Probabilistic-valued decomposable set functions with respect to triangle functions

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Abstract. Real world applications often require dealing with the situations in which the exact numerical values of the (sub)measure of a set may not be provided, but at least some probabilistic assignment still could be done. Also, several concepts in uncertainty processing are linked to the processing of distribution functions. In the framework of generalized measure theory we introduce the probabilistic-valued decomposable set functions which are related to triangle functions as natural candidates for the "addition" in an appropriate probabilistic metric space. Several set functions, among them the classical (sub)measures, previously defined τ_T-submeasures, $\tau_{L,A}$-submeasures as well as recently introduced Shen's (sub)measures are described and investigated in a unified way. Basic properties and characterizations of τ-decomposable (sub)measures are also studied and numerous extensions of results from the above mentioned papers are provided.

1 Introduction

Consider a grant agency providing a financial support for research in some area. From the set of all grant applications only the "successful" ones (depending on some internal rules of agency) will receive certain amount of money. So, we have only a probabilistic information about the measure of the set of "successful" grant applications. Of course, the knowledge of this information depends on many different aspects: the total budget of money to be divided, the internal rules of the agency, the quality of reviewers (if any), etc. Further examples are provided by lotteries, or guessing results when we have a probabilistic information about the (counting) measure of possibilities to win the prize. A closely related concept can be found in Moore's interval mathematics [16], where the use of intervals in data processing is due to measurement inaccuracy and due to rounding. Here intervals can be considered in distribution function form linked to random variables uniformly distributed over the relevant intervals. These model examples resemble the original idea of Menger of probabilistic metric spaces. In his work, Menger proposed to replace a positive number by a distance distribution function. This fact was motivated by thinking of situations where the exact distance between two objects may not be provided, but some probability assignment is still possible. Thus, the importance/diameter/measure of a set might be represented by a distance distribution function. Recently, probabilistic approaches were successfully applied to modelling uncertain logical arguments [10], to approximations of incomplete data [6], to inference rules playing an important role in non-monotonic reasoning [4], or to cluster structure ensemble [25].

In the paper by Hutník and Mesiar [11] the notion of *τ_T-submeasure* was defined as a certain (non-additive) set function γ on a ring Σ of subsets of a non-empty set Ω, taking values in the set Δ^+ of distribution functions of non-negative random variables satisfying $\gamma_\emptyset = \varepsilon_0$, the "antimonotonicity" property $\gamma_E \geq \gamma_F$ whenever $E, F \in \Sigma$ with $E \subseteq F$, and "subadditivity" property of the form

$$\gamma_{E\cup F}(x+y) \geq T(\gamma_E(x), \gamma_F(y)), \quad E, F \in \Sigma, x, y > 0, \tag{1}$$

with T being a left-continuous t-norm. Here, ε_0 is the distribution function of Dirac random variable concentrated at point 0. As is shown in [11], such τ_T-submeasures can be seen as fuzzy number-valued submeasures. In this case the value γ_E may be seen as a non-negative LT-fuzzy number, see [3], where $\tau_T(\gamma_E, \gamma_F)$ corresponds to the T-sum of fuzzy numbers γ_E and γ_F. Also,

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each τ_M-submeasure γ with the minimum t-norm $M(x,y)=\min\{x,y\}$ can be represented by means of a non-decreasing system $(\eta_\alpha)_{\alpha\in[0,1]}$ of numerical submeasures as follows

$$\gamma_E(x)=\sup\big\{\alpha\in[0,1];\,\eta_\alpha(E)\le x\big\},\quad E\in\Sigma.$$

This representation resembles the horizontal representation $(S_\alpha)_{\alpha\in[0,1]}$ of a fuzzy subset S.

The study of probabilistic-valued set functions continued in papers [8] and [9], where a more general concept has been used. In fact, in [9] a generalization of τ_T-submeasures was suggested involving suitable operations L, which replace the standard addition $+$ on $\overline{\mathbb{R}}_+$. On the other hand, since t-norms are rather special operations on the unit interval $[0,1]$, the paper [8] deals with a number of possible generalizations based on aggregation functions in general, studying certain properties of the corresponding probabilistic (sub)measures and their (sub)measure spaces.

Recently, in [23] Shen has defined and has studied a class of probabilistic (sub)measures which does not seem to fit into the concept of the above mentioned results of Hutník and Mesiar. In fact, Shen's definition of a probabilistic-valued $\top$-decomposable supmeasure[2] $\mathfrak{M}:\Sigma\to\Delta^+$ with the "subadditivity" property

$$\mathfrak{M}_{E\cup F}(t)\ge\top(\mathfrak{M}_E(t),\mathfrak{M}_F(t)),\quad E,F\in\Sigma,t>0,\tag{2}$$

with $\top$ being a t-norm, corresponds to the notion of $\tau_{\max,T}$-submeasure defined in [9], however deeper contextual understanding was still unclear from that paper. Thus, in this paper we provide an even deeper insight into all the mentioned notions which are special cases of a *probabilistic-valued set function with respect to a triangle function*. Recall that a triangle function τ is a binary operation on Δ^+ such that the triple $(\Delta^+,\tau,\le)$ forms a commutative, partially ordered semigroup with neutral element ε_0.

More precisely, the notion of τ_T-submeasure is related to the triangle function $\tau=\tau_T$ given by

$$\tau_T(G,H)(x)=\sup_{u+v=x}T(G(u),H(v)),\quad G,H\in\Delta^+\tag{3}$$

with T being a left-continuous t-norm. Thus, the "subadditivity" property (1) resembles the "probabilistic analogue" of the triangle inequality in the Menger probabilistic metric space (under T), see [22]. Moreover, $\tau_{L,T}$-submeasures defined in [9, Definition 2.3] are related to the (triangle) function

$$\tau_{L,T}(G,H)(x)=\sup_{L(u,v)=x}T(G(u),H(v)),\quad G,H\in\Delta^+,\tag{4}$$

with a suitable operation L on $\overline{\mathbb{R}}_+$. What is more, Shen's considerations are related to the pointwisely defined (triangle) function

$$\tau_\top(G,H)(t)=\top(G(t),H(t)),\quad G,H\in\Delta^+,$$

and the "subadditivity" property (2) is related to the triangle inequality of the corresponding probabilistic metric space. So, we can see that triangle functions are the main ingredients connecting all the mentioned notions of (sub)measure. Thus, considering a general triangle function τ on Δ^+, we define and study certain properties of τ-decomposable (sub)measures on a ring Σ of subsets of $\Omega\ne\emptyset$ in this general setting.

