An integral with respect to probabilistic-valued decomposable measures

Lenka HALČINOVÁ, Ondrej HUTNÍK[1]

Abstract. Several concepts of approximate reasoning in uncertainty processing are linked to the processing of distribution functions. In this paper we make use of probabilistic framework of approximate reasoning by proposing a Lebesgue-type approach to integration of non-negative real-valued functions with respect to probabilistic-valued decomposable (sub)measures. Basic properties of the corresponding probabilistic integral are investigated in detail. It is shown that certain properties, among them linearity and additivity, depend on the properties of the underlying triangle function providing (sub)additivity condition of the considered (sub)measure. It is demonstrated that the introduced integral brings a new tool in approximate reasoning and uncertainty processing with possible applications in several areas.

1 Introduction

A numerical (i.e., real-valued) measure is a generalization of the concept of length, area and volume. Natural properties of these concepts are described by non-negativity and additivity of considered set functions. The celebrated Lebesgue measure theory goes even further assuming a countable additivity or σ-additivity, which provides a natural background for probability theory. However, probability theory based on Lebesgue measure and integral is too restrictive in many diverse fields of mathematical, economical and engineering sciences, especially in approximate reasoning, for instance in several kinds of decision procedures when considering interactions, etc. Thus, there is a need for non-additive measure and integral theory.

In practice, many physical measurements can be best modelled by the concept of numerical measure. However, due to the presence of noise and error in many measurements it is eligible to replace the (additive) measures with the set functions assigning to each measurable subset a distribution function, rather than a non-negative real number. Thus, the probabilistic point of view comes again to the game: the importance/diameter/measure of a set might be represented by a distribution function. This resembles the original idea of Menger replacing a distance function $d : \Omega \times \Omega \to \mathbb{R}_+$ with a distribution function $F_{p,q} : \mathbb{R} \to [0, 1]$ wherein for any number x the value $F_{p,q}(x)$ describes the probability that the distance between the points p and q in Ω is less than x. Recently, probabilistic approaches were successfully applied to various areas in approximate reasoning as well, e.g., to modelling uncertain logical arguments [9], to approximations of incomplete data [4], or to theory of rough sets [15]. Furthermore, a closely related concept can be found in Moore's interval mathematics [17], where the use of intervals in data processing is due to measurement inaccuracy and due to rounding. Observe that intervals can be considered in distribution function form linked to random variables uniformly distributed over the relevant intervals. Thus, several concepts of approximate reasoning in uncertainty processing are linked to the processing of distribution functions.

The idea behind the above mentioned observations was elaborated in [10] leading to the notion of τ_T-*submeasure*: a set function γ defined on a ring Σ of subsets of a non-empty set Ω taking values in the set Δ^+ of distribution functions of non-negative random variables satisfying "initial" condition $\gamma_\emptyset = \varepsilon_0$, "antimonotonicity" property $\gamma_E \geq \gamma_F$ whenever $E, F \in \Sigma$ with $E \subseteq F$, and "subadditivity" property of the form

$$\gamma_{E\cup F}(x + y) \geq T(\gamma_E(x), \gamma_F(y)), \quad E, F \in \Sigma,\, x, y > 0,$$

with T being a left-continuous t-norm. Here, ε_0 is the distribution function of Dirac random variable concentrated at point 0. Naturally, the value $\gamma_E(x)$ describes the probability that

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the numerical (sub)measure of a set E is less than x. Moreover, a fuzzy-number-theoretical interpretation is interesting: the value γ_E may be seen as a non-negative LT-fuzzy number, where $\tau_T(\gamma_E, \gamma_F)$ corresponds to the T-sum of fuzzy numbers γ_E and γ_F with

$$\tau_T(G,H)(x) := \sup_{u+v=x} T(G(u), H(v)), \quad G, H \in \Delta^+, \tag{1}$$

as a special case of triangle functions on Δ^+, i.e., certain semigroups on Δ^+, see Section 2.

The study of probabilistic-valued set functions continued in papers [6] and [7], where a more general concept has been proposed. Motivated by the paper of Shen [20] we have recently introduced in [5] a general framework of τ-decomposable set functions which covers all the mentioned approaches and treat them in a unified way. Indeed, a set function $\gamma : \Sigma \to \Delta^+$ is said to be a *τ-decomposable measure* if $\gamma_\emptyset = \varepsilon_0$ and $\gamma_{E\cup F} = \tau(\gamma_E, \gamma_F)$ with τ being a triangle function on Δ^+. In fact, this definition resembles the well-known definitions of S-decomposable and $\oplus$-decomposable numerical measures studied in the classical and pseudo-analysis theory by many authors. Here, a triangle function τ is a natural choice for "aggregation" of γ_E and γ_F in order to compare them with $\gamma_{E\cup F}$. For a τ-decomposable measure γ we expect that the value of γ at $E \cup F$ is the same as its value at $F \cup E$ for disjoint sets $E, F \in \Sigma$, from which follows that $\tau(\gamma_E, \gamma_F) = \tau(\gamma_F, \gamma_E)$, i.e., τ has to be commutative. Moreover, from the natural equality $\gamma_{(E\cup F)\cup G} = \gamma_{E\cup(F\cup G)}$ (as distance distribution functions) we obtain $\tau(\tau(\gamma_E, \gamma_F), \gamma_G) = \tau(\gamma_E, \tau(\gamma_F, \gamma_G))$, i.e., τ has to be associative. Since $\gamma_E = \gamma_{E\cup\emptyset} = \tau(\gamma_E, \gamma_\emptyset) = \tau(\gamma_E, \varepsilon_0)$, then ε_0 has to be a neutral element of τ. Indeed, (Δ^+, τ) has to be a semigroup. Also, monotonicity of τ comes into play to provide monotonicity of the set function γ. We summarize that the probabilistic-valued (sub)measures defined and studied in [2], [6], [7], [10] and [20] are only special cases of τ-decomposable set functions w.r.t. a different choice of triangle function τ.

A source of motivation for the present paper may be found in [2] where the authors define a probabilistic-valued measure, the notion of probabilistic integral of a measurable function and the corresponding L^p-spaces. In doing so their considerations are related to (in our terminology) τ_M-decomposable measures defined on a σ-algebra of subsets of $\Omega \neq \emptyset$. So, in this paper we follow the pattern introduced in [2] to define the probabilistic integral with respect to an arbitrary τ-decomposable (sub)measure. On one hand, we describe basic properties mentioned in [2] in detail, on the other hand we investigate further features of the integral.

The paper is organized as follows. Section 2 contains preliminary notions, such as distance distribution functions, their topological and algebraic structure, aggregation functions and triangle functions as well as their relationships with probabilistic metric spaces. The important facts about probabilistic-valued set functions with a number of concrete examples can be found in Section 3. In Section 4 we define the probabilistic integral $\int_E f \, \mathrm{d}\gamma$ of a non-negative function $f : \Omega \to [0, +\infty[$ w.r.t. a τ-decomposable measure $\gamma : \Sigma \to \Delta^+$ (with τ being *an arbitrary distributive triangle function* on Δ^+) on a set $E \in \Sigma$ using a construction similar to that of the Lebesgue integral together with a description of its basic properties. Section 5 deals with a different characterization of the integral using sequences of integrals of simple functions pointwisely converging to the integrand. An interesting feature of the integral is its linearity: the integral of (pointwise) sum of two non-negative measurable functions is a "sum" (in a probabilistic sense) of their integrals if and only if γ is a τ_M-decomposable measure w.r.t. the triangle function τ_M given by (1) with M being the minimum t-norm $M(x, y) = \min\{x, y\}$. This property certifies the importance of τ_M-decomposable measure and the corresponding integral in paper [2] when considering probabilistic L^p-spaces. Moreover, we study a τ-decomposable measure induced by the integral of some fixed function. Furthermore, it is deduced that the classical integration theory with respect to a positive measure is naturally included in our general case. The introduced integral brings a new tool in approximate reasoning and uncertainty processing with possible applications in several areas, for example in multicriteria decision support, as it is demonstrated in Concluding Remarks.

2 Preliminaries

In order to make the exposition self-contained, we remind the reader the basic notions and constructions used in this paper.

Distribution functions and triangle functions Let Δ be the family of all distribution functions on the extended real line $\overline{\mathbb{R}} := [-\infty, +\infty]$, i.e., $F : \overline{\mathbb{R}} \to [0,1]$ is non-decreasing, left continuous on the real line $\mathbb{R}$ with $F(-\infty) = 0$ and $F(+\infty) = 1$. The elements of Δ are partially ordered by the usual pointwise order $G \leq H$ if and only if $G(x) \leq H(x)$ for all $x \in \overline{\mathbb{R}}$.

A *distance distribution function* is a distribution function whose support is a subset of $\overline{\mathbb{R}}_+ := [0, +\infty]$, i.e., a distribution function $F : \overline{\mathbb{R}} \to [0,1]$ with $F(0) = 0$. The class of all distance distribution functions will be denoted by Δ^+. Distance distribution functions are a proper tool for measuring distances in probabilistic metric spaces.

