr{B}_t} S\left(\inf_{u \in A} h_{m,f}(u), \lambda(A)\right) \\
&\geq \sup_{t \in [0,1]} S\left(h_{m,f}(t), \sup_{A \in \mathscr{B}_t} \lambda(A)\right) \\
&= \sup_{t \in [0,1]} S\left(h_{m,f}(t), t\right) = \sup_{t \in [0,1]} S\left(t, h_{m,f}(t)\right) \\
&= \mathbf{I}_S(m, f).
\end{aligned}$$

Let us consider an arbitrary $\varepsilon \in (0,1]$. Then for each $t \in [\varepsilon, 1]$ we get $h_{m,f}(t - \varepsilon) \geq h_{m,f}(t^-)$, and similarly as above we have

$$\begin{aligned}
\mathbf{I}_S(\lambda, h_{m,f}) &= \sup_{t \in [0,1]} \sup_{A \in \mathscr{B}_t} S\left(\inf_{u \in A} h_{m,f}(u), \lambda(A)\right) \leq \sup_{t \in [0,1]} S\left(h_{m,f}(t^-), t\right) \\
&= \sup_{t \in [\varepsilon,1]} S\left(h_{m,f}(t^-), t\right) \vee \sup_{t \in [0,\varepsilon)} S\left(h_{m,f}(t^-), t\right) \\
&\leq \sup_{t \in [\varepsilon,1]} S\left(h_{m,f}(t^-), t\right) \vee \sup_{t \in [0,\varepsilon)} S(1, t) = \sup_{t \in [\varepsilon,1]} S\left(h_{m,f}(t^-), t\right) \vee \varepsilon \\
&\leq \sup_{t \in [\varepsilon,1]} S\left(h_{m,f}(t - \varepsilon), t\right) \vee \varepsilon = \sup_{t \in [\varepsilon,1]} S\left(h_{m,f}(t - \varepsilon), t - \varepsilon + \varepsilon\right) \vee \varepsilon \\
&= \sup_{t \in [\varepsilon,1]} S\left(t - \varepsilon + \varepsilon, h_{m,f}(t - \varepsilon)\right) \vee \varepsilon \\
&\leq \sup_{t \in [\varepsilon,1]} \left[S\left(t - \varepsilon, h_{m,f}(t - \varepsilon)\right) + S\left(\varepsilon, h_{m,f}(t - \varepsilon)\right)\right] \vee \varepsilon \\
&\leq \sup_{t - \varepsilon \in [0,1]} \left[S\left(t - \varepsilon, h_{m,f}(t - \varepsilon)\right) + S(\varepsilon, 1)\right] \vee \varepsilon \\
&= \sup_{t - \varepsilon \in [0,1]} \left[S\left(t - \varepsilon, h_{m,f}(t - \varepsilon)\right) + \varepsilon\right] \vee \varepsilon \\
&= \sup_{t - \varepsilon \in [0,1]} S\left(t - \varepsilon, h_{m,f}(t - \varepsilon)\right) + \varepsilon = \mathbf{I}_S(m, f) + \varepsilon.
\end{aligned}$$

Since ε is arbitrarily small, from the proven inequalities we conclude the desired equality $\mathbf{I}_S(m, f) = \mathbf{I}_S(\lambda, h_{m,f})$. $\square$

Remark 2.6 Commutative semicopulas provide a "good" generalization of triangular norms (t-norms). Here we show that commutativity of a semicopula cannot be abandoned from the assumption of Theorem 2.5. Take

$$f(x) = \begin{cases} a, & x \in A, \emptyset \neq A \subset X, \\ 0, & \text{elsewhere}, \end{cases} \qquad m(B) = \begin{cases} 0, & B = \emptyset, \\ 1, & B = X, \\ b, & \text{elsewhere}, \end{cases}$$

where $a, b \in]0,1[$ are such that $b \neq a$. Easily,

$$h_{m,f}(t) = \begin{cases} 1, & t = 0, \\ b, & t \in]0,a], \\ 0, & t \in]a,1], \end{cases} \quad \text{and} \quad h_{\lambda,h_{m,f}}(t) = \begin{cases} 1, & t = 0, \\ a, & t \in]0,b], \\ 0, & t \in]b,1]. \end{cases}$$

Consequently, it follows that

$$\mathbf{I}_S(m,f) = \sup_{t\in[0,1]} S(t, h_{m,f}(t)) = S(a,b) \text{ and } \mathbf{I}_S(\lambda, h_{m,f}) = S(b,a).$$

Considering an arbitrary non-symmetric semicopula S, e.g. the Marshall-Olkin family

$$S_{\alpha,\beta}(x,y) = \begin{cases} x^{1-\alpha} y, & x^\alpha \geq y^\beta, \\ x y^{1-\beta}, & x^\alpha < y^\beta \end{cases}$$

with $\alpha, \beta \in [0,1]$ and $\alpha \neq \beta$, we conclude $\mathbf{I}_S(m,f) \neq \mathbf{I}_S(\lambda, h_{m,f})$.

Remark 2.7 From the proof of our transformation theorem it is not clear whether the concavity of horizontal sections can be dropped from its assumptions. We conjecture that this theorem should hold for all continuous and commutative semicopulas, but we do not have any proof of this conjecture. Moreover, we do not have any counterexample that would disprove this conjecture either.