In the next section, the short overview of basic notions and definitions is given. In Section 3, we introduce the basic object of our study: a τ-decomposable set function with values in distance distribution functions and provide a number of concrete examples. Several properties of such set functions are then investigated in Section 4 and the results related to the probabilistic Hausdorff distance are provided in Section 6 generalizing the recent results of Shen [23].

[2] *supmeasure* in the terminology of Shen corresponds to *submeasure* in our terminology, see [23, Definition 4.1(v)]

2 Basic notions and definitions

In order to make the exposition self-contained, we remind the reader the basic notions and constructions used in this paper.

Distribution functions Let Δ be the family of all distribution functions on the extended real line $\overline{\mathbb{R}} := [-\infty, +\infty]$, i.e., $F : \overline{\mathbb{R}} \to [0,1]$ is non-decreasing, left continuous on the real line $\mathbb{R}$ with $F(-\infty) = 0$ and $F(+\infty) = 1$. A *distance distribution function* is a distribution function $F : \overline{\mathbb{R}} \to [0,1]$ with $F(0) = 0$. The class of all distance distribution functions will be denoted by Δ^+.

For a distance distribution function G and a non-negative constant $c \in \overline{\mathbb{R}}_+$, we define the multiplication of G by a constant c as follows

$$(c \odot G)(x) := \begin{cases} \varepsilon_0(x), & c \in \{0, +\infty\}, \\ G\left(\frac{x}{c}\right), & \text{otherwise.} \end{cases} \tag{5}$$

Clearly, $c \odot G \in \Delta^+$.

A *triangle function* is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$, which is symmetric, associative, non-decreasing in each variable, and has ε_0 as the identity, where ε_0 is the distribution function of Dirac random variable concentrated at point 0. More precisely, for $a \in [-\infty, +\infty[$ we put

$$\varepsilon_a(x) := \begin{cases} 1 & \text{for } x > a, \\ 0 & \text{otherwise.} \end{cases}$$

For more details on triangle functions we recommend the overview paper [20]. The most important triangle functions are those obtained from certain aggregation functions, especially t-norms.

Aggregation functions A binary *aggregation function* $A : [0,1]^2 \to [0,1]$ is a non-decreasing function in both variables with the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. The class of all binary aggregation functions will be denoted by $\mathscr{A}$. For more details on aggregation functions we recommend the monograph [5].

A *triangular norm*, shortly a t-norm, is a commutative lattice ordered semi-group on $[0,1]$ with identity 1. The most important ones are the minimum t-norm $M(x,y) := \min\{x,y\}$, the product t-norm $\Pi(x,y) := xy$, the Łukasiewicz t-norm $W(x,y) := \max\{x+y-1, 0\}$, and the drastic product t-norm

$$D(x,y) := \begin{cases} \min\{x,y\} & \text{for } \max\{x,y\} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For more information about t-norms and their properties, see [14, 22]. Throughout this paper $\mathscr{T}$ denotes the class of all t-norms.

Binary operations on non-negative reals Let us denote by $\mathscr{L}$ the set of all binary operations L on $\overline{\mathbb{R}}_+$ such that

(i) L is commutative and associative;

(ii) L is jointly strictly increasing, i.e., for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}_+$ with $u_1 < u_2$, $v_1 < v_2$ holds $L(u_1, v_1) < L(u_2, v_2)$;

(iii) L is continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;

(iv) L has 0 as its neutral element.

Observe that $L \in \mathscr{L}$ is a jointly increasing pseudo-addition on $\overline{\mathbb{R}}_+$ in the sense of [24]. The usual (class of) examples of operations in $\mathscr{L}$ are

$$K_\alpha(x,y) := (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}, \quad \alpha > 0,$$
$$K_\infty(x,y) := \max\{x,y\}.$$

In general, L belongs to $\mathscr{L}$ if and only if there is a (possibly empty) system $(]a_k,b_k[)_{k\in K}$ of pairwise disjoint open subintervals of $]0,+\infty[$, and a system $(\ell_k)_{k\in K}$ of increasing bijections $\ell_k : [a_k,b_k] \to \overline{\mathbb{R}}_+$ such that

$$L(x,y) = \begin{cases} \ell_k^{-1}(\ell_k(x)+\ell_k(y)) & \text{if } (x,y) \in]a_k,b_k[^2, \\ \max\{x,y\} & \text{otherwise.} \end{cases}$$

For more details see [14]. Further, for $(L,A) \in \mathscr{L} \times \mathscr{A}$ define the function

$$\tau_{L,A}(G,H)(x) := \sup_{L(u,v)=x} A(G(u),H(v)), \quad G,H \in \Delta^+. \tag{6}$$

However, $\tau_{L,A}$ need not be associative in general, thus need not be a triangle function, but it has good properties on Δ^+. As is shown in [20], the appropriate choice for A is a semi-copula S — a binary aggregation function on $[0,1]$ with 1 as its neutral element. Indeed, the left-continuity of S guarantees that $\tau_{L,S}$ is a binary operation on Δ^+, cf. [20, Lemma 7.1]. Also, for any semi-copula S,

$$\tau_{K_\infty,S}(G,H)(x) = S(G(x),H(x))$$

is the operation pointwisely induced by S on Δ^+. For more information about (triangular) functions in connection with various aggregation functions and their properties, see [20].