A *triangle function* is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ which is symmetric, associative, non-decreasing in each variable and has ε_0 as the identity, where ε_0 is the distribution function of Dirac random variable concentrated at point 0. More precisely, for $a \in [-\infty, +\infty[$ we put

$$\varepsilon_a(x) := \begin{cases} 1 & \text{for } x > a, \\ 0 & \text{otherwise.} \end{cases}$$

The order on Δ^+ induces an order on the set of triangle functions, i.e., $\tau_1 \leq \tau_2$ if and only if $\tau_1(G,H)(x) \leq \tau_2(G,H)(x)$ for each $x \in \mathbb{R}_+$ and each $G, H \in \Delta^+$. For more details on triangle functions we recommend an overview paper [18]. The most important triangle functions are those obtained from certain aggregation functions, especially t-norms.

A *triangular norm*, shortly a t-norm, is a commutative lattice ordered semi-group on $[0,1]$ with identity 1. The most important are the minimum t-norm $M(x,y) := \min\{x,y\}$, the product t-norm $\Pi(x,y) := xy$, the Łukasiewicz t-norm $W(x,y) := \max\{x+y-1, 0\}$, and the drastic product t-norm

$$D(x,y) := \begin{cases} \min\{x,y\} & \text{for } \max\{x,y\} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For more information about t-norms and their properties we refer to books [14, 19]. Throughout this paper denotes the class of all t-norms.

Binary operations Let us denote by the set of all binary operations L on $\overline{\mathbb{R}}_+$ such that

(i) L is commutative and associative;

(ii) L is jointly strictly increasing, i.e., for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}_+$ with $u_1 < u_2$, $v_1 < v_2$ holds $L(u_1, v_1) < L(u_2, v_2)$;

(iii) L is continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;

(iv) L has 0 as its neutral element.

Note that $L \in$ is a jointly increasing pseudo-addition on $\overline{\mathbb{R}}_+$ in the sense of [22]. The usual (class of) examples of operations in are

$$K_\alpha(x,y) := (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}, \quad \alpha > 0,$$
$$K_\infty(x,y) := \max\{x,y\}.$$

Probabilistic metric spaces Triangular norms and triangle functions were originally introduced in the context of probabilistic metric spaces, cf. [19]. Recall that a *probabilistic metric space* (PM-space, for short) is a non-empty set Ω together with a family of probability functions $F_{p,q} \in \Delta^+$ satisfying

$$F_{p,q} = \varepsilon_0 \text{ if and only if } p = q,$$
$$F_{p,q} = F_{q,p},$$

and the "probabilistic analogue" of the triangle inequality expressed by

$$F_{p,r} \geq \tau(F_{p,q}, F_{q,r}) \tag{2}$$

with τ being a triangle function on Δ^+, which holds for all $p, q, r \in \Omega$. The inequality (2) depends on a triangle function. In his original formulation [16] Menger gave as a generalized triangle inequality the following

$$F_{p,r}(x+y) \geq T(F_{p,q}(x), F_{q,r}(y)) \text{ for all } x, y \geq 0,$$

where $T \in$ (a left-continuous one). This corresponds to the triangle function (1). Thus, the triple $(\Omega, , \tau_T)$ is called a *Menger PM-space* (under T). Considering suitable operations L replacing the standard addition $+$ on $\overline{\mathbb{R}}_+$ we obtain a larger class of Menger PM-spaces $(\Omega, , \tau_{L,T})$ with triangle function $\tau_{L,T}$ given by

$$\tau_{L,T}(G,H)(x) := \sup_{L(u,v)=x} T(G(u), H(v)), \quad G, H \in \Delta^+, L \in . \tag{3}$$

Linear and metric structure on Δ^+ For a distance distribution function G and a non-negative constant $c \in \mathbb{R}_+$ we define the multiplication of G by a constant c as follows

$$(c \odot G)(x) := \begin{cases} \varepsilon_0(x), & c = 0, \\ G\left(\frac{x}{c}\right), & \text{otherwise.} \end{cases} \tag{4}$$

Clearly, $c \odot G \in \Delta^+$. An addition of distance distribution functions may be defined in the sense of addition via triangular function τ, i.e., we put

$$(G \oplus_\tau H)(x) := \tau(G,H)(x). \tag{5}$$

Clearly, $G \oplus_\tau H \in \Delta^+$ for each triangular function τ. We usually omit a triangle function from the subscript $\oplus_\tau$ and write just $\oplus$ when no possible confusion may arise. By associativity of τ we may introduce the "sum" of n functions $G_1, \ldots, G_n \in \Delta^+$ as follows

$$\bigoplus_{k=1}^{n} G_k := \tau\left(G_1, \bigoplus_{k=2}^{n} G_k\right).$$

The following two "laws" will be important for our purposes: for each $a, b \in \mathbb{R}_+$ and each $G \in \Delta^+$ it holds

$$(a \cdot b) \odot G = a \odot (b \odot G) = b \odot (a \odot G).$$

A triangle function τ such that for each $c \in \mathbb{R}_+$ and each $G, H \in \Delta^+$ it holds

$$c \odot (G \oplus_\tau H) = (c \odot G) \oplus_\tau (c \odot H)$$

will be called a *distributive triangle function.* In fact, this property depends only on $c \in]0, +\infty[$, because for each triangle function τ it holds $\tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0$. The set of all distributive triangle functions on Δ^+ will be denoted by $\mathfrak{D}(\Delta^+)$.

The set Δ^+ may be endowed with different metrics. We consider a mapping $d_S : \Delta^+ \times \Delta^+ \to [0,1]$ given by

$$d_S(G,H) = \inf_{h>0} \left\{ G(x-h) - h \leq H(x) \leq G(x+h) + h;\, x \in \left[0, \frac{1}{h}\right) \right\},$$

which is called the *Sibley metric* (also a modified Lévy metric) on Δ^+, see [19]. Immediately, for $G, H \in \Delta^+$ such that $G \leq H$ it holds $d_S(G, \varepsilon_0) \leq d_S(H, \varepsilon_0)$. A well-known fact is that this metric metrizes the topology of weak convergence: a sequence $(G_n)_1^\infty \in \Delta^+$ is said to be *weakly convergent* to $G \in \Delta^+$, usually written as $G_n \xrightarrow{w} G$, if the sequence $(G_n(x))_1^\infty$ converges to $G(x)$ for every point x of continuity of G. Moreover,

$$G_n \xrightarrow{w} G \quad \text{if and only if} \quad d_S(G_n, G) \to 0.$$

Let us recall that the convergence in every point of continuity of the function G fails to be equivalent to the convergence in any point of $]0, +\infty[$, see [19].

Remark 2.1. If $\{G_i : i \in I\}$ is a family of functions from Δ^+, then a pointwise supremum of this family is always a distance distribution function. On the other hand, the function $G : \overline{\mathbb{R}} \to [0, 1]$ defined as a pointwise infimum

$$G(x) = \inf\{G_i(x) : i \in I\},\ x \in \mathbb{R},$$

is a non-decreasing function, but it is not necessarily left-continuous on $\mathbb{R}$. Taking the left-limit

$$\mathfrak{G}(x) := \lim_{x' \nearrow x} G(x) = \sup_{x' < x} G(x), \quad x \in \mathbb{R},$$

the function $\mathfrak{G}$ belongs to Δ^+ and $\mathfrak{G} = \inf_{i \in I} G_i$ is the infimum of the family $\{G_i\}$ in the ordered set $(\Delta^+, \leq)$, see [3].

Finally, a triangle function is *continuous*, if it is continuous in the metric space (Δ^+, d_S). Since $L \in$ and M is a continuous t-norm, then the operation $\oplus_{\tau_{L,M}}$ is continuous, see e.g. [18, Theorem 7.13].

3 Probabilistic-valued decomposable set functions w.r.t. a triangle function: basic facts and examples

Now we introduce the basic notion of probabilistic decomposable (sub)measure in its general form. For better readability we also use the following conventions:

(i) for a probabilistic-valued set function $\gamma : \Sigma \to \Delta^+$ we write $\gamma_E(x)$ instead of $\gamma(E)(x)$;

(ii) since Δ^+ is the set of all distribution functions with support $\overline{\mathbb{R}}_+$, we state the expression for a mapping $\gamma : \Sigma \to \Delta^+$ just for positive values of x. In case $x \leq 0$ we always suppose $\gamma_\cdot(x) = 0$.

Definition 3.1. Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ with $\gamma_\emptyset = \varepsilon_0$ is said to be a *τ-decomposable submeasure*, if $\gamma_{E \cup F} \geq \tau(\gamma_E, \gamma_F)$ for each disjoint sets $E, F \in \Sigma$. If in the preceding inequality equality holds, then γ is said to be a *τ-decomposable measure* on Σ.

Characterization and basic observations about τ-decomposable (sub)measures are summarized in the following proposition, for the proofs we refer to [5].