2.2 Null value of integral $\mathbf{I}_S$ versus null value of function

The following property of the integral $\mathbf{I}_S$ is related to the concept of zero function almost everywhere with respect to the set function m: we say that a *measurable function f is zero m-almost everywhere* (we write $f = 0$ m-a.e.), if

$$m(\{x \in X; f(x) \neq 0\}) = 0.$$

For zero functions almost everywhere we expect that the corresponding integral will have the zero value regardless of which semicopula we consider. On the other hand, the zero value almost everywhere of the function might not be the necessary condition for the zero value of the integral also in the case when continuous measures are considered.

Example 2.8 Let $X = [0,1]$ and λ be the Lebesgue measure on all Borel subsets $\mathscr{B}(X)$ in X. For $f(x) = x$ on X it holds $h_{\lambda,f}(t) = 1 - t$, and therefore

$$\mathbf{I}_W(\lambda, f) = \sup_{t\in[0,1]} W(t, 1-t) = 0.$$

However, $\lambda(\{x \in [0,1]; f(x) \neq 0\}) = 1$, thus $f \neq 0$ λ-a.e.

The following theorem provides the answer to the question, for which semicopulas S and set functions m the zero value of integral $\mathbf{I}_S$ is equivalent to the zero value almost everywhere of the function. This property is known for the case of the Sugeno integral, see e.g. [25, Theorem 9.2] and for the Shilkret integral as well, see [22].

Theorem 2.9 *Let $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$, m be continuous from below and $S \in \mathfrak{S}$ without zero divisors. Then the following assertions are equivalent:*

(i) $f = 0$ *m-a.e.;*

(ii) $\boldsymbol{I}_S(m, f) = 0$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since $m(\{x \in X; f(x) \neq 0\}) = 0$ and $h_{m,f}$ is non-increasing, then for every $t \in]0, 1]$ we have $h_{m,f}(t) \leq m(\{x \in X; f(x) \neq 0\}) = 0$. The non-negativity of $h_{m,f}$ yields $h_{m,f}(t) = 0$ for every $t \in]0, 1]$. Then

$$\mathbf{I}_S(m, f) = \sup_{t\in[0,1]} S(t, h_{m,f}(t)) = \sup_{t\in[0,1]} S(t, 0) = 0.$$

(ii) $\Rightarrow$ *(i)* Firstly, we show that $\mathbf{I}_S(m, f) = 0$ implies $h_{m,f}(t) = 0$ for every $t \in]0, 1]$. Suppose that there exists $t_0 \in]0, 1]$ such that $h_{m,f}(t_0) = \alpha > 0$. Since $h_{m,f}$ is non-increasing, then for every $t \in]0, t_0]$ it holds $h_{m,f}(t) \geq h_{m,f}(t_0) = \alpha$, therefore for every $t \in]0, t_0]$ it holds $S(t, h_{m,f}(t)) \geq S(t, \alpha)$. Consequently,

$$\mathbf{I}_S(m, f) = \sup_{t\in[0,1]} S(t, h_{m,f}(t)) \geq \sup_{t\in]0,t_0]} S(t, h_{m,f}(t)) \geq \sup_{t\in]0,t_0]} S(t, \alpha) > 0,$$

because S has only trivial zero divisors. However, the inequality $\mathbf{I}_S(m, f) > 0$ is in contradiction with the assumption of the zero value of the integral.

Further, we denote $A = \{x \in X; f(x) \neq 0\}$. Then

$$A = \bigcup_{n=1}^{\infty} A_n, \quad \text{where } A_n = \left\{x \in X; f(x) \geq \frac{1}{n}\right\}.$$

Hence we have that A is the limit of the non-decreasing sequence of sets A_n, i.e., $A_n \nearrow A$. Since m is continuous from below, we have

$$m(A) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} h_{m,f}\left(\frac{1}{n}\right) = 0.$$

Therefore, $f = 0$ m-a.e. □

Remark 2.10 The previous theorem does not hold in the case when the considered set function is not continuous from below. Let $X =]0, 1]$ and the set function m be defined on all Borel subsets $\mathscr{B}(X)$ in X by

$$m(A) = \begin{cases} 1, & A = X \\ 0, & \text{elsewhere.} \end{cases}$$

Obviously, m is continuous from above, but it is not continuous from below. For the function $f(x) = x$, $x \in X$, it holds $h_{m,f}(t) = 0$ for every $t \in]0, 1]$, and therefore, $\mathbf{I}_S(m, f) = 0$ for arbitrary semicopula S. However, $m(X) = 1 \neq 0$.

Remark 2.11 A different proof of Theorem 2.9 given in [6, Theorem 4] uses also continuity from above of the set function m, i.e., m is a fuzzy measure in the assumption of Theorem 4 in [6]. As we can see from our proof, this assumption is not required for equivalence of the zero value of a function and the zero value of its integral. Our proof also gives a good background for investigating one further property of the integral $\mathbf{I}_S$ discussed in Section 2.3.