Probabilistic metric spaces The function (6), at least for a special choice of L and A, naturally arises in the context of probabilistic metric spaces. Recall that a *probabilistic metric space* (PM-space, for short) is a non-empty set Ω together with a family $\mathscr{F}$ of probability functions $F_{p,q}(x)$ (interpreted as the probability that distance between elements p,q of Ω is less than x) with $F_{p,q}(0) = 0$ satisfying

(i) $F_{p,q} = \varepsilon_0$ if and only if $p = q$;

(ii) $F_{p,q} = F_{q,p}$;

(iii) $F_{p,r} \geq \tau(F_{p,q},F_{q,r})$,

where τ is a triangle function on Δ^+. A particular case includes the triangle inequality

$$F_{p,r}(L(x,y)) \geq A(F_{p,q}(x),F_{q,r}(y)) \tag{7}$$

which holds for all $p,q,r \in \Omega$ and all real x,y with a suitable binary aggregation function A and operation L. In particular, the triple $(\Omega,\mathscr{F},\tau_{L,T})$ with $\tau_{L,T}$ given by (6) for $A = T$ (a left-continuous t-norm) is called an *L-Menger PM-space* (under T). For $L = K_\infty$ we get a non-Archimedean Menger PM-space $(\Omega,\mathscr{F},\tau_{K_\infty,T})$ with $T \in \mathscr{T}$. Note that if $\mathscr{F}$ satisfies $F_{p,p} = \varepsilon_0$ for each $p \in \Omega$ and properties (ii) and (iii), then the triple $(\Omega,\mathscr{F},\tau)$ will be called a *probabilistic pseudo-metric space* (PpM-space, for short). In general, different triangle functions lead to PM-spaces with different geometric and topological properties.

3 Probabilistic-valued decomposable set functions with respect to a triangle function

As we have already mentioned in the introduction, the natural origin of probabilistic-valued set functions comes from the fact that they work in such situations in which we have only

a *probabilistic information* about the measure of a set. For example, if rounding of reals is considered, then the uniform distributions over intervals describe our information about the measure of a set. A similar situation is discussed in the framework of information measures in [13]. Thus, the probabilistic-valued (sub)measures represent the concept of (sub)measures probabilistically rather than deterministically.

In what follows we introduce the basic notion of a probabilistic decomposable (sub)measure in its general form. For better readability we also use the following conventions:

(i) for a probabilistic-valued set function $\gamma : \Sigma \to \Delta^+$ we write $\gamma_E(x)$ instead of $\gamma(E)(x)$;

(ii) since Δ^+ is the set of all distribution functions with the support $\overline{\mathbb{R}}_+$, we state the expression for a mapping $\gamma : \Sigma \to \Delta^+$ just for positive values of x. In case $x \leq 0$ we always suppose $\gamma_\cdot(x) = 0$.

Definition 3.1. Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ with $\gamma_\emptyset = \varepsilon_0$ is said to be a *τ-decomposable submeasure*, if $\gamma_{E\cup F} \geq \tau(\gamma_E, \gamma_F)$ for each choice of disjoint sets $E, F \in \Sigma$. If in the preceding inequality equality holds, then γ is said to be a *τ-decomposable measure* on Σ.

Remark 3.2. In fact, the mapping $\gamma : \Sigma \to \Delta^+$ is a probabilistic-valued set function, where the value of γ_E at x may be interpreted as the probability that a numerical (sub)measure of the set E is less than x. A triangle function τ is a natural choice for the "aggregation" of γ_E and γ_F in order to compare them with $\gamma_{E\cup F}$. With regard to a τ-decomposable measure, we naturally expect that $\gamma_{E\cup F}$ is the same distance distribution function as is $\gamma_{F\cup E}$ for disjoint sets $E, F \in \Sigma$, from which it follows that $\tau(\gamma_E, \gamma_F) = \tau(\gamma_F, \gamma_E)$, i.e., τ has to be commutative. Moreover, from the natural equality $\gamma_{(E\cup F)\cup G} = \gamma_{E\cup(F\cup G)}$ we obtain $\tau(\tau(\gamma_E, \gamma_F), \gamma_G) = \tau(\gamma_E, \tau(\gamma_F, \gamma_G))$, i.e., τ has to be associative. Since $\gamma_E = \gamma_{E\cup\emptyset} = \tau(\gamma_E, \gamma_\emptyset) = \tau(\gamma_E, \varepsilon_0)$, then ε_0 has to be neutral element of τ. The role of monotonicity of τ (a non-decreasing function in each place) will be examined in what follows.

Theorem 3.3. *Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. Then each τ-decomposable measure γ is "antimonotone" on Σ, i.e., $\gamma_E \geq \gamma_F$ whenever $E, F \in \Sigma$ such that $E \subseteq F$.*

Proof. From the monotonicity of the triangular function τ with ε_0 as identity we easily have

$$\gamma_F = \gamma_{E\cup(F\setminus E)} = \tau(\gamma_E, \gamma_{F\setminus E}) \leq \tau(\gamma_E, \varepsilon_0) = \gamma_E$$

whenever E and F are in Σ such that $E \subseteq F$. □

Remark 3.4. The probabilistic interpretation of this property is as follows: the probability that a numerical measure of the set E is less than x is greater than or equal to the probability that a numerical measure of the set F is less than x.

The "antimonotonicity" property does not hold for arbitrary τ-decomposable submeasures $\gamma : \Sigma \to \Delta^+$. Therefore, in [8, Definition 2.1] we have considered the notion of a probabilistic submeasure w.r.t. a function $\tau = \tau_{L,A}$, being a $\tau_{L,A}$-decomposable antimonotone submeasure on Σ where the "subadditivity" property $\gamma_{E\cup F} \geq \tau_{L,A}(\gamma_E, \gamma_F)$ holds for all sets $E, F \in \Sigma$. Therefore we state the following observation.

Theorem 3.5. *Let τ be a triangle function on Δ^+ and Σ be a ring of subsets of $\Omega \neq \emptyset$. If $\gamma : \Sigma \to \Delta^+$ is a τ-decomposable antimonotone submeasure, then the inequality $\gamma_{E\cup F} \geq \tau(\gamma_E, \gamma_F)$ holds for all sets $E, F \in \Sigma$.*

Proof. For $E \cap F = \emptyset$ the inequality holds by Definition 3.1. Let $E, F \in \Sigma$ and $E \cap F \neq \emptyset$. Then the inequality

$$\gamma_{E\cup F} = \gamma_{E\cup(F\setminus E)} \geq \tau(\gamma_E, \gamma_{F\setminus E}) \geq \tau(\gamma_E, \gamma_F)$$

follows from the antimonotonicity of γ and monotonicity of triangle functions. □

Now we provide several examples of τ-decomposable measures, which are based on various constructions and are related to different classes of aggregation functions, which appear in the definition of the underlying triangle function. Note that the "numerical (sub)measure" means a real-valued set function with the (usual) (sub)additivity property.