Proposition 3.2. *Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. Then*

(i) γ is a τ-decomposable measure if and only if $\tau(\gamma_{E \cup F}, \gamma_{E \cap F}) = \tau(\gamma_E, \gamma_F)$ for each $E, F \in \Sigma$;

(ii) each τ-decomposable measure γ is "antimonotone" on Σ, i.e., $\gamma_E \geq \gamma_F$ whenever $E, F \in \Sigma$ such that $E \subseteq F$;

(iii) if γ is a τ-decomposable antimonotone submeasure, then the inequality $\gamma_{E \cup F} \geq \tau(\gamma_E, \gamma_F)$ holds for arbitrary sets $E, F \in \Sigma$.

The definition of a τ-decomposable measure may be strengthen by considering σ-additive τ-decomposable measures. Indeed, for a triangle function τ introduce the notation

$$\bigoplus_{n=1}^{\infty} G_n := \lim_{n\to\infty} \bigoplus_{k=1}^{n} G_k$$

with $G_k \in \Delta^+$ for each $k \in \mathbb{N}$. Then γ is said to be a *probabilistic-valued σ-additive τ-decomposable measure* on a σ-ring Σ, if

$$\gamma_{\bigcup_{n=1}^{\infty} E_n} = \bigoplus_{n=1}^{\infty} \gamma_{E_n}$$

whenever $E_n \in \Sigma$ for each $n \in \mathbb{N}$.

Definition 3.3. We say that a probabilistic-valued set function γ is *continuous from below*, if $\gamma_{E_n} \xrightarrow{w} \gamma_E$ whenever $E_n \nearrow E$, i.e., $E_n \subseteq E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = E$.

The role of continuity in approximate reasoning is emphasized e.g. in paper [12]. In the whole paper we will consider the limit operation taken in the weak topology on Δ^+, i.e., the limit $\lim_{n\to\infty} G_n = G$ is always understood in the metric space (Δ^+, d_S). Then the following result holds.

Lemma 3.4. *Let τ be a triangle function on Δ^+ and γ be a σ-additive τ-decomposable measure on a σ-ring Σ. If $E_n \nearrow E$, then γ is continuous from below.*

Proof. For each $n \in \mathbb{N}$ put $F_n = E_n \setminus E_{n-1}$, where $E_0 := \emptyset$. Clearly, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$ and $\bigcup_{k=1}^{n} F_k = E_n$. So, $\gamma_{E_n} = \gamma_{\bigcup_{k=1}^{n} F_k} = \bigoplus_{k=1}^{n} \gamma_{F_k}$ and it follows

$$\gamma_E = \gamma_{\bigcup_{n=1}^{\infty} E_n} = \gamma_{\bigcup_{n=1}^{\infty} F_n} = \bigoplus_{n=1}^{\infty} \gamma_{F_n} = \lim_{n\to\infty} \bigoplus_{k=1}^{n} \gamma_{F_k} = \lim_{n\to\infty} \gamma_{E_n},$$

which completes the proof. □

Let us present some examples of probabilistic-valued (sub)measures. The first one claims that each numerical measure may be regarded as a probabilistic-valued decomposable measure. It is, in fact, highly expected, because similar situation appears in the case of metric spaces and PM-spaces: each metric space (Ω, d) is a Menger PM-space $(\Omega, , \tau_T)$ with $F_{p,q} = \varepsilon_{d(p,q)}$ and a left-continuous t-norm T, cf. [14, Remark 9.25]. So, probabilistic-valued measures offer a wider framework than that of the classical measures and are general enough to cover even wider statistical situations. The importance of such measures can also be traced in other areas, e.g., stochastic differential equations.

Example 3.5. Let m be a finite numerical measure on Σ. If m is L-decomposable, i.e., $m(E \cup F) = L(m(E), m(F))$ with $L \in$, then for any $\Phi \in \Delta^+$ the set function $\gamma^\Phi : \Sigma \to \Delta^+$ defined by $\gamma^\Phi_E := m(E) \odot \Phi$ with $E \in \Sigma$ is a $\tau_{L,M}$-decomposable measure on Σ, where $\tau_{L,M}$ is given by (3) with the minimum t-norm M. Indeed, the additivity property $\gamma^\Phi_{E\cup F} = \tau_{L,M}\left(\gamma^\Phi_E, \gamma^\Phi_F\right)$ follows from [19, Section 7.7]: the equality

$$\tau\left(c_1 \odot G, c_2 \odot G\right) = L(c_1, c_2) \odot G \tag{6}$$

holds for each $c_1, c_2 \in \mathbb{R}_+$ and each $G \in \Delta^+$ if and only if $\tau = \tau_{L,M}$. Moreover, see [1],

$$\tau\left(c_1 \odot G, c_2 \odot G\right) \geq (\leq)\, L(c_1, c_2) \odot G \quad \Leftrightarrow \quad \tau \geq (\leq)\, \tau_{L,M}.$$

These properties will be crucial when investigating linearity of the introduced probabilistic integral in Section 4.

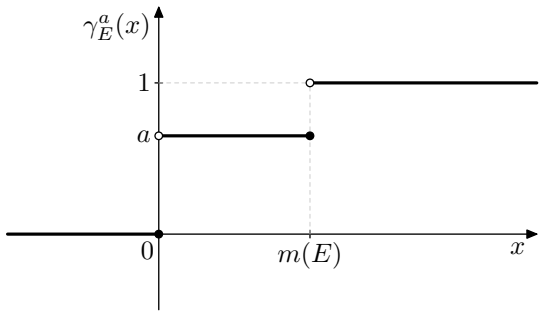

Figure 1: Graph of probabilistic measure γ_E^a from Example 3.6

Example 3.6. A special case of Example 3.5 will be useful for us later to demonstrate certain properties of the introduced integral in Section 4. Indeed, for $a \in [0,1]$ let $\lambda^a \in \Delta^+$ be defined as follows

$$\lambda^a(x) = a\chi_{]0,1]}(x) + \chi_{]1,+\infty]}(x) = \begin{cases} 0, & x \leq 0, \\ a, & x \in]0,1] \\ 1, & x > 1, \end{cases} \tag{7}$$

where χ_S is the characteristic function of a set S. Clearly, $\lambda^0 = \varepsilon_1$ and $\lambda^1 = \varepsilon_0$. Moreover, the set $\Lambda = \{\lambda^a; a \in [0,1]\}$ is linearly ordered, i.e., $\lambda^a \leq \lambda^b$ whenever $a \leq b$. For a finite numerical (additive) measure m on a ring Σ put $\gamma_E^a := m(E) \odot \lambda^a$ for $E \in \Sigma$, i.e., we have

$$\gamma_E^a(x) = a\chi_{]0,m(E)]}(x) + \chi_{]m(E),+\infty]}(x) = \begin{cases} 0, & x \leq 0, \\ a, & x \in]0,m(E)] \\ 1, & x > m(E). \end{cases} \tag{8}$$

Graph of γ_E^a for some $a \in [0,1]$ and certain value of measure $m(E)$ is illustrated in Fig. 1. Then γ^a is a τ_M-decomposable measure. For $A \subseteq [0,1]$ we denote by Γ_A the following set of all such probabilistic τ_M-decomposable measures.

Example 3.7. *Shen's $\top$-probabilistic decomposable measures*, cf. [20]: this class of measures $\mathfrak{M} : \Sigma \to \Delta^+$ of the form $\mathfrak{M}_{E\cup F}(t) = \top(\mathfrak{M}_E(t), \mathfrak{M}_F(t))$ for disjoint $E, F \in \Sigma$ corresponds to the class of τ-decomposable measures w.r.t. the pointwisely defined function $\Pi_\top : \Delta^+ \times \Delta^+ \to \Delta^+$ of the form

$$\Pi_\top(G,H)(t) = \top(G(t), H(t)), \quad G, H \in \Delta^+, \tag{9}$$

with $\top$ being a left-continuous t-norm. Left-continuity of $\top$ is a necessary and sufficient condition for $\Pi_\top$ being a triangle function, cf. [18, Theorem 5.2].

Example 3.8. Let $\mu : \Sigma \to [0,+\infty)$ be a set function with $\mu(\emptyset) = 0$ and for $E \in \Sigma$ put $\gamma_E = \varepsilon_{\mu(E)}$. Then

(i) if μ is additive, then γ is a τ_*-decomposable measure, where

$$\tau_*(G,H)(x) := (G * H)(x) = \begin{cases} 0, & x = 0, \\ \int_0^x G(x-t)\,\mathrm{d}H(t), & x \in]0,+\infty[, \\ 1, & x = +\infty, \end{cases}$$

is the convolution of $G, H \in \Delta^+$ and the integral is meant in the sense of Lebesgue-Stieltjes; moreover, γ is a τ_D-decomposable measure (i.e., for non-continuous triangle function τ_D with D being the drastic product), and it is a τ_T-decomposable measure for an arbitrary continuous $T \in$ (i.e., for continuous triangle function τ_T);

(iii) if μ is L-decomposable with $L \in$, and $T \in$ is continuous, then γ is a $\tau_{L,T}$-decomposable measure;

(iv) if μ is K_∞-decomposable, then γ is $\Pi_\top$-decomposable measure for each $\top \in$, where $\Pi_\top$ is given by (9), i.e., γ is a Shen's $\top$-probabilistic decomposable measure (in the terminology and notation of [20]).