In the classical theory of (additive) measure and integral, the integrals of functions which are equal almost everywhere are always equal. This assertion does not hold in the non-additive theory of measure and integral, therefore we are interested in the validity of this equality for (S)-universal integral. We recall that the set function $m : \mathscr{A} \to [0,1]$ is said to be *null-additive*, see [25], if $m(A \cup B) = m(A)$ for all $A, B \in \mathscr{A}$ with $m(B) = 0$. Similarly, we say that f_1 and f_2 are equal m-almost everywhere (we write $f_1 = f_2$ m-a.e.), if

$$m(\{x \in X; f_1(x) \neq f_2(x)\}) = 0.$$

Theorem 2.12 *Let $S \in \mathfrak{S}$ and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:*

(i) m is null-additive;

(ii) for all $f_1, f_2 \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_1 = f_2$ m-a.e. it holds $\boldsymbol{I}_S(m, f_1) = \boldsymbol{I}_S(m, f_2)$.

Proof. $(i) \Rightarrow (ii)$ From the null-additivity of m it yields

$$h_{m,f_1}(t) \leq m(\{x \in X; f_1(x) \geq t\} \cup \{x \in X; f_1(x) \neq f_2(x)\}) \leq h_{m,f_2}(t)$$

for every $t \in [0,1]$. Since the converse inequality also holds (by the interchanging of f_1 and f_2), then $(m, f_1) \sim (m, f_2)$. Hence by definition of universal integral the equality $\mathbf{I}_S(m, f_1) = \mathbf{I}_S(m, f_2)$ holds.

$(ii) \Rightarrow (i)$ Consider $A, B \in \mathscr{A}$ with $m(B) = 0$. Firstly, suppose that $m(A) = 1$. Then from the monotonicity of the set function m we have $m(A \cup B) \geq m(A) = 1$, therefore $m(A \cup B) = m(A) = 1$, thus m is null-additive. If $m(A) < 1$, by contradiction we show that $m(A \cup B) = m(A)$ also holds. Let $m(A \cup B) > m(A)$. Put $f_1 = \mathbf{1}_A$ a $f_2 = \mathbf{1}_{A \cup B}$. Then

$$m(\{x \in X; f_1(x) \neq f_2(x)\}) = m(B \setminus A) \leq m(B) = 0$$

implies that $f_1 = f_2$ m-a.e. Consequently, by the assumption it holds $\mathbf{I}_S(m, f_1) = \mathbf{I}_S(m, f_2)$. However, from the definition of the universal integral we have $\mathbf{I}_S(m, f_1) = m(A)$ and $\mathbf{I}_S(m, f_2) = m(A \cup B) > m(A)$, but this is a contradiction with the assumption of the equality of these integrals. $\square$

Remark 2.13 A detailed analysis of the proof of Theorem 2.12 yields that this theorem holds for an *arbitrary universal integral* $\mathbf{I} : \mathscr{D}_{[0,1]} \to [0,1]$, and may be extended to an arbitrary universal integral $\mathbf{I} : \mathscr{D}_{[0,+\infty]} \to [0,+\infty]$.

Naturally, the equality of integrals $\mathbf{I}_S(m, f_1) = \mathbf{I}_S(m, f_2)$ does not ensure the equality of functions m-a.e. even in the case of the additive measure m.

For example, for $X = [0,1]$, the Lebesgue measure λ on $\mathscr{B}(X)$ and the function $f(x) = \sqrt{x}$ we get $\mathbf{I}_W(\lambda, f) = \frac{1}{4} = \mathbf{I}_W(\lambda, g)$, where

$$g(x) = \begin{cases} \frac{1}{4}, & x \in \{z \in X; f(z) < \frac{1}{4}\} \\ f(x), & \text{elsewhere} \end{cases}.$$

However, $\lambda(\{x \in X; f(x) \neq g(x)\}) = \frac{1}{16}$, therefore we obtain the following elementary result.

Proposition 2.14 *Let $S \in \mathfrak{S}$ and $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. If $\mathbf{I}_S(m, f) = a$, then $\mathbf{I}_S(m, g) = a$, where*

$$g(x) = \begin{cases} a, & x \in \{z \in X; f(z) < a\}, \\ f(x), & \textit{elsewhere.} \end{cases}$$

Proof. Since $f \leq g$, from the monotonicity of $\mathbf{I}_S$ we have $a = \mathbf{I}_S(m, f) \leq \mathbf{I}_S(m, g)$. Furthermore, the inclusion

$$\{S(t, h_{m,g}(t));\ t \in [0,1]\} \subseteq \{S(t, h_{m,f}(t));\ t \in [0,1]\} \cup \{a\}$$

yields the inequality

$$\sup\{S(t, h_{m,g}(t));\ t \in [0,1]\} \leq \sup\{S(t, h_{m,f}(t));\ t \in [0,1]\} \cup \{a\} = a,$$

and therefore $\mathbf{I}_S(m, g) \leq a = \mathbf{I}_S(m, f)$. □

2.3 Characterization of the Sugeno integral

From the proof of Theorem 2.9 it follows that the property

$$\mathbf{I}_S(m, f) = 0 \;\Leftrightarrow\; h_{m,f}(t) = 0 \;\text{ for every }\; t \in]0,1]$$