Example 3.6. Let m be a numerical (additive) measure on Σ. If m is L-decomposable, i.e., $m(E \cup F) = L(m(E), m(F))$ with $L \in \mathscr{L}$ and disjoint sets $E, F \in \Sigma$, then for any $\Phi \in \Delta^+$ the set function $\gamma^\Phi : \Sigma \to \Delta^+$ defined by $\gamma^\Phi_E := m(E) \odot \Phi$ is a $\tau_{L,M}$-decomposable measure on Σ, where $\tau_{L,M}$ is given by (4) with the minimum t-norm $T = M$. Indeed, "additivity" property follows from [22, Section 7.7]: the function $\tau_{L,M}$ is the only triangular function with the property

$$\tau_{L,M}\left(c_1 \odot H, c_2 \odot H\right) = L(c_1, c_2) \odot H, \qquad c_1, c_2 \in \overline{\mathbb{R}}_+, H \in \Delta^+.$$

Notice that γ^Φ need not be a $\tau_{L,T}$-decomposable measure for $T \neq M$.

Example 3.7. *Shen's $\top$-probabilistic decomposable measures*, cf. [23]: this class of measures $\mathfrak{M} : \Sigma \to \Delta^+$ of the form $\mathfrak{M}_{E\cup F}(t) = \top(\mathfrak{M}_E(t), \mathfrak{M}_F(t))$ for disjoint $E, F \in \Sigma$ corresponds to the class of τ-decomposable measures w.r.t. the triangle function $\Pi_\top : \Delta^+ \times \Delta^+ \to \Delta^+$ of the form

$$\Pi_\top(G, H)(t) = \top(G(t), H(t)), \quad G, H \in \Delta^+, \tag{8}$$

with $\top$ being a t-norm, or, equivalently, to a $\tau_{\max,T}$-decomposable measure. According to [20, Theorem 5.2] the left-continuity of the t-norm $\top$ is a necessary and sufficient condition for $\Pi_\top$ being a triangle function. Note that Shen does not consider the left-continuity of $\top$ in his definition of a $\top$-probabilistic decomposable measure, cf. [23, Definition 4.1]. Moreover, Shen's considerations are made on a σ-algebra instead of a ring of subsets of $\Omega \neq \emptyset$ in order to provide a countable extension of $\top$-decomposable measures.

Example 3.8. The convolution of distance distribution functions is a binary operation $*$ on Δ^+ given by

$$\tau_*(G, H)(x) := (G * H)(x) = \begin{cases} 0, & x = 0, \\ \int_0^x G(x-t)\,\mathrm{d}H(t), & x \in]0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

for all $G, H \in \Delta^+$, where the integral is meant in the sense of Lebesgue-Stieltjes. According to [20, Theorem 13.4 and Definition 13.1] τ_* is a triangle function on Δ^+. The corresponding τ_*-decomposable (sub)measure γ provides an extension of the notion of (sub)measure to (sub)measures, which can be used in the Wald spaces — those involving the convolution $*$ of distance distribution functions, but also in a wider class of PM-spaces.

If m is a numerical measure on Σ, then the set function $\gamma : \Sigma \to \Delta^+$ given by $\gamma_E = \varepsilon_{m(E)}$ for $E \in \Sigma$ is a τ_*-decomposable measure, because

$$\gamma_{E\cup F} = \varepsilon_{m(E\cup F)} = \varepsilon_{m(E)+m(F)} = \tau_*(\varepsilon_{m(E)}, \varepsilon_{m(F)}) = \tau_*(\gamma_E, \gamma_F),$$

cf. [20, Theorem 14.1(f)].

The latter example of the mapping $\gamma_E = \varepsilon_{m(E)}$ is also interesting from the viewpoint of other triangle functions. It provides the reason for claiming that *each numerical (additive) measure can be regarded as a probabilistic-valued decomposable measure.* Thus, τ-decomposable measures are extensions of the classical measure.

Example 3.9. Let $\mu : \Sigma \to [0, +\infty)$ be a set function with $\mu(\emptyset) = 0$ and for $E \in \Sigma$ put $\gamma_E = \varepsilon_{\mu(E)}$. Due to [20, Theorem 14.1] we have the following results:

(i) if μ is additive, then γ is a τ_D-decomposable measure;

(ii) if μ is additive and $T \in \mathscr{T}$ is continuous, then γ is a τ_T-decomposable measure;

(iii) if μ is L-decomposable with $L \in \mathscr{L}$, and $T \in \mathscr{T}$ is continuous, then γ is a $\tau_{L,T}$-decomposable measure;

(iv) if μ is L-decomposable with $L \in \mathscr{L}$ and Q is a symmetric quasi-copula with an associative dual quasi-copula $\overline{Q}$, then γ is a $\rho_{L,Q}$-decomposable measure, where

$$\rho_{L,Q}(G,H)(x) = \inf_{L(u,v)=x} \overline{Q}(G(u),H(v)), \quad G,H \in \Delta^+;$$

(v) if μ is K_∞-decomposable, then γ is a $\Pi_\top$-decomposable measure for each $\top \in \mathscr{T}$, where $\Pi_\top$ is given by (8), i.e., γ is a Shen's $\top$-probabilistic decomposable measure. Note that if μ is additive, then γ need not be a $\Pi_\top$-decomposable measure. For instance, taking $\top = M$ (the minimum t-norm) we get

$$\varepsilon_{\mu(E)+\mu(F)}(x) \neq M(\varepsilon_{\mu(E)}(x), \varepsilon_{\mu(F)}(x))$$

whenever $\mu(E) \neq \mu(F) \neq \emptyset$.

4 Basic properties and constructions of decomposable (sub)measures

Clearly, each τ-decomposable measure on Σ is a τ-decomposable submeasure on Σ. The following result provides a characterization of the class of τ-decomposable measures on Σ. In fact, it provides a generalization of [23, Theorem 4.1] to an arbitrary triangle function.