Further examples of probabilistic (sub)measures w.r.t. different triangle functions may be found in [6] as well as in [5].

4 Probabilistic integral: a general construction

In what follows let $\Omega^{\overline{\mathbb{R}}_+}$ be a set of all functions $f : \Omega \to \overline{\mathbb{R}}_+$. Motivating by the concept of "probabilistic integral" introduced in [2] we aim to investigate here in detail a little bit more general definition of integral

$$\int_E f \, \mathrm{d}\gamma,$$

where

- $E \in \Sigma$ with Σ being a ring of subsets of a non-empty set Ω;
- $f \in \Omega^{\overline{\mathbb{R}}_+}$ is a measurable function (in the sense described below);
- $\gamma : \Sigma \to \Delta^+$ is a τ-decomposable measure with respect to a distributive triangle function τ.

In what follows (for the purpose of this section) we always suppose that γ is a probabilistic-valued τ-decomposable measure w.r.t. to an arbitrary distributive triangle function τ and all functions are supposed to be measurable. The introduction of the γ-integral follows the pattern known from the classical Lebesgue integral: first it is defined for measurable step functions and then for non-negative measurable functions using the fact that each such functions can be approximated uniformly by a monotone sequence of step functions.

Definition 4.1. A function $f \in \Omega^{\overline{\mathbb{R}}_+}$ is called *simple*, if it attains a finite number of values $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$ on pairwise disjoint sets $E_1, E_2, \ldots, E_n \in \Sigma$. For such a function we write $f = \sum\limits_{i=1}^{n} x_i \chi_{E_i}$. A set of all simple functions from $\Omega^{\overline{\mathbb{R}}_+}$ will be denoted by .

Clearly, if f and g are simple functions and $c \in \mathbb{R}_+$, then $f + g$ and cf are simple functions as well. The simplest example of a simple function is the characteristic function of a measurable set $F \in \Sigma$, i.e.,

$$\chi_F(x) := \begin{cases} 1, & x \in F, \\ 0, & x \notin F. \end{cases}$$

Its γ-integral w.r.t. a τ-decomposable measure γ on Σ with τ being a distributive triangle function is then defined by the equality

$$\int_E \chi_F \, \mathrm{d}\gamma := \gamma_{E \cap F}.$$

For $f \in$ we put

$$\int_E f \, \mathrm{d}\gamma = \int_E \left(\sum_{i=1}^{n} x_i \chi_{E_i} \right) \mathrm{d}\gamma := \bigoplus_{i=1}^{n} x_i \odot \gamma_{E \cap E_i}.$$

Clearly, the γ-integral of a simple function is a distance distribution function. The correctness of the introduced definition of integral will be examined in the following lemma which enlightens why distributivity of a triangle function is crucial in the construction of the integral.

Lemma 4.2. *Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ be a τ-decomposable measure on Σ. If $f \in$ is of the form $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{R}_+$ and $E_i \in \Sigma$, $i = 1, \dots, n$, are pairwise disjoint sets, and $g \in$ is of the form $g = \sum_{j=1}^{m} \beta_j \chi_{F_j}$ where $\beta_j \in \mathbb{R}_+$ and $F_j \in \Sigma$, $j = 1, \dots, m$, are pairwise disjoint sets, then*

$$\int_E f \,\mathrm{d}\gamma = \int_E g \,\mathrm{d}\gamma$$

provided that $f = g$.

Proof. We may consider $\alpha_i \neq 0$ and $\beta_j \neq 0$ for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, because multiplication of a distance distribution function by zero produces the neutral element ε_0, see (4). Since E_i, resp. F_j are disjoint, then $E_i = \bigcup_{j=1}^{m} (E_i \cap F_j)$ for each $i = 1, \dots, n$ as well as $F_j = \bigcup_{i=1}^{n} (F_j \cap E_i)$ for each $j = 1, \dots, m$. Also, if $E_i \cap F_j \neq \emptyset$, then $\alpha_i = \beta_j$. Hence, $\alpha_i \odot \gamma_{E_i \cap F_j} = \beta_j \odot \gamma_{E_i \cap F_j}$ (for $E_i \cap F_j = \emptyset$ the both sides are equal to ε_0). Finally, from τ-decomposability of γ we get

$$\begin{aligned}
\int_E f \,\mathrm{d}\gamma &= \bigoplus_{i=1}^{n} \alpha_i \odot \gamma_{E \cap E_i} = \bigoplus_{i=1}^{n} \alpha_i \odot \gamma_{E \cap \left(\bigcup_{j=1}^{m} (E_i \cap F_j) \right)} = \bigoplus_{i=1}^{n} \alpha_i \odot \bigoplus_{j=1}^{m} \gamma_{E \cap (E_i \cap F_j)} \\
&= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \alpha_i \odot \gamma_{E \cap (E_i \cap F_j)} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \beta_j \odot \gamma_{E \cap (E_i \cap F_j)} = \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \beta_j \odot \gamma_{E \cap (E_i \cap F_j)} \\
&= \bigoplus_{j=1}^{m} \beta_j \odot \bigoplus_{i=1}^{n} \gamma_{E \cap (E_i \cap F_j)} = \bigoplus_{j=1}^{m} \beta_j \odot \gamma_{E \cap \left(\bigcup_{i=1}^{n} (E_i \cap F_j) \right)} = \bigoplus_{j=1}^{m} \beta_j \odot \gamma_{E \cap F_j} \\
&= \int_E g \,\mathrm{d}\gamma
\end{aligned}$$

which completes the proof. □

Because of this, the integral of a simple function w.r.t. a probabilistic-valued τ-decomposable measure γ with τ being a distributive triangle function is well-defined.

Example 4.3. Let γ^a be given by (8) for some $a \in [0, 1]$. If $f \in$, then the γ^a-integral of f on $E \in \Sigma$ is a distribution function of the form

$$\int_E f \,\mathrm{d}\gamma^a = \int_E \left(\sum_{i=1}^{n} x_i \chi_{E_i} \right) \mathrm{d}\gamma^a = r \odot \lambda^a,$$

where $r = \sum_{i=1}^{n} x_i \mu(E \cap E_i)$ with $x_i \in \mathbb{R}_+$ and E_i, $i = 1, 2, \dots, n$, being pairwise disjoint sets from Σ. The resulting form of integral follows from addition $\oplus_{\tau_M}$ of distribution functions in Menger PM-space (under the minimum t-norm M), see the equality (6). Note that τ_M is a distributive triangle function. Coefficient r may be expressed in the form $r = \int_E f \,\mathrm{d}\mu$ of the classical Lebesgue integral.

More generally, if we consider the operation $\oplus_\tau$ with $\tau = \tau_{L,M}$ given by (3) for $L \in$, the γ^a-integral of f on $E \in \Sigma$ is the function $r \odot \lambda^a$ with

$$r = \underset{i=1}{\overset{n}{}} r_i := L(r_1, L(r_2, \dots, L(r_{n-1}, r_n) \dots)), \quad r_i = x_i \, \mu(E \cap E_i).$$

Let f be a simple non-negative function on Ω. Since $\int_E 0 \,\mathrm{d}\gamma = \varepsilon_0$ and $H \leq \varepsilon_0$ for each $H \in \Delta^+$, then we immediately get $\int_E f \,\mathrm{d}\gamma \leq \varepsilon_0$. So, an interesting feature of the introduced integral of simple functions is the following "order reversing" property.

Theorem 4.4. *Let $\tau \in \mathfrak{D}(\Delta^+)$, γ be a τ-decomposable measure on Σ and $f, g \in$. If $f \leq g$ then $\int_E f \,\mathrm{d}\gamma \geq \int_E g \,\mathrm{d}\gamma$ on $E \in \Sigma$.*

Proof. Let $E \in \Sigma$ and $f, g \in$ such that $f \leq g$. Then there exist pairwise disjoint sets $E_1, \ldots, E_n \in \Sigma$ such that both functions may be expressed in the form

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \quad g = \sum_{i=1}^{n} \beta_i \chi_{E_i}$$

with $\alpha_i, \beta_i \in \mathbb{R}_+$ and $0 \leq \alpha_i \leq \beta_i < +\infty$. From monotonicity of distance distribution functions it follows $\alpha_i \odot F \geq \beta_i \odot F$ for each $F \in \Delta^+$. Hence,

$$\int_E f \, \mathrm{d}\gamma = \bigoplus_{i=1}^{n} \alpha_i \odot \gamma_{E \cap E_i} \geq \bigoplus_{i=1}^{n} \beta_i \odot \gamma_{E \cap E_i} = \int_E g \, \mathrm{d}\gamma,$$

completing the proof. □

The "order reversing" property of the introduced integral of a simple function motivates the following definition of the integral of a non-negative measurable function. Measurability will be understood in the following sense: a function $f \in \Omega^{\overline{\mathbb{R}}_+}$ is measurable w.r.t. Σ, if there exists a sequence $(f_n)_1^\infty \in$ such that $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in \Omega$. For a measurable function $f \in \Omega^{\overline{\mathbb{R}}_+}$ we denote by $_{f,E}$ the set of all simple functions $\mathfrak{f}$ with $\mathfrak{f}(t) \leq f(t)$ for each $t \in E$.