is fulfilled for each semicopula S without zero divisors. It is easy to verify that for every $t \in [0,1[$ the equivalence $\mathbf{I}_S(m, f) = 1 \;\Leftrightarrow\; h_{m,f}(t) = 1$ also holds. More generally, for a binary monotone set function m, i.e., for a monotone set function with values in the set $\{0,1\}$, this equivalence holds even for all semicopulas S and all functions $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. On the other hand, for the Sugeno integral it is known that for every $t \in [0,1]$ the equivalence $\mathbf{I}_M(m, f) \geq t \;\Leftrightarrow\; h_{m,f}(t) \geq t$ holds, see [25, Lemma 9.5(4)]. Naturally we may ask, whether this property may be generalized to arbitrary (semi)copula S. However, the following example shows that it is not always true for semicopulas without zero divisors either.

Example 2.15 Consider $X = [0,1]$ with the Lebesgue measure λ defined on $\mathscr{B}(X)$ from X. For $f(x) = x$ on X we have $h_{\lambda,f}(t) = 1-t$, therefore the Shilkret integral $\mathbf{I}_\Pi$ has the value

$$\mathbf{I}_\Pi(\lambda, f) = \sup_{t \in [0,1]} \Pi(t, 1-t) = \frac{1}{4}.$$

So, we can see that the implication $h_{\lambda,f}(1/2) = 1/2 \Rightarrow \mathbf{I}_\Pi(\lambda, f) \geq 1/2$ is not true.

Since the implication $h_{m,f}(t) \geq t \Rightarrow \mathbf{I}_S(m,f) \geq t$ does not hold in general, the following theorem provides the answer to the question for which semicopulas this converse implication holds. We show that this property holds only for the Sugeno integral $\mathbf{I}_M$, therefore this property becomes its characteristic feature.

Theorem 2.16 *Let $S \in \mathfrak{S}$. Then the following assertions are equivalent:*

(i) $S = M$;

(ii) *for all* $(X, \mathscr{A}) \in \mathscr{S}$, *for all* $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ *and* $t \in [0,1]$ *it holds* $\mathbf{I}_S(m,f) \geq t \;\Leftrightarrow\; h_{m,f}(t) \geq t$.

Proof. $(i) \Rightarrow (ii)$ See [25, Lemma 9.5(4)].

$(ii) \Rightarrow (i)$ Let for all $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ hold

$$\mathbf{I}_S(m,f) \geq t \Leftrightarrow h_{m,f}(t) \geq t \tag{4}$$

and $S \neq M$, i.e., there exists $(a,b) \in]0,1[^2$ such that $S(a,b) < M(a,b)$. Since the equivalence (4) holds for every capacity m and function f, for the pair of numbers $a, b \in]0,1[$ consider the function and measure from Remark 2.6. Without loss of generality we may suppose that $b \geq a$. As we have shown in Remark 2.6, for an arbitrary semicopula $S \neq M$ we have

$$\mathbf{I}_S(m,f) = \sup_{t \in [0,1]} S(t, h_{m,f}(t)) = S(a,b) < M(a,b) = a.$$

On the other hand, $h_{m,f}(a) = b \geq a$, which is a contradiction. $\square$

Fortunately, one implication always holds.

Theorem 2.17 *Let $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $S \in \mathfrak{S}$. If for $t \in [0,1]$ the inequality $\mathbf{I}_S(m,f) > t$ holds, then $h_{m,f}(t) > t$.*

Proof. For $t = 0$ the validity of the theorem is obvious, therefore it is enough to consider $t \in]0,1]$. Let $h_{m,f}(t) \leq t$, and, firstly, consider the case $z \in]0,t]$. Then

$$S(z, h_{m,f}(z)) \leq M(z, h_{m,f}(z)) \leq z \leq t.$$

For the case $z \in]t,1]$ we have

$$S(z, h_{m,f}(z)) \leq M(z, h_{m,f}(z)) \leq h_{m,f}(z) \leq h_{m,f}(t) \leq t.$$

Consequently, for every $t \in [0,1]$ the inequality

$$\mathbf{I}_S(m,f) = \sup_{z \in [0,1]} S(z, h_{m,f}(z)) \leq t$$

holds. $\square$

Remark 2.18 It is easy to verify that if for $t \in [0,1]$ it holds $h_{m,f}(t) < t$, then we can always obtain only non-strict inequality $\mathbf{I}_S(m,f) \leq t$.

2.4 Translatability of integral $\mathbf{I}_S$

From the fact that $\mathbf{I}_S$ is a universal integral, for each semicopula S it follows that $\mathbf{I}_S(m, \alpha) = \alpha$ for each $\alpha \in [0, 1]$. This yields that

$$\mathbf{I}_S(m, \alpha + \beta) = \alpha + \beta = \mathbf{I}_S(m, \alpha) + \mathbf{I}_S(m, \beta)$$

for each $\alpha, \beta \in [0, 1]$, $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ and an arbitrary semicopula S. Note that in what follows we always consider such functions f, g for which $\mathrm{Ran}(f+g) \subseteq [0, 1]$ (also $\alpha, \beta \in [0, 1]$ such that $\alpha+\beta \in [0, 1]$). In general, the integral $\mathbf{I}_S$ is non-linear since it covers the Sugeno and Shilkret case. As it is proved in [25, Theorem 9.2(6)], the Sugeno integral obeys the property

$$\mathbf{I}_M(m, f + \alpha) \leq \mathbf{I}_M(m, f) + \alpha.$$

This property is true for the Shilkret integral as well. For a general semicopula we may prove the following result.