Theorem 4.1. *Let τ be a triangle function on Δ^+. Then γ is a τ-decomposable measure on Σ if and only if $\tau(\gamma_{E\cup F}, \gamma_{E\cap F}) = \tau(\gamma_E, \gamma_F)$ for all $E, F \in \Sigma$.*

Proof. "$\Leftarrow$" If $E \cap F = \emptyset$, then $\gamma_{E\cap F} = \varepsilon_0$ and

$$\tau(\gamma_E, \gamma_F) = \tau(\gamma_{E\cup F}, \gamma_{E\cap F}) = \tau(\gamma_{E\cup F}, \varepsilon_0) = \gamma_{E\cup F}.$$

"$\Rightarrow$" Since $E \cup F = (E \cap F) \cup (E \setminus F) \cup (F \setminus E)$, then we have

$$\begin{aligned}\tau(\gamma_{E\cap F}, \gamma_{E\cup F}) &= \tau(\gamma_{E\cap F}, \tau(\gamma_{E\cap F}, \tau(\gamma_{E\setminus F}, \gamma_{F\setminus E})) = \tau(\gamma_{E\cap F}, \tau(\tau(\gamma_{E\cap F}, \gamma_{E\setminus F}), \gamma_{F\setminus E}))\\ &= \tau(\gamma_{E\cap F}, \tau(\gamma_E, \gamma_{F\setminus E})) = \tau(\gamma_{E\cap F}, \tau(\gamma_E, \gamma_{F\setminus E})) = \tau(\gamma_E, \tau(\gamma_{E\cap F}, \gamma_{F\setminus E}))\\ &= \tau(\gamma_E, \gamma_F)\end{aligned}$$

which completes the proof. □

Now we are interested in generating new decomposable (sub)measures from given ones. For this reason we will need the following notion, cf. [21, Definition 6.1]: Let $(X, \leq)$ be a partially ordered set and let f and g be two binary operations on X. Then f *dominates* g, written $f \gg g$, if, for all $x, y, u, v \in X$, it holds

$$f(g(x,y), g(u,v)) \geq g(f(x,u), f(y,v)).$$

It is a well known fact that dominance is an antisymmetric and reflexive relation, but is not transitive on the set of triangle functions, cf. [21, Corollary 6.1]. Note that the idea of dominance appears in the previous article [19] concerning the aggregation functions, as well as in the research study [1] on triangular norms.

A triangle function τ, such that for each $c \in \overline{\mathbb{R}}_+$ and all $G, H \in \Delta^+$ it holds

$$c \odot \tau(G,H) = \tau(c \odot G, c \odot H),$$

will be called a *distributive triangle function*. Recall that the operation $\odot$ is defined by (5). The standard examples of distributive triangle functions are τ_T and $\Pi_\top$.

Theorem 4.2. *Let τ, ϑ be two triangle functions on Δ^+ and $\gamma^1, \gamma^2 : \Sigma \to \Delta^+$ be τ-decomposable measures. Then*

(i) if τ is distributive, the set function $\gamma := c \odot \gamma^1$ is a τ-decomposable measure for each $c \in \overline{\mathbb{R}}_+$;

(ii) the set function $\zeta := \tau(\gamma^1, \gamma^2)$ is a τ-decomposable measure;

(iii) the set function $\lambda := \vartheta(\gamma^1, \gamma^2)$ is a τ-decomposable submeasure if and only if $\vartheta \gg \tau$.

Proof. Consider $E, F \in \Sigma$ such that $E \cap F = \emptyset$. It is easy to verify that $\gamma_\emptyset = \zeta_\emptyset = \varepsilon_0$.

(i) Immediately, by the distributivity of τ, we get

$$\gamma_{E\cup F} = c \odot \gamma^1_{E\cup F} = c \odot \tau(\gamma^1_E, \gamma^1_F) = \tau(c \odot \gamma^1_E, c \odot \gamma^1_F) = \tau(\gamma_E, \gamma_F).$$

(ii) It follows from the associativity and the symmetry of triangle functions that

$$\begin{aligned}\zeta_{E\cup F} &= \tau(\gamma^1_{E\cup F}, \gamma^2_{E\cup F}) = \tau(\tau(\gamma^1_E, \gamma^1_F), \gamma^2_{E\cup F}) = \tau(\gamma^1_E, \tau(\gamma^2_{E\cup F}, \gamma^1_F)) = \tau(\gamma^1_E, \tau(\tau(\gamma^2_E, \gamma^2_F), \gamma^1_F)) \\ &= \tau(\gamma^1_E, \tau(\gamma^2_E, \tau(\gamma^2_F, \gamma^1_F))) = \tau(\tau(\gamma^1_E, \gamma^2_E), \tau(\gamma^1_F, \gamma^2_F))) = \tau(\zeta_E, \zeta_F).\end{aligned}$$

(iii) If $\lambda := \vartheta(\gamma^1, \gamma^2)$ is a τ-decomposable submeasure, then $\lambda_{E\cup F} \geq \tau(\lambda_E, \lambda_F)$, i.e.,

$$\vartheta(\gamma^1_{E\cup F}, \gamma^2_{E\cup F}) \geq \tau(\vartheta(\gamma^1_E, \gamma^2_E), \vartheta(\gamma^1_F, \gamma^2_F)).$$

Since

$$\vartheta(\gamma^1_{E\cup F}, \gamma^2_{E\cup F}) = \vartheta(\tau(\gamma^1_E, \gamma^1_F), \tau(\gamma^2_E, \gamma^2_F)),$$

therefore

$$\vartheta(\tau(\gamma^1_E, \gamma^1_F), \tau(\gamma^2_E, \gamma^2_F)) \geq \tau(\vartheta(\gamma^1_E, \gamma^2_E), \vartheta(\gamma^1_F, \gamma^2_F)), \tag{9}$$

which means that $\vartheta \gg \tau$.

On the other hand, if $\vartheta \gg \tau$, then the inequality (9) holds and the equalities

$$\begin{aligned}&\vartheta(\tau(\gamma^1_E, \gamma^1_F), \tau(\gamma^2_E, \gamma^2_F)) = \vartheta(\gamma^1_{E\cup F}, \gamma^2_{E\cup F}) = \lambda_{E\cup F}, \\ &\tau(\vartheta(\gamma^1_E, \gamma^2_E), \vartheta(\gamma^1_F, \gamma^2_F)) = \tau(\lambda_E, \lambda_F)\end{aligned}$$

imply the result. □

As we can see, a τ-sum of two τ-decomposable measures produces a τ-decomposable measure. The same is true for (antimonotone) τ-decomposable submeasures. Replacing a triangle function by a more general aggregation operator (on Δ^+) may also produce a desired set function.