Definition 4.5. Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ be a τ-decomposable measure on Σ. We say that a function $f \in \Omega^{\overline{\mathbb{R}}_+}$ is *γ-integrable* on a set $E \in \Sigma$, if there exists $H \in \Delta^+$ such that $\int_E \mathfrak{f} \, \mathrm{d}\gamma \geq H$ for each $\mathfrak{f} \in {}_{f,E}$. In this case we put

$$\int_E f \, \mathrm{d}\gamma := \inf \left\{ \int_E \mathfrak{f} \, \mathrm{d}\gamma; \, \mathfrak{f} \in {}_{f,E} \right\}.$$

Remark 4.6. We emphasize the fact that the values of a simple function are to be finite non-negative real numbers, however a γ-integrable function is extended non-negative real-valued. Indeed, if $f(x) = +\infty$, then by Halmos [8, §20, Theorem B] $f_n(x) = n$ for every n. Thus, putting

$$\varepsilon_\infty(x) := \begin{cases} 1 & \text{for } x = +\infty, \\ 0 & \text{otherwise,} \end{cases}$$

and extending the multiplication $\odot$ by $+\infty$ as follows

$$(c \odot G)(x) := \varepsilon_\infty(x), \quad c = +\infty,$$

the integral of such a function f is obviously equal to ε_∞. This enables to integrate extended non-negative real-valued functions.

Since all properties of simple (integrable) functions and their integrals remain valid also for general integrable functions and their integrals, in theorems we shall use the following formulation: ...(simple) γ-integrable functions... On one hand, by this formulation we want to emphasize that the theorem has an importance by itself for simple functions, while on the other hand, we want to emphasize that the theorem is valid for general integrable functions. We prove them first for simple functions and at the same time we point out their proofs for general integrable functions.

Theorem 4.7. *Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ be a τ-decomposable measure on Σ. Let $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ be (simple) γ-integrable functions on the corresponding sets and put $: f \mapsto \int_\cdot f \, \mathrm{d}\gamma$. Then γ -integral is*

(i) an antimonotone operator, i.e., $(f) \geq (g)$ whenever $f \leq g$;

(ii) positively homogeneous, i.e., for each $c \in \mathbb{R}_+$ it holds $(c \cdot f) = c \odot (f)$.

Proof. (i) For simple functions $f \leq g$ the inequality $(f) \geq (g)$ follows from Theorem 4.4. Let $f, g \in \Omega^{\mathbb{R}_+}$ be such that $f \leq g$. Then

$$\left\{ \int_E \mathfrak{f} \, \mathrm{d}\gamma; \ \mathfrak{f} \in {}_{f,E} \right\} \subseteq \left\{ \int_E \mathfrak{g} \, \mathrm{d}\gamma; \ \mathfrak{g} \in {}_{g,E} \right\},$$

from it follows

$$\inf \left\{ \int_E \mathfrak{f} \, \mathrm{d}\gamma; \ \mathfrak{f} \in {}_{f,E} \right\} \geq \inf \left\{ \int_E \mathfrak{g} \, \mathrm{d}\gamma; \ \mathfrak{g} \in {}_{g,E} \right\}.$$

This implies the required inequality.

(ii) Let $c \in \mathbb{R}_+$. For a simple function f of the form $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ the function $c \cdot f$ is also simple of the form $c \cdot f = \sum_{i=1}^{n} c x_i \chi_{E_i}$. Then

$$\int_E c \cdot f \, \mathrm{d}\gamma = \bigoplus_{i=1}^{n} c x_i \odot \gamma_{E \cap E_i} = \bigoplus_{i=1}^{n} c \odot (x_i \odot \gamma_{E \cap E_i}) = c \odot \bigoplus_{i=1}^{n} x_i \odot \gamma_{E \cap E_i} = c \odot \int_E f \, \mathrm{d}\gamma,$$

where the third equality is due to distributivity of τ. Then for a general γ-integrable function f we get

$$\int_E c \cdot f \, \mathrm{d}\gamma = \inf \left\{ \int_E c \cdot \mathfrak{f} \, \mathrm{d}\gamma; \ \mathfrak{f} \in {}_{f,E} \right\} = c \odot \inf \left\{ \int_E \mathfrak{f} \, \mathrm{d}\gamma; \ \mathfrak{f} \in {}_{f,E} \right\} = c \odot \int_E f \, \mathrm{d}\gamma,$$

which completes the proof. □

Now we examine the γ-integral with respect to the structure of measures as distance distribution functions: homogeneity and monotonicity.

Theorem 4.8. *Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ^i be τ-decomposable measures on Σ with $i \in \{1, 2\}$. If $f \in \Omega^{\overline{\mathbb{R}}_+}$ is a (simple) γ^i-integrable function on $E \in \Sigma$ for each $i \in \{1, 2\}$, then*

(i) for each $c \in \mathbb{R}_+$ the function f is $c \odot \gamma^1$-integrable on $E \in \Sigma$, and it holds

$$\int_E f \, \mathrm{d}(c \odot \gamma^1) = c \odot \int_E f \, \mathrm{d}\gamma^1;$$

(ii) if $\gamma^1_E \leq \gamma^2_E$, then

$$\int_E f \, \mathrm{d}\gamma^1 \leq \int_E f \, \mathrm{d}\gamma^2.$$

Proof. First observe that if γ^i are τ-decomposable measures on Σ for $i \in \{1, 2\}$, then by [5, Theorem 4.2] the set function $\gamma := c \odot \gamma^1$, $c \in \mathbb{R}_+$ is a τ-decomposable measure on Σ.

(i) Since for a simple function $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ we get

$$\int_E f \, \mathrm{d}\gamma = \bigoplus_{i=1}^{n} x_i \odot \left(c \odot \gamma^1_{E \cap E_i} \right) = \bigoplus_{i=1}^{n} c x_i \odot \gamma^1_{E \cap E_i} = \int_E c \cdot f \, \mathrm{d}\gamma^1 = c \odot \int_E f \, \mathrm{d}\gamma^1,$$

where the last equality follows from Theorem 4.7(ii), for a general non-negative function f the result holds by Theorem 4.7(ii) as well.

(ii) From monotonicity of operation $\oplus_\tau$ for a simple function $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ it holds

$$\int_E f \, \mathrm{d}\gamma^1 = \bigoplus_{i=1}^{n} x_i \odot \gamma^1_{E \cap E_i} \leq \bigoplus_{i=1}^{n} x_i \odot \gamma^2_{E \cap E_i} = \int_E f \, \mathrm{d}\gamma^2.$$

If f is a non-negative integrable function, then

$$\int_E f \,\mathrm{d}\gamma^1 = \inf\left\{\int_E \mathfrak{f}\,\mathrm{d}\gamma^1;\ \mathfrak{f} \in {}_{f,E}\right\} \leq \inf\left\{\int_E \mathfrak{f}\,\mathrm{d}\gamma^2;\ \mathfrak{f} \in {}_{f,E}\right\} = \int_E f \,\mathrm{d}\gamma^2,$$

which completes the proof. □

Similarly as in the Lebesgue measure and integral theory we introduce sets which may be "neglected".

Definition 4.9. Let $\gamma : \Sigma \to \Delta^+$ be a probabilistic-valued set function. A set $E \in \Sigma$ is said to be γ-*null*, if $\gamma_E = \varepsilon_0$. A set of all γ-null sets will be denoted by ${}_\gamma$.

Clearly, for each τ-decomposable (sub)measure γ w.r.t. a triangle function τ the set ${}_\gamma$ is always non-empty, because $\emptyset \in {}_\gamma$. Also, the union of γ-null sets is a γ-null set. Indeed, if E_i are γ-null sets, then

$$\gamma_{\cup_{i=1}^n E_i} = \bigoplus_{i=1}^n \gamma_{E_i} = \bigoplus_{i=1}^n \varepsilon_0 = \varepsilon_0.$$

From antimonotonicity of a τ-decomposable measure it follows that each subset of γ-null set is again a γ-null set.

Example 4.10. Put $\gamma_E = \varepsilon_{\mu(E)}$. If $\Omega = \mathbb{N}$, Σ contains of all finite subsets of Ω and μ is a counting measure on Σ, then the only γ-null set is empty-set. On the other hand, for the Lebesgue measure μ on a ring Σ the set ${}_\gamma$ contains of all Lebesgue μ-measurable null sets from Σ.