Theorem 2.19 *Let $S \in \mathfrak{S}$ be such that for each $z \in [0, 1]$ its horizontal section $h_{S,z}$ is concave. Then for all $(X, \mathscr{A}) \in \mathscr{S}$, $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha \in [0, 1]$ such that $f + \alpha \in [0, 1]$ it holds*

$$\mathbf{I}_S(m, f + \alpha) \leq \mathbf{I}_S(m, f) + \alpha.$$

Proof. By Theorem 2.2 and the inequality (3) we have

$$\begin{aligned}
\mathbf{I}_S(m, f + \alpha) &= \sup_{A\in\mathscr{A}} S\left(\inf_{x\in A}(f(x) + \alpha), m(A)\right) = \sup_{A\in\mathscr{A}} S\left(\inf_{x\in A} f(x) + \alpha, m(A)\right) \\
&\leq \sup_{A\in\mathscr{A}} \left(S\left(\inf_{x\in A} f(x), m(A)\right) + S\left(\alpha, m(A)\right)\right) \\
&\leq \sup_{A\in\mathscr{A}} S\left(\inf_{x\in A} f(x), m(A)\right) + \sup_{A\in\mathscr{A}} S\left(\alpha, m(A)\right) \\
&= \mathbf{I}_S(m, f) + \alpha,
\end{aligned}$$

which proves the result. $\square$

Remark 2.20 The concavity of horizontal sections is only a sufficient condition for the validity of the integral inequality in Theorem 2.19. First, we show that for all $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ the integral $\mathbf{I}_W$, with W being the Łukasiewicz semicopula, has the form

$$\mathbf{I}_W(m, f) = \sup_{t\in[0,1]} (t + h_{m,f}(t) - 1).$$

Easily, for each $t \in [0, 1]$ we have

$$W(t, h_{m,f}(t)) = \max\{t + h_{m,f}(t) - 1, 0\} \geq t + h_{m,f}(t) - 1,$$

thus, taking the supremum over $[0, 1]$ yields the inequality

$$\mathbf{I}_W(m, f) \geq \sup_{t\in[0,1]} (t + h_{m,f}(t) - 1).$$

On the other hand, from the inclusion

$$\{t + h_{m,f}(t) - 1;\ t \in [0,1]\} \supseteq \{\max\{t + h_{m,f}(t) - 1, 0\};\ t \in [0,1]\}$$

it follows that taking the supremum over $[0,1]$ we get the reverse inequality $\mathbf{I}_W(m,f) \leq \sup_{t\in[0,1]} (t + h_{m,f}(t) - 1)$.

Second, we show that for the semicopula W the conclusion of Theorem 2.19 still holds even with the equality. Indeed,

$$\begin{aligned}\mathbf{I}_W(m, f+\alpha) &= \sup_{t\in[0,1]} (t + h_{m,f+\alpha}(t) - 1) = \sup_{t\in[0,1-\alpha]} (t + \alpha + h_{m,f}(t) - 1) \\ &= \alpha + \sup_{t\in[0,1-\alpha]} (t + h_{m,f}(t) - 1) = \alpha + \sup_{t\in[0,1]} (t + h_{m,f}(t) - 1) \\ &= \alpha + \mathbf{I}_W(m,f).\end{aligned}$$

However, as it is well-known, W is the only convex copula, see e.g. [9], thus it does not fulfill the inequality (3). In fact, it is enough to choose for instance $x = y = z = \frac{1}{2}$ to disprove the inequality.

The inequality (3) plays an important role when considering both transformation theorem as well as translatability of integral $\mathbf{I}_S$. However, the concavity of horizontal sections of a semicopula S is only a sufficient condition for the inequality (3) to hold. Independently of the subject of this paper we may ask for a characterization of all semicopulas which satisfy the inequality. As far as we know the problem of this kind has never been solved in the available literature.

Open problem 2.21 *To characterize all the semicopulas S for which the inequality (3) is fulfilled.*

In connection with this problem we arrive to the question of a characterization of translation invariant property of the integral $\mathbf{I}_S$ with respect to constants. Easily, for constant functions, an arbitrary semicopula S and monotone set functions m the above mentioned property holds trivially. We have also shown in Remark 2.20 that the class of semicopulas with this property is non-empty.

Open problem 2.22 *Characterize the class of semicopulas S, for which the equality $\mathbf{I}_S(m, f+\alpha) = \mathbf{I}_S(m,f) + \alpha$ holds for all $(X,\mathscr{A}) \in \mathscr{S}$ and all $(m,f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$.*

2.5 Linearity, homogeneity and maxitivity of integral $\mathbf{I}_S$

Since $\mathbf{I}_S$ covers the well-known cases of non-linear integrals of Sugeno and Shilkret, thus, it is natural to solve the linearity question: *under which conditions the functional $\mathbf{I}_S$ is linear for a fixed $S \in \mathfrak{S}$?* It is a well-known fact that the Sugeno integral is linear only for small classes of measures, see the paper [17], thus the following result (provided in not very well-known paper of Kim [12]) is not surprising. As we can see, the linearity of integral $\mathbf{I}_S$ is true only for trivial cases of constant functions, resp. binary measures.