Recall that an (n-ary) *aggregation operator* on Δ^+ is a mapping $\alpha : (\Delta^+)^n \to \Delta^+$, which is non-decreasing in each place with the boundary condition $\alpha(\varepsilon_0, \ldots, \varepsilon_0) = \varepsilon_0$. Also, for domination between aggregation operators we refer to [19]: we say that an (n-ary) aggregation operator α *dominates* an (m-ary) aggregation operator β, we write $\alpha \gg \beta$, if for all $H_{i,j} \in \Delta^+$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the following property holds:

$$\alpha\big(\beta(H_{1,1}, \ldots, H_{m,1}), \ldots, \beta(H_{1,n}, \ldots H_{m,n})\big) \geq \beta\big(\alpha(H_{1,1}, \ldots, H_{1,n}), \ldots, \alpha(H_{m,1}, \ldots, H_{m,n})\big).$$

Theorem 4.3. *Let τ_i be triangle functions on Δ^+ and $\gamma^i : \Sigma \to \Delta^+$ be τ_i-decomposable submeasures for $i = 1, \ldots, n$, $n \in \mathbb{N}$. If τ is a triangle function on Δ^+ such that $\tau \leq \tau_i$ for each $i = 1, \ldots, n$, and α is an (n-ary) aggregation operator on Δ^+ such that $\alpha \gg \tau$, then $\gamma := \alpha(\gamma^1, \cdots \gamma^n)$ is a τ-decomposable submeasure on Σ.*

Proof. Immediately, for all $E, F \in \Sigma$ we have

$$\begin{aligned}\gamma_{E\cup F} &= \alpha(\gamma^1_{E\cup F}, \ldots, \gamma^n_{E\cup F}) \geq \alpha(\tau_1(\gamma^1_E, \gamma^1_F), \ldots, \tau_n(\gamma^n_E, \gamma^n_F)) \\ &\geq \alpha(\tau(\gamma^1_E, \gamma^1_F), \ldots, \tau(\gamma^n_E, \gamma^n_F)) \geq \tau(\alpha(\gamma^1_E, \cdots, \gamma^n_E), \alpha(\gamma^1_F, \cdots, \gamma^n_F)) \\ &= \tau(\gamma_E, \gamma_F).\end{aligned}$$

Finally, by boundary condition we get $\gamma_\emptyset = \alpha(\gamma^1_\emptyset, \ldots, \gamma^n_\emptyset) = \alpha(\varepsilon_0, \ldots, \varepsilon_0) = \varepsilon_0$. □

Remark 4.4. Especially, it is well-known that the arithmetic mean AM on $[0,1]$ dominates W, cf. [18], thus Π_{AM} dominates $\tau_{L,W}$ for an arbitrary $L \in \mathscr{L}$, cf. [21]. So, the Π_{AM}-aggregation of $\tau_{L,W}$-decomposable submeasures, i.e., the set function $\gamma := \Pi_{AM}(\gamma^1, \ldots, \gamma^n)$, is again a $\tau_{L,W}$-decomposable submeasure.

The following construction is based on the observation taken from a construction of product PM-spaces for the case of finite products, cf. [21]. For more details about pseudo-metrics generated by probabilistic-valued decomposable (sub)measures see Section 5.

Theorem 4.5. *Let τ_i be a (finite) family of triangle functions on Δ^+ and $\gamma^i : \Sigma_i \to \Delta^+$ be τ_i-decomposable antimonotone submeasures for $i = 1, \ldots n$, $n \in \mathbb{N}$. Let $(\Sigma_i, \rho^i, \tau_i)$ be a family of PpM-spaces with $\rho^i_{E_i,F_i} := \gamma^i_{E_i \triangle F_i}$, and α be an (n-ary) aggregation operator on Δ^+. If there exists a triangle function τ on Δ^+ such that $\alpha \gg \tau$ and that $\tau \leq \tau_i$ for every $i = 1, \ldots, n$, then the triple (Σ, ρ, τ) is a PpM-space, where ρ is defined on $\Sigma := \prod_{i=1}^n \Sigma_i$ by*

$$\rho_{E,F} := \alpha(\rho^1_{E_1,F_1}, \ldots, \rho^n_{E_n,F_n}).$$

Proof. We prove that ρ is a pseudo-metric on Σ w.r.t. τ. Clearly, $\rho_{E,E} = \varepsilon_0$ for each $E \in \Sigma$ and $\rho_{E,F} = \rho_{F,E}$ for all $E, F \in \Sigma$. Taking $E, F, G \in \Sigma$ we obtain

$$\begin{aligned}
\tau(\rho_{E,F}, \rho_{F,G}) &= \tau(\alpha(\rho^1_{E_1,F_1}, \ldots, \rho^n_{E_n,F_n}), \alpha(\rho^1_{F_1,G_1}, \ldots, \rho^n_{F_n,G_n})) \\
&\leq \alpha(\tau(\rho^1_{E_1,F_1}, \rho^1_{F_1,G_1}), \ldots, \tau(\rho^n_{E_n,F_n}, \rho^n_{F_n,G_n})) \\
&\leq \alpha(\tau_1(\rho^1_{E_1,F_1}, \rho^1_{F_1,G_1}), \ldots, \tau_n(\rho^n_{E_n,F_n}, \rho^n_{F_n,G_n})) \\
&\leq \alpha(\rho^1_{E_1,G_1}, \ldots, \rho^n_{E_n,G_n}) = \rho_{E,G},
\end{aligned}$$

which completes the proof. □

5 Mappings and structures induced by decomposable (sub)measures

The aim of this section is to extend results from [9] to a general case of τ-decomposable (sub)measures instead of $\tau_{L,T}$-submeasures considered therein.