Example 4.11. Put $\gamma_E^\Phi = \mu(E) \odot \Phi$ for an arbitrary $\Phi \in \Delta^+$. For $\Phi = \varepsilon_0$ all the sets from a ring Σ of subsets of $\Omega \neq \emptyset$ belong to ${}_{\gamma^\Phi}$ for any finite measure μ on Σ.

Theorem 4.12. *Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ be a τ-decomposable measure on Σ. If $E \in {}_\gamma$, then $\int_E f \,\mathrm{d}\gamma = \varepsilon_0$ for any (simple) γ-integrable function $f \in \Omega^{\overline{\mathbb{R}}_+}$.*

Proof. For a simple function of the form $f = \sum\limits_{i=1}^n x_i \chi_{E_i}$ we have

$$\int_E f \,\mathrm{d}\gamma = \int_E \left(\sum_{i=1}^n x_i \chi_{E_i}\right) \mathrm{d}\gamma = \bigoplus_{i=1}^n x_i \odot \gamma_{E \cap E_i}.$$

By antimonotonicity of γ and $E \cap E_i \subseteq E$ we get $\varepsilon_0 \geq \gamma_{E \cap E_i} \geq \gamma_E = \varepsilon_0$. Thus, $\gamma_{E \cap E_i} = \varepsilon_0$, and then

$$\int_E f \,\mathrm{d}\gamma = \bigoplus_{i=1}^n x_i \odot \gamma_{E \cap E_i} = \bigoplus_{i=1}^n x_i \odot \varepsilon_0 = \varepsilon_0.$$

For a general non-negative γ-integrable function $f \in \Omega^{\overline{\mathbb{R}}_+}$ there is a function $H \in \Delta^+$ such that $\int_E \mathfrak{f}\,\mathrm{d}\gamma \geq H$ for each $\mathfrak{f} \in {}_{f,E}$ and $\int_E f \,\mathrm{d}\gamma = \inf\{\int_E \mathfrak{f}\,\mathrm{d}\gamma;\ \mathfrak{f} \in {}_{f,E}\}$. Since $\int_E \mathfrak{f}\,\mathrm{d}\gamma = \varepsilon_0$ for each $\mathfrak{f} \in {}_{f,E}$, then $\int_E f \,\mathrm{d}\gamma = \varepsilon_0$. □

Corollary 4.13. Let $\tau \in \mathfrak{D}(\Delta^+)$ and γ be a τ-decomposable measure on Σ. If ${}_\gamma = \Sigma$, then $\int_E f \,\mathrm{d}\gamma = \varepsilon_0$ for any (simple) γ-integrable function $f \in \Omega^{\overline{\mathbb{R}}_+}$.

5 Integral as a limit of integrals of simple functions

In this section consider a σ-ring Σ of subsets of $\Omega \neq \emptyset$. Motivated by Halmos [8, §20, Theorem B], each non-negative (extended) real-valued function f is the pointwise limit of a sequence $(f_n)_1^\infty$ of non-decreasing simple functions, it is possible to define integral via uniform approximation of a non-negative function by a sequence of pointwise converging monotone sequence of simple functions. However, we have to assume here that a triangle function τ is continuous

and a probabilistic-valued measure γ is continuous from below (or, it is sufficient to consider its σ-additivity, see Lemma 3.4). In what follows $\mathfrak{CD}(\Delta^+)$ is the set of all continuous distributive triangle functions on Δ^+.

Lemma 5.1. *Let $\tau \in \mathfrak{CD}(\Delta^+)$, γ be a τ-decomposable measure on Σ continuous from below and $f \in$. If $E_n \nearrow E$, then*

$$\lim_{n\to\infty} \int_{E_n} f \,\mathrm{d}\gamma = \int_E f \,\mathrm{d}\gamma.$$

Proof. Let the simple function f be of the form $\sum\limits_{i=1}^{m} x_i \cdot \chi_{G_i}$, where $x_i \in [0, +\infty)$, and $G_i \in \Sigma$ are pairwise disjoint sets. From Lemma 3.4 and from continuity of triangle function τ it follows

$$\lim_{n\to\infty} \int_{E_n} f \,\mathrm{d}\gamma = \lim_{n\to\infty} \int_{E_n} \left(\sum_{i=1}^{m} x_i \cdot \chi_{G_i} \right) \mathrm{d}\gamma = \lim_{n\to\infty} \bigoplus_{i=1}^{m} x_i \odot \gamma_{E_n \cap G_i} = \bigoplus_{i=1}^{m} \lim_{n\to\infty} x_i \odot \gamma_{E_n \cap G_i}$$
$$= \bigoplus_{i=1}^{m} x_i \odot \left(\lim_{n\to\infty} \gamma_{E_n \cap G_i} \right) = \bigoplus_{i=1}^{m} x_i \odot \gamma_{E \cap G_i} = \int_E f \,\mathrm{d}\gamma,$$

which completes the proof. □

Now we are able to prove that the γ-integral of a measurable function f may be written as limit of integrals of simple functions f_n pointwisely converging to f.

Theorem 5.2. *Let $\tau \in \mathfrak{CD}(\Delta^+)$, γ be a τ-decomposable measure on Σ continuous from below, and $f \in \Omega^{\overline{\mathbb{R}}_+}$ be a γ-integrable function on a set $E \in \Sigma$. Then there exists a non-decreasing sequence $(f_n)_1^\infty \in {}_{f,E}$ converging pointwisely to f such that*

$$\int_E f \mathrm{d}\gamma = \lim_{n\to\infty} \int_E f_n \mathrm{d}\gamma.$$

Proof. By Halmos [8, §20, Theorem B] each non-negative (extended) real-valued function f is the pointwise limit of a sequence $(f_n)_1^\infty$ of non-decreasing simple functions. Since $\left(\int_E f_n \,\mathrm{d}\gamma\right)_1^\infty$ is a non-increasing sequence of distance distribution functions, which is bounded below by distance distribution function $\int_E f \mathrm{d}\gamma$, existence of the limit follows. From monotonicity of integral we immediately get the inequality $\lim\limits_{n\to\infty} \int_E f_n \mathrm{d}\gamma \geq \int_E f \mathrm{d}\gamma$. Thus, it suffices to show the reverse inequality.

On the set $E \in \Sigma$ consider an arbitrary simple function $\mathfrak{f} \in$ such that $\mathfrak{f} \leq f$. For a fixed $t \in (0,1)$ put $E_n := \{x \in E; f_n(x) \geq t \cdot \mathfrak{f}(x)\}$. Clearly, $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \ldots$ and $\bigcup\limits_{n=1}^{\infty} E_n = E$. Thus,

$$\int_E f_n \mathrm{d}\gamma \leq \int_{E_n} f_n \mathrm{d}\gamma \leq \int_{E_n} t \cdot \mathfrak{f} \,\mathrm{d}\gamma = t \odot \int_{E_n} \mathfrak{f} \,\mathrm{d}\gamma,$$

where the last equality is due to positive homogeneity of the integral, see Theorem 4.7. Then Lemma 5.1 yields $\lim\limits_{n\to\infty} \int_{E_n} \mathfrak{f} \,\mathrm{d}\gamma = \int_E \mathfrak{f} \,\mathrm{d}\gamma$. Therefore,

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma \leq \lim_{n\to\infty} t \odot \int_{E_n} \mathfrak{f} \,\mathrm{d}\gamma = t \odot \lim_{n\to\infty} \int_{E_n} \mathfrak{f} \,\mathrm{d}\gamma = t \odot \int_E \mathfrak{f} \,\mathrm{d}\gamma.$$

Since $t \in (0,1)$, then $\lim\limits_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma \leq \int_E \mathfrak{f} \,\mathrm{d}\gamma$. Taking the infimum of all such simple functions we get

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma \leq \inf \left\{ \int_E \mathfrak{f} \,\mathrm{d}\gamma;\ \mathfrak{f} \in {}_{f,E} \right\} = \int_E f \,\mathrm{d}\gamma.$$

□

The following theorem examines the linearity of the integral **which depends on a choice of triangle function τ.**

Theorem 5.3. *Let $\tau \in \mathfrak{CD}(\Delta^+)$ and γ be a τ-decomposable measure on Σ continuous from below. If $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ are (simple) γ-integrable functions on $E \in \Sigma$ and $: f \mapsto \int_{\cdot} f \,\mathrm{d}\gamma$, then the γ-integral is $\oplus_\tau$ -linear, i.e., $(f+g) = (f) \oplus_\tau (g)$ if and only if $\tau = \tau_M$. In general, $(f+g) \geq (f) \oplus_\tau (g)$ if and only if $\tau \geq \tau_M$.*

Proof. Let $f, g \in$. Then there exist pairwise disjoint sets $B_1, \ldots, B_p \in \Sigma$ such that both functions may be expressed in the form $f = \sum\limits_{k=1}^{p} x_k \chi_{B_k}$, $g = \sum\limits_{k=1}^{p} y_k \chi_{B_k}$. Thus, the function $f+g$ is also a simple function of the form $f+g = \sum\limits_{k=1}^{p} z_k \chi_{B_k}$ with $z_k = x_k + y_k$, $k = 1, 2, \ldots, p$.