Proposition 2.23 *Let $m \in \mathscr{M}^1_{(X,\mathscr{A})}$, $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $S \in \mathfrak{S}$. Then the following assertions are equivalent:*

(i) m *is a binary probability measure;*

(ii) *for every* $\alpha, \beta \in [0,1]$ *such that* $\alpha f + \beta g \in [0,1]$ *it holds* $\boldsymbol{I}_S(m, \alpha f + \beta g) = \alpha \boldsymbol{I}_S(m, f) + \beta \boldsymbol{I}_S(m, g)$.

It is also well-known that the Shilkret integral has the property of homogeneity, i.e., for all $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ it holds

$$(\forall \alpha \in [0,1]) \quad \mathbf{I}_\Pi(m, \alpha \cdot f) = \alpha \cdot \mathbf{I}_\Pi(m, f).$$

Since Π is the only semicopula S which is positively homogeneous with respect to one variable, i.e., for each $x, y, \alpha \in [0,1]$ it holds $\alpha S(x,y) = S(\alpha x, y)$, or $\alpha S(x,y) = S(x, \alpha y)$, see [9, Proposition 3], it is easy to see that the following characterization of the Shilkret integral is true.

Proposition 2.24 *Let* $S \in \mathfrak{S}$. *Then the following assertions are equivalent:*

(i) $S = \Pi$;

(ii) *for all* $(X, \mathscr{A}) \in \mathscr{S}$, *all* $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ *and* $\alpha \in [0,1]$ *it holds* $\mathbf{I}_S(m, \alpha \cdot f) = \alpha \cdot \mathbf{I}_S(m, f)$.

The standard linearity is based on the common addition and multiplication of real numbers. In the fuzzy framework, there are several other operations applied on reals (usually on $[0,1]$) among them a distinguished role of idempotent operations of max (modifying the addition) and $M = \min$ (modifying the multiplication) are underlined. The *idempotent linearity* of a functional $I : \mathscr{F}^{[0,1]}_{(X,\mathscr{A})} \to [0,1]$ means that the equality

$$I\big(\max(M(\alpha, f), M(\beta, g))\big) = \max\big(M(\alpha, I(f)), M(\beta, I(g))\big)$$

holds for all $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha, \beta \in [0,1]$. Clearly, the idempotent linearity of a functional I implies its maxitivity and M-homogeneity, i.e.,

$$I(\max(f,g)) = \max(I(f), I(g))$$

and

$$I(M(\alpha, f)) = M(\alpha, I(f))$$

for all $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha \in [0,1]$. García and Álvarez in [23, Proposition 3.11] proved that for each semicopula S the resulting universal integral $\mathbf{I}_S$ is max-homogeneous, i.e., for all $(X, \mathscr{A}) \in \mathscr{S}$ and $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ it holds

$$(\forall \alpha \in [0,1]) \quad \mathbf{I}_S(m, \max(\alpha, f)) = \max(\alpha, \mathbf{I}_S(m, f)).$$

Moreover, it is well-known that the Sugeno integral is not only max-homogeneous, but also M-homogeneous: the latter property becomes its characterization, i.e., the Sugeno integral is the only (S)-universal integral, which is M-homogeneous for each measurable space $(X, \mathscr{A})$ and each pair $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Indeed, it may be formulated in the following proposition.

Proposition 2.25 *Let* $S \in \mathfrak{S}$. *Then the following assertions are equivalent:*

(i) $S = M$;

(ii) *for all* $(X, \mathscr{A}) \in \mathscr{S}$, *all* $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ *and* $\alpha \in [0,1]$ *it holds* $\mathbf{I}_S(m, M(\alpha, f)) = M(\alpha, \mathbf{I}_S(m, f))$.

Observe that for all $(X, \mathscr{A}) \in \mathscr{S}$, $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha \in [0,1]$ we have

$$\mathbf{I}_S(m, M(\alpha, f)) = \sup_{t \in [0,\alpha]} S(t, h_{m,f}(t)).$$

Using Proposition 2.25 we may easily prove the following result. Recall that $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ is said to be *maxitive* whenever $m(A \cup B) = \max(m(A), m(B))$ for each $A, B \in \mathscr{A}$.