Theorem 5.1. *Let τ be a triangle function on Δ^+ and $\gamma : \Sigma \to \Delta^+$ be a τ-decomposable antimonotone submeasure. Define the mapping $\rho : \Sigma \times \Sigma \to \Delta^+$ by the formula $\rho_{E,F} := \gamma_{E \triangle F}$ with $E, F \in \Sigma$. Then the triple (Σ, ρ, τ) is a PpM-space (under τ).*

Proof. Clearly, $\rho_{E,E} = \gamma_{E \triangle E} = \varepsilon_0$ for each $E \in \Sigma$. Similarly, the symmetry is trivial. It is enough to prove the triangle inequality. By Theorem 3.5, consider the arbitrary sets $E, F, G \in \Sigma$. Since

$$E \triangle G \subset (E \triangle F) \cup [(G \setminus (E \cup F)) \cup ((E \cap F) \setminus G)],$$

and

$$(G \setminus (E \cup F)) \cup ((E \cap F) \setminus G) \subset F \triangle G,$$

then antimonotonicity of γ yields

$$\begin{aligned}
\rho_{E,G} = \gamma_{E \triangle G} &\geq \gamma_{(E \triangle F) \cup [(G \setminus (E \cup F)) \cup ((E \cap F) \setminus G)]} \geq \tau\big(\gamma_{E \triangle F}, \gamma_{(G \setminus (E \cup F)) \cup ((E \cap F) \setminus G)}\big) \\
&\geq \tau(\gamma_{E \triangle F}, \gamma_{F \triangle G}) = \tau(\rho_{E,F}, \rho_{F,G}),
\end{aligned} \tag{10}$$

which means that ρ is a pseudo-metric with respect to the triangle function τ. □

Remark 5.2. Since $E \triangle G = (E \triangle F) \triangle (F \triangle G)$, then it is easy to observe that ρ is translation invariant, i.e., $\rho_{E,G} = \gamma_{E \triangle G} = \gamma_{(E \triangle F) \triangle (F \triangle G)} = \rho_{E \triangle F, F \triangle G}$.

For a fixed triangle function τ denote by $\Gamma_\tau(\Sigma,\gamma)$ *the set of all pseudo-metrics* ρ *generated by* τ*-decomposable antimonotone submeasures* γ on Σ, i.e., $\rho_{E,F}=\gamma_{E\triangle F}$ with $E,F\in\Sigma$. Define a relation $\preceq$ on $\Gamma_\tau(\Sigma,\gamma)$ as follows

$$\rho\preceq\varrho\Leftrightarrow\rho_{E,F}\geq\varrho_{E,F}\ \text{ for all }\ E,F\in\Sigma.$$

Clearly, $\preceq$ is a partial order on $\Gamma_\tau(\Sigma,\gamma)$ and $\nu_{E,F}=\varepsilon_0$ for all $E,F\in\Sigma$ is an element such that $\nu\preceq\rho$ for every $\rho\in\Gamma_\tau(\Sigma,\gamma)$.

Let ϑ be a triangle function on Δ^+ such that $\vartheta\gg\tau$. We define a binary operation $\oplus_\vartheta$ on $\Gamma_\tau(\Sigma,\gamma)$ such that

$$(\rho\oplus_\vartheta\varrho)_{E,F}:=\vartheta(\rho_{E,F},\varrho_{E,F}),\quad E,F\in\Sigma,$$

for all $\rho,\varrho\in\Gamma_\tau(\Sigma,\gamma)$. We show that $\rho\oplus_\vartheta\varrho\in\Gamma_\tau(\Sigma,\gamma)$. Indeed, for an arbitrary $E\in\Sigma$ we have

$$(\rho\oplus_\vartheta\varrho)_{E,E}=\vartheta(\rho_{E,E},\varrho_{E,E})=\vartheta(\varepsilon_0,\varepsilon_0)=\varepsilon_0$$

and the symmetry $(\rho\oplus_\vartheta\varrho)_{E,F}=(\rho\oplus_\vartheta\varrho)_{F,E}$ is obvious for all $E,F\in\Sigma$. Also, by the triangle inequality (10) and domination of ϑ over τ we have

$$\begin{aligned}(\rho\oplus_\vartheta\varrho)_{E,G}&=\vartheta(\rho_{E,G},\varrho_{E,G})\geq\vartheta(\tau(\rho_{E,F},\rho_{F,G}),\tau(\varrho_{E,F},\varrho_{F,G}))\\&\geq\tau(\vartheta(\rho_{E,F},\varrho_{E,F}),\vartheta(\rho_{F,G},\varrho_{F,G}))\\&=\tau((\rho\oplus_\vartheta\varrho)_{E,F},(\rho\oplus_\vartheta\varrho)_{F,G}),\end{aligned}$$

where $E,F,G\in\Sigma$. Thus $\rho\oplus_\vartheta\varrho$ is a pseudo-metric on Σ. Note that for all $\rho\in\Gamma_\tau(\Sigma,\gamma)$ and $\vartheta\gg\tau$ it holds

$$(\nu\oplus_\vartheta\rho)_{E,F}=\vartheta(\nu_{E,F},\rho_{E,F})=\vartheta(\varepsilon_0,\rho_{E,F})=\rho_{E,F}$$

for all $E,F\in\Sigma$. The operation $\oplus_\vartheta$ is clearly commutative and associative. Thus we have the following result.

Proposition 5.3. *Let* $\vartheta\gg\tau$. *The triple* $(\Gamma_\tau(\Sigma,\gamma),\oplus_\vartheta,\preceq)$ *is a partially ordered commutative semigroup with the neutral element* ν.

A relationship between the relation $\preceq$ and the operation $\oplus_\vartheta$ is described in the following theorem.

Theorem 5.4. *Let* $\vartheta\gg\tau$. *Then for all* $\rho,\varrho,\sigma\in\Gamma_\tau(\Sigma,\gamma)$ *the relationship* $\rho\oplus_\vartheta\sigma\preceq\varrho\oplus_\vartheta\sigma$ *holds whenever* $\rho\preceq\varrho$.

Proof. The relation $\rho\preceq\varrho$ means that $\rho_{E,F}\geq\varrho_{E,F}$ for all $E,F\in\Sigma$. From the monotonicity of ϑ we have $\vartheta(\rho_{E,F},\sigma_{E,F})\geq\vartheta(\varrho_{E,F},\sigma_{E,F})$ and the proof is complete. □

Recall that Π_M (the pointwise defined operation on Δ^+ given by (8) with the minimum t-norm M) dominates each triangle function τ, cf. [21, Proposition 6.2]. Also, a *semilattice* is an idempotent commutative semigroup. A semilattice is *bounded* if it includes a neutral element.

Theorem 5.5. *The ordered pair* $(\Gamma_\tau(\Sigma,\gamma),\oplus_{\Pi_M})$ *is a bounded semilattice with the properties*

(i) $\rho\preceq\varrho$ *if and only if* $\rho\oplus_{\Pi_M}\varrho=\varrho$;

(ii) $(\sigma\oplus_\vartheta\rho)\oplus_{\Pi_M}(\sigma\oplus_\vartheta\varrho)\preceq\sigma\oplus_\vartheta(\rho\oplus_{\Pi_M}\varrho)$

for all $\rho,\varrho,\sigma\in\Gamma_\tau(\Sigma,\gamma)$ *provided that* $\vartheta\gg\tau$.