(a) Let us consider $\tau = \tau_M$. Then

$$\int_E (f+g)\,\mathrm{d}\gamma = \bigoplus_{k=1}^{p} (x_k + y_k) \odot \gamma_{E \cap B_k} = \left(\bigoplus_{k=1}^{p} x_k \odot \gamma_{E \cap B_k} \right) \oplus_\tau \left(\bigoplus_{k=1}^{p} y_k \odot \gamma_{E \cap B_k} \right)$$
$$= \int_E f \,\mathrm{d}\gamma \oplus_\tau \int_E g \,\mathrm{d}\gamma,$$

where the second equality holds if and only if $\tau = \tau_M$, see Example 3.5. Now we show that $f+g$ is γ-integrable on $E \in \Sigma$ provided that $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ are γ-integrable on $E \in \Sigma$. By definition there exist $\varphi \in \Delta^+$ and $\psi \in \Delta^+$ such that $\int_E \mathfrak{f} \,\mathrm{d}\gamma \geq \varphi$ for each $\mathfrak{f} \in {}_{f,E}$ and $\int_E \mathfrak{g} \,\mathrm{d}\gamma \geq \psi$ for each $\mathfrak{g} \in {}_{g,E}$, respectively. Then $\mathfrak{f} + \mathfrak{g} \in {}_{f+g,E}$ and

$$\int_E (\mathfrak{f} + \mathfrak{g})\,\mathrm{d}\gamma = \int_E \mathfrak{f} \,\mathrm{d}\gamma \oplus_\tau \int_E \mathfrak{g} \,\mathrm{d}\gamma \geq \varphi \oplus_\tau \psi$$

for each $\mathfrak{f} + \mathfrak{g} \in {}_{f+g,E}$, i.e., $f+g$ is γ-integrable on $E \in \Sigma$. Since $\tau = \tau_M$ is continuous, because M is a continuous t-norm, by Theorem 5.2 there exist non-decreasing sequences $(f_n)_1^\infty \in {}_{f,E}$ and $(g_n)_1^\infty \in {}_{g,E}$ pointwisely converging to f and g, respectively, such that

$$\int_E (f+g)\,\mathrm{d}\gamma = \lim_{n\to\infty} \int_E (f_n + g_n)\,\mathrm{d}\gamma = \lim_{n\to\infty} \left(\int_E f_n \,\mathrm{d}\gamma \bigoplus \int_E g_n \,\mathrm{d}\gamma \right)$$
$$= \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma \bigoplus \lim_{n\to\infty} \int_E g_n \,\mathrm{d}\gamma = \int_E f \,\mathrm{d}\gamma \bigoplus \int_E g \,\mathrm{d}\gamma.$$

(b) If $\tau \geq \tau_M$, then for $f, g \in$ we get

$$\int_E (f+g)\,\mathrm{d}\gamma \geq \left(\bigoplus_{k=1}^{p} x_k \odot \gamma_{E \cap B_k} \right) \oplus_\tau \left(\bigoplus_{k=1}^{p} y_k \odot \gamma_{E \cap B_k} \right) = \int_E f \,\mathrm{d}\gamma \oplus_\tau \int_E g \,\mathrm{d}\gamma,$$

where $\oplus_\tau$-suplinearity of γ-integral of simple functions (the first inequality) is equivalent to $\tau \geq \tau_M$, see Example 3.5. The rest of the proof is similar to the part (a). □

We illustrate the $\oplus_{\tau_M}$-linearity property of the integral considering a class of probabilistic measures from Γ_A.

Example 5.4. Let $\gamma^{a_i} \in \Gamma_A$ be probabilistic measures defined on pairwise disjoint sets $E_i \in \Sigma$, where $i = 1, 2, \ldots, n$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then the integral of function $f = x_0 \chi_E$ for $x_0 \in \mathbb{R}_+$ (a constant function on the set $E \in \Sigma$) computed on the sets $E_1, E_2, \ldots, E_n$ is a function given by

$$H = \bigoplus_{i=1}^{n} \int_{E_i} f \,\mathrm{d}\gamma^{a_i} = a_1 \chi_{]0,r_1]} + a_2 \chi_{]r_1,r_2]} + \ldots a_n \chi_{]r_{n-1},r_n]} + \chi_{]r_n,+\infty]},$$

where

$$r_i = s_1 + \cdots + s_i \quad \text{a} \quad s_i = \sum_{k=1}^{i} x_0 \mu(E \cap E_k) = \sum_{k=1}^{i} \int_{E_k} f \mathrm{d}\mu.$$

Graph of this function is illustrated in Fig. 2.

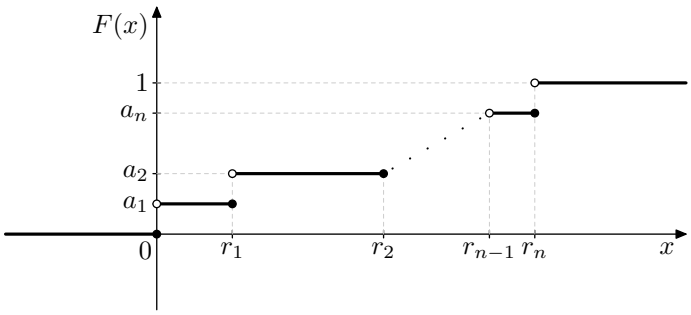

Figure 2: Result of integration from Example 5.4

In accordance with [2] for a function $G \in \Delta^+$ and a constant $a \in [0,1]$ introduce the notation

$$G_a^- := \inf\{x \in \mathbb{R};\ a \le G(x)\},$$
$$G_a^+ := \sup\{x \in \mathbb{R};\ a \ge G(x)\}.$$

Then the distance distribution function H which is the result of integration in Example 5.4 may be written as

$$H = \bigoplus_{i=1}^{n} \int_{E_i} f \,\mathrm{d}\gamma^{a_i} = \sum_{a \in A} a \chi_{]H_a^-, H_a^+]}.$$

The resulting form follows from observation that if $a = a_i$ for some $i = 1, 2, \ldots, n$, then $H_a^- < H_a^+$, otherwise $H_a^- = H_a^+$.

Theorem 5.5. *Let $\tau \in \mathfrak{CD}(\Delta^+)$ and γ be τ-decomposable measures on Σ continuous from below with $i \in \{1,2\}$. If $f \in \Omega^{\overline{\mathbb{R}}_+}$ is a (simple) γ^i-integrable function on $E \in \Sigma$ for $i \in \{1,2\}$, then f is $\gamma^1 \oplus_\tau \gamma^2$-integrable on $E \in \Sigma$, and it holds*

$$\int_E f \,\mathrm{d}(\gamma^1 \oplus_\tau \gamma^2) = \int_E f \,\mathrm{d}\gamma^1 \oplus_\tau \int_E f \,\mathrm{d}\gamma^2.$$

Proof. First observe that if γ^i are τ-decomposable measures on Σ for $i \in \{1,2\}$, then by [5, Theorem 4.2] the set function $\zeta := \gamma^1 \oplus_\tau \gamma^2$ is τ-decomposable measure on Σ as well.

Let f be a simple function of the form $f = \sum_{i=1}^{n} x_i \chi_{E_i}$. Then distributivity and associativity of τ yields

$$\begin{aligned}\int_E f \,\mathrm{d}\zeta &= \bigoplus_{i=1}^{n} x_i \odot (\gamma^1_{E\cap E_i} \oplus_\tau \gamma^2_{E\cap E_i}) = \bigoplus_{i=1}^{n} (x_i \odot \gamma^1_{E\cap E_i}) \oplus_\tau (x_i \odot \gamma^2_{E\cap E_i}) \\ &= \left(\bigoplus_{i=1}^{n} x_i \odot \gamma^1_{E\cap E_i}\right) \oplus_\tau \left(\bigoplus_{i=1}^{n} x_i \odot \gamma^2_{E\cap E_i}\right) = \int_E f \,\mathrm{d}\gamma^1 \oplus_\tau \int_E f \,\mathrm{d}\gamma^2.\end{aligned}$$

Let us show that it holds for a non-negative integrable function f. By Theorem 5.2 there is a non-decreasing sequence $(f_n)_1^\infty \in_{f,E}$ pointwisely converging to f such that

$$\begin{aligned}\int_E f \,\mathrm{d}\zeta &= \lim_{n\to\infty} \left(\int_E f_n \,\mathrm{d}\gamma^1 \bigoplus \int_E f_n \,\mathrm{d}\gamma^2\right) = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma^1 \bigoplus \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\gamma^2 \\ &= \int_E f \,\mathrm{d}\gamma^1 \bigoplus \int_E f \,\mathrm{d}\gamma^2,\end{aligned}$$

where the continuity of $\oplus_\tau$ has been used. □

As in the case of the Lebesgue integral, the γ-integral can be seen as an extension of a τ-decomposable measure.