Theorem 2.26 *Let* $S \in \mathfrak{S}$ *and* $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ *be maxitive. Then the following assertions are equivalent:*

(i) $S = M$;

(ii) $\mathbf{I}_S$ *is idempotent linear.*

Proof. *(i)* $\Rightarrow$ *(ii)* Consider $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha, \beta \in [0,1]$. Put $k = \max(M(\alpha, f), M(\beta, g))$. Then for each $t \in [0,1]$ we get

$$h_{m,k}(t) = m(F_t \cup G_t) = \max(m(F_t), m(G_t)) = \max(h_{m,M(\alpha,f)}(t), h_{m,M(\beta,g)}(t)),$$

where $F_t = \{x \in X; M(\alpha, f(x)) \geq t)\}$ and $G_t = \{x \in X; M(\beta, g(x)) \geq t)\}$. Then

$$\begin{aligned}
\mathbf{I}_M(m, k) &= \sup_{t \in [0,1]} M\big(t, \max(h_{m,M(\alpha,f)}(t), h_{m,M(\beta,g)}(t))\big) \\
&= \sup_{t \in [0,1]} \max\big(M(t, h_{m,M(\alpha,f)}(t)), M(t, h_{m,M(\beta,g)}(t))\big) \\
&\leq \max\left(\sup_{t \in [0,1]} M(t, h_{m,M(\alpha,f)}(t)), \sup_{t \in [0,1]} M(t, h_{m,M(\beta,g)}(t)) \right) \\
&= \max\big(\mathbf{I}_M(m, M(\alpha, f)), \mathbf{I}_M(m, M(\beta, g))\big).
\end{aligned}$$

On the other hand, from the monotonicity of the integral we have

$$\mathbf{I}_M(m, k) \geq \mathbf{I}_M(m, M(\alpha, f)) \quad \text{and} \quad \mathbf{I}_M(m, k) \geq \mathbf{I}_M(m, M(\beta, g)),$$

and thus $\mathbf{I}_M(m, k) \geq \max\big(\mathbf{I}_M(m, M(\alpha, f)), \mathbf{I}_M(m, M(\beta, g))\big)$. This proves the equality

$$\mathbf{I}_M(m, \max(M(\alpha, f), M(\beta, g))) = \max\big(\mathbf{I}_M(m, M(\alpha, f)), \mathbf{I}_M(m, M(\beta, g))\big).$$

Then Proposition 2.25 completes the proof.

(ii) $\Rightarrow$ *(i)* Since idempotent linearity of $\mathbf{I}_S$ implies its M-homogeneity, Proposition 2.25 yields that $S = M$. $\square$

Example 2.27 Let $X = [0,1[$ and λ be the Lebesgue measure on $\mathscr{B}(X)$. For the function $f(x) = \sqrt{x}$ on X and the drastic product D it holds

$$M(\alpha, \mathbf{I}_D(\lambda, f)) = 0 = \mathbf{I}_D(\lambda, M(\alpha, f))$$

for every $\alpha \in [0,1]$. For every $\alpha \in [0,1]$ it also holds

$$D(\alpha, \mathbf{I}_D(\lambda, f)) = 0 = \mathbf{I}_D(\lambda, D(\alpha, f)).$$

Example 2.28 Let $X = [0,1]$ and λ be the Lebesgue measure on $\mathscr{B}(X)$. For the function $f(x) = x$ on X we have $\mathbf{I}_W(\lambda, f) = 0$, therefore $W(\alpha, \mathbf{I}_W(\lambda, f)) = 0$ for every $\alpha \in [0,1]$. However, $\mathbf{I}_W(\lambda, W(\alpha, f)) = \alpha - 1$ for every $\alpha \in [0,1]$, therefore $\mathbf{I}_W(\lambda, W(\alpha, f)) \neq W(\alpha, \mathbf{I}_W(\lambda, f))$ for $\alpha \in [0,1[$. On the other hand, it holds

$$M(\alpha, \mathbf{I}_W(\lambda, f)) = 0 = \mathbf{I}_W(\lambda, M(\alpha, f))$$

for every $\alpha \in [0,1]$.

The given examples motivate us to formulate the following open problem. We conjecture that the class of semicopulas under consideration will contain only the given examples of (semi)copulas M and Π.

Open problem 2.29 *To characterize the class of semicopulas S for which the property*

$$(\forall \alpha \in [0,1]) \quad \mathbf{I}_S(m, S(\alpha, f)) = S(\alpha, \mathbf{I}_S(m, f))$$

holds for all $(X, \mathscr{A}) \in \mathscr{S}$ and all $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$.

Since the Sugeno integral is the only one which is M-homogeneous, for another semicopulas this property may hold only under additional conditions. The following necessary and suficient condition for the validity of M-homogeneity for the integral $\mathbf{I}_S$ was proved in [13]. For a semicopula S we define the so-called *inf-function* $\mathbf{s} : [0,1] \to [0,1]$ by $\mathbf{s}(x) := \inf\{y;\ S(x,y) = x\}$.

Theorem 2.30 *Let $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $S \in \mathfrak{S}$ be continuous, inf-function $\mathbf{s}$ be increasing and $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Then the following assertions are equivalent:*

(i) $\mathbf{s}(a) \leq \lim_{t \to a^-} h_{m,f}(t)$, *where* $a = \mathbf{I}_S(m, f)$;

(ii) $(\forall \alpha \in [0,1]) \quad \mathbf{I}_S(m, M(\alpha, f)) = M(\alpha, \mathbf{I}_S(m, f))$.

The next important property of the Sugeno integral is its comonotone maxitivity, i.e., for a pair of comonotone functions $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ it holds

$$\mathbf{I}_M(m, \max(f, g)) = \max(\mathbf{I}_M(m, f), \mathbf{I}_M(m, g)). \tag{5}$$

Recall that functions f, g are comonotone if $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in X$. Therefore, we may naturally ask for which class of semicopulas this equality holds.