Theorem 5.6. *Let $\tau \in \mathfrak{CD}(\Delta^+)$ and γ be a τ-decomposable measure on Σ continuous from below. For each (simple) γ-integrable function $f \in \Omega^{\overline{\mathbb{R}}_+}$ the set function $\nu^f : \Sigma \to \Delta^+$ defined by*

$$\nu^f_E := \int_E f \,\mathrm{d}\gamma, \quad E \in \Sigma,$$

is a τ-decomposable measure.

Proof. Clearly, $\nu^f_\emptyset = \varepsilon_0$. Let $E, F \in \Sigma$ be disjoint sets. For a simple function $f = \sum\limits_{i=1}^{n} x_i \chi_{E_i}$ we write

$$\begin{aligned}
\nu^f_{E\cup F} &= \int_{E\cup F} f \,\mathrm{d}\gamma = \bigoplus_{i=1}^{n} x_i \odot \gamma_{(E\cup F)\cap E_i} = \bigoplus_{i=1}^{n} x_i \odot \gamma_{(E\cap E_i)\cup(F\cap E_i)} \\
&= \bigoplus_{i=1}^{n} x_i \odot (\gamma_{E\cap E_i} \oplus_\tau \gamma_{F\cap E_i}) = \left(\bigoplus_{i=1}^{n} x_i \odot \gamma_{E\cap E_i}\right) \oplus_\tau \left(\bigoplus_{i=1}^{n} x_i \odot \gamma_{F\cap E_i}\right) \\
&= \nu^f_E \oplus_\tau \nu^f_F.
\end{aligned}$$

For a general γ-integrable function f by Theorem 5.2 there exists a non-decreasing sequence $(f_n)_1^\infty \in {}_{f,E\cup F}$ such that

$$\nu^f_{E\cup F} = \lim_{n\to\infty} \nu^{f_n}_{E\cup F} = \lim_{n\to\infty} \left(\nu^{f_n}_E \oplus_\tau \nu^{f_n}_F\right) = \left(\lim_{n\to\infty} \nu^{f_n}_E\right) \oplus_\tau \left(\lim_{n\to\infty} \nu^{f_n}_F\right) = \nu^f_E \oplus_\tau \nu^f_F,$$

which completes the proof. □

Remark 5.7. According to Proposition 3.2(ii) the γ-integral is antimonotone, i.e., the inequality $\nu^f_E \geq \nu^f_F$ holds for each (simple) γ-integrable function f and each $E, F \in \Sigma$ such that $E \subseteq F$. Moreover, the set function ν^f and the operator are related by the formula $\nu^f_E = (f \cdot \chi_E)$. Theorem 5.6 shows that the γ-integral is indeed a proper extension of the underlying measure γ because of $\nu^{\chi_E} = \gamma_E$ for each $E \in \Sigma$.

Problem 5.8. *Under which conditions on a probabilistic-valued set function γ, triangle function τ and functions $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ the following equalities hold*

$$\int_E f \,\mathrm{d}\nu^g = \int_E g \,\mathrm{d}\nu^f = \int_E f \cdot g \,\mathrm{d}\gamma, \quad E \in \Sigma?$$

Definition 5.9. Functions $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ are said to be *equal a.e.* on a set S, we write $f = g$ a.e., if $f(x) = g(x)$ for all $x \in S \setminus E$, where $E \in {}_\gamma$.

Theorem 5.10. *Let $\tau \in \mathfrak{CD}(\Delta^+)$, γ be a τ-decomposable measure on Σ continuous from below, and $f, g \in \Omega^{\overline{\mathbb{R}}_+}$ such that $f = g$ a.e. on a set $E \in \Sigma$. Then $\int_E f \,\mathrm{d}\gamma = \int_E g \,\mathrm{d}\gamma$.*

Proof. Let F be a γ-null set and $f(x) = g(x)$ for all $x \in E \setminus F$. Since by Theorem 5.6 the γ-integral is a τ-decomposable measure, then

$$\begin{aligned}
\int_E f \,\mathrm{d}\gamma &= \int_{(E\setminus F)\cup F} f \,\mathrm{d}\gamma = \int_{(E\setminus F)} f \,\mathrm{d}\gamma \oplus_\tau \int_F f \,\mathrm{d}\gamma = \int_{E\setminus F} f \,\mathrm{d}\gamma = \int_{E\setminus F} g \,\mathrm{d}\gamma = \int_{E\setminus F} g \,\mathrm{d}\gamma \oplus \varepsilon_0 \\
&= \int_{(E\setminus F)} g \,\mathrm{d}\gamma \oplus \int_F g \,\mathrm{d}\gamma = \int_E g \,\mathrm{d}\gamma,
\end{aligned}$$

which completes the proof. □

Concluding remarks

Origin of probabilistic-valued set functions comes from the fact that they work in such situations in which we have only a *probabilistic information* about measure of a set (recall a similar situation in the framework of information measures as discussed in [13]). For example, in Moore's interval analysis, if rounding of reals is considered, then the uniform distributions over intervals describe our information about the measure of a set. Another such probabilistic information occurs in biometric decision-making where the information is often obtained from biometric sensors as well as from the analysis of historical data. In this paper we have discussed an approach to the investigation of a Lebesgue-type integral w.r.t. probabilistic-valued set functions. Although the introduced probabilistic integral is based on a τ-decomposable measure, all the results may easily be rewritten for a τ-decomposable submeasure leading to a certain non-additive integral. Some further research of the integral will be concentrated on the problems of convergences for sequences of measurable functions, convergences of the corresponding probabilistic-valued integrals as well as their possible applications in different mathematical theories.

The applicability of our integral in approximate reasoning will be illustrated on a simple example. Consider a universe $\Omega = \{1, \dots, n\}$ of criteria (and then $\Sigma = 2^\Omega$ is the power set of Ω). Consider a fixed distributive triangle function τ. To define properly a probabilistic-valued $\oplus$-decomposable measure (where $\oplus$ is defined by τ), it is enough to have the knowledge about measures of singletons of Ω. For each such singleton $E_i = \{i\}$, let γ_{E_i} be a distance distribution function of a random variable uniformly distributed over non-negative interval $[a_i, b_i]$, i.e., $\gamma_{E_i} = F_{[a_i,b_i]}$. For any alternative $\mathbf{a}$, its score vector $\mathbf{x}$ in $[0,1]^n$ is assigned. Now, we can apply our integral to determine a probabilistic-valued utility function,

$$U(\mathbf{a}) = \int \mathbf{x} \, \mathrm{d}\gamma.$$

For the sake of simplicity, we illustrate only one distinguished case: let $T = M$ be the strongest minimum t-norm, and let $\tau = \tau_M$. Then τ is a continuous distributive distance distribution function, and

$$U(\mathbf{a}) = \int \mathbf{x} \, \mathrm{d}\gamma = F_{[\alpha,\beta]},$$

where $\alpha = \sum x_i \cdot a_i$ and $\beta = \sum x_i \cdot b_i$. As we can see, in this case we have recovered standard Moore's weighted arithmetic mean with interval-valued means.

It is worth to be mentioned that a different approach to probabilistic-integral may be provided, e.g. via the Choquet one. For a partition $D = \{a_0, a_1, \dots, a_n\}$ with $0 = a_0 < a_1 < \dots < a_n < +\infty$ we define a corresponding simple function $f_D : \Omega \to [0, +\infty[$ as follows

$$f_D = \sum_{i=1}^{n} (a_i - a_{i-1}) \chi_{E_i},$$

where $E_i = \{x \in \Omega; f_D(x) \geq a_i\}$. Then a Choquet-like integral of a simple function f_D on a set $E \in \Sigma$ w.r.t. a τ-decomposable measure is defined by the equality

$$(C) \int_E f_D \, \mathrm{d}\gamma := \bigoplus_{i=1}^{n} (a_i - a_{i-1}) \odot \gamma_{E \cap E_i}.$$

The integral of a non-negative measurable function f may be defined as

$$(C) \int_E f \, \mathrm{d}\gamma = \lim_{D \in} (C) \int_E f_D \, \mathrm{d}\gamma$$

provided that the limit (in the sense of Moore-Smith) exists. Here, is the set of all partitions of $\mathbb{R}_+$ with the order induced by inclusion. Note that this approach may be understood as a modification of Aumann integral, or, of Choquet-type integral based on interval-valued measures as discussed e.g. in [11].

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Lenka Halčinová, Ondrej Hutník
Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University in Košice,
Current address: Jesenná 5, SK 040 01 Košice, Slovakia,
E-mail addresses: lenka.halcinova@student.upjs.sk
ondrej.hutnik@upjs.sk