Open problem 2.31 *Let $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $f, g \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ be comonotone functions. Characterize all the semicopulas S for which*

$$\mathbf{I}_S(m, \max(f, g)) = \max(\mathbf{I}_S(m, f), \mathbf{I}_S(m, g)).$$

The property (5) may be understood as a commuting of Sugeno integral and maximum operation for comonotone functions. Inspired by the results of paper [20] we may formulate the following problem.

Open problem 2.32 *Let $S \in \mathfrak{S}$ be fixed. To describe all the commuting binary operators with the integral $\mathbf{I}_S$ (under the condition of comonotonicity of functions f, g).*

3 An application of integral $\mathbf{I}_S$

Consider a funding agency providing a financial support for research in an area. From the set of all research proposals only the successful ones (depending on some internal rules of the agency) will receive a certain amount of money. In fact, we have only a probabilistic information about the measure of the set of "successful" research proposals. Of course, the knowledge of this information depends on many different aspects: total budget to be divided, the internal rules of the agency, the quality of reviewers (if any), etc. So, now we illustrate the applicability of the integral $\mathbf{I}_S$ in this setting of a democratic decision-making process to allocate research resources.

Assume that the quality of a research proposal is judged by an expert following three quality factors denoted by q_1, q_2 and q_3 (for instance, the quality of scientific publications of the principal investigator, the relevance of scientific goals of a the grant proposal, the applicability of results, etc.). Importance m of the quality factors (e.g., internal rules of the agency) is given as follows.

	$\{\}$	$\{q_1\}$	$\{q_2\}$	$\{q_3\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_2, q_3\}$	$\{q_1, q_2, q_3\}$
m	0	0,7	0,3	0,1	0,9	0,9	0,4	1

Assume that an expert is invited to judge each quality factor of the four grant proposals (denoted by A_i, $i \in \{1, 2, 3, 4\}$) and his scores f of quality factors are given in the table.

	A_1	A_2	A_3	A_4
$f(q_1)$	0,9	0,85	0,9	0,95
$f(q_2)$	0,8	0,85	0,95	0,75
$f(q_3)$	0,95	0,74	0,78	0,8

Depending on the context of the problem for the synthetic evaluation of the quality of each grant proposal, we may choose a different semicopula S to "aggregate" the information included in both tables via integral $\mathbf{I}_S$. The resulting evaluations of A_i for $i \in \{1, 2, 3, 4\}$ using various semicopulas are summarized in Table 1. Here, $C : [0, 1]^2 \to [0, 1]$ is a (non-commutative) copula given by

$$C(x, y) = xy + x^2y(1 - x)(1 - y).$$

If we consider that a funding agency will provide a financial support to those grant proposals which obtained at least the score 0,9, we can see that only two research proposals will obtain the support when using the strongest Sugeno integral. For other integrals the situation is even worser (more pesimistic). When

	A_1	A_2	A_3	A_4
$\mathbf{I}_M(m,f)$	0,9	0,85	0,9	0,8
$\mathbf{I}_C(m,f)$	0,817	0,775	0,817	0,75
$\mathbf{I}_\Pi(m,f)$	0,81	0,765	0,81	0,75
$\mathbf{I}_W(m,f)$	0,8	0,75	0,8	0,75
$\mathbf{I}_D(m,f)$	0,8	0,74	0,78	0,75

Table 1: Evaluations of grant proposals using various (semi)copulas

considering different internal rules of the agency, the situation may change. Also, a comparison of obtained values of one research proposal is also interesting: proposals A_1 and A_2 have been evaluated equally except for the drastic product where the low value of quality factor q_3 prevails.

Concluding remarks

We have investigated the basic properties of semicopula-based universal integrals $\mathbf{I}_S$. We have provided a characterization of two prominent cases of the considered integrals, namely the Sugeno integral $\mathbf{I}_M$ which is the only M-homogeneous integral in this class satisfying $\mathbf{I}_S(m,f) \geq t \Leftrightarrow h_{m,f}(t) \geq t$ for $t \in [0,1]$, and the Shilkret integral $\mathbf{I}_\Pi$ which is the only homogeneous functional in this class (i.e., these characterizations hold for arbitrary measurable spaces, arbitrary monotone measures and arbitrary measurable functions).

During the last period many papers dealing with the generalization of classical (additive) integral inequalities for universal (non-additive) integrals have appeared in the literature, e.g. [2, 11, 18]. As far as we know some of them are also devoted to the seminormed integral, see e.g. [1, 3, 6, 21]. Thus, further research may follow this direction. Nevertheless, in the second part [5] of this two-paper set we continue our research with a detailed investigation of convergence theorems for semicopula-based universal integrals $\mathbf{I}_S$.

Since both Sugeno and Shilkret integrals, with respect to monotone measures, are useful tools in decision support systems, we expect applications in multicriteria decision area and optimization tasks where (semi)copulas can express the dependence between scores and weights of criteria as it was demonstrated in the last section. It should be interesting to compare the results obtained in [10] with the values computed using the integral $\mathbf{I}_S$ with respect to various (semi)copulas S.

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