Proof. Since $\nu\in\Gamma_\tau(\Sigma,\gamma)$ is the neutral element, then $(\Gamma_\tau(\Sigma,\gamma),\oplus_{\Pi_M})$ is a bounded semilattice.

(i) If $\rho\preceq\varrho$, then $\rho_{E,F}\geq\varrho_{E,F}$ for all $E,F\in\Sigma$, thus $\rho\oplus_{\Pi_M}\varrho=\varrho$. The opposite statement is obvious.

(ii) For the proof of the second part, it is enough to observe that $\rho\preceq\rho\oplus_{\Pi_M}\varrho$ and $\varrho\preceq\rho\oplus_{\Pi_M}\varrho$. By Theorem 5.4, we have $\sigma\oplus_\vartheta\rho\preceq\sigma\oplus_\vartheta(\rho\oplus_{\Pi_M}\varrho)$ and $\sigma\oplus_\vartheta\varrho\preceq\sigma\oplus_\vartheta(\rho\oplus_{\Pi_M}\varrho)$. Since $(\Gamma_\tau(\Sigma,\gamma),\oplus_{\Pi_M})$ is a semilattice, the property (ii) follows. □

6 Decomposable measures and the probabilistic Hausdorff distance

As it was already mentioned in this paper, Shen introduced in [23] a class of probabilistic decomposable measures w.r.t. the triangle function $\Pi_\top$ given by (8). Now, we extend the results related to the probabilistic Hausdorff distance. For this reason, we will recall all the necessary notions from [23].

Let $(\Omega, \mathscr{F}, \tau)$ be a PM-space.
(i) The *probabilistic diameter* of a non-empty subset E of Ω is a mapping $D_E : [0, +\infty) \to [0, 1]$ defined by

$$D_E(t) := \sup_{s<t} \inf_{p,q\in E} H_{p,q}(s).$$

(ii) A set E is said to be *probabilistically bounded* if $\sup_t D_E(t) = 1$. The collection of all probabilistically bounded subsets of Ω will be denoted by $PB(\Omega)$.
(iii) Let $p \in \Omega$ and $F \in PB(\Omega)$. The *probabilistic distance* from p to F is defined as

$$d_{p,F}(t) := \begin{cases} 0, & t = 0, \\ \sup\limits_{s<t} \sup\limits_{q\in F} H_{p,q}(s), & t \in \mathbb{R}_+, \end{cases}$$

with the convention $d_{p,\emptyset} = 1 - \varepsilon_0$.
(iv) Given $E, F \in PB(\Omega)$, the *probabilistic distance* from E to F is defined as

$$d_{E,F}(t) := \begin{cases} 0, & t = 0, \\ \sup\limits_{s<t} \inf\limits_{p\in E} \sup\limits_{q\in F} H_{p,q}(s), & t \in \mathbb{R}_+, \end{cases}$$

with the convention $d_{\emptyset,F} = \varepsilon_0$.

Definition 6.1. Let $(\Omega, \mathscr{F}, \tau)$ be a PM-space and $E, F \in PB(\Omega)$. The probabilistic Hausdorff distance between E and F is a mapping $\mathscr{H}_{E,F} : [0, +\infty) \to [0, 1]$ defined by

$$\mathscr{H}_{E,F}(t) := \begin{cases} 0, & t = 0, \\ \sup\limits_{s<t} M\left(\inf\limits_{p\in E} \sup\limits_{q\in F} H_{p,q}(s), \inf\limits_{q\in F} \sup\limits_{p\in E} H_{p,q}(s)\right), & t \in \mathbb{R}_+. \end{cases}$$

From now on let $\Sigma := P(\Omega)$ be the power set of Ω. Using the probabilistic Hausdorff distance we may define a mapping $\Lambda^{\mathscr{H}} : \Sigma \to \Delta^+$ by

$$\Lambda^{\mathscr{H}}_X := \mathscr{H}_{E^c,\Omega},$$

where E^c is the complement of a set $E \in \Sigma$.

Theorem 6.2. *Let $(\Omega, \mathscr{F}, \tau)$ be a PM-space. Then $\Lambda^{\mathscr{H}}$ is an antimonotone set function with $\Lambda^{\mathscr{H}}_\emptyset = \varepsilon_0$ and*

$$\Lambda^{\mathscr{H}}_{E\cup F} \le \Pi_M\left(\Lambda^{\mathscr{H}}_E, \Lambda^{\mathscr{H}}_F\right), \quad E, F \in \Sigma.$$

Proof. From definitions above, it is obvious that $\Lambda^{\mathscr{H}}_\emptyset = \varepsilon_0$. Also, for $E, F \in \Sigma$ such that $E \subseteq F$, we have $F^c \subseteq E^c$, and therefore

$$\Lambda^{\mathscr{H}}_E(t) = \mathscr{H}_{E^c,\Omega}(t) = \sup_{s<t} \inf_{q\in\Omega} \sup_{p\in E^c} H_{p,q}(s) \ge \sup_{s<t} \inf_{q\in\Omega} \sup_{p\in F^c} H_{p,q}(s) = \mathscr{H}_{F^c,\Omega}(t) = \Lambda^{\mathscr{H}}_F(t)$$

for each $t \ge 0$ which proves antimonotonicity of $\Lambda^{\mathscr{H}}$.

Finally, the inclusions $E \subseteq E \cup F$, $F \subseteq E \cup F$ and antimonotonicity of $\Lambda^{\mathscr{H}}$ yields

$$\Lambda^{\mathscr{H}}_{E\cup F}(t) \le M\left(\Lambda^{\mathscr{H}}_E(t), \Lambda^{\mathscr{H}}_F(t)\right) = \Pi_M\left(\Lambda^{\mathscr{H}}_E(t), \Lambda^{\mathscr{H}}_F(t)\right)$$

for each $t \ge 0$, i.e., $\Lambda^{\mathscr{H}}_{E\cup F} \le \Pi_M\left(\Lambda^{\mathscr{H}}_E, \Lambda^{\mathscr{H}}_F\right)$. □

Remark 6.3. Equivalently, we may say that $\Lambda^{\mathscr{H}}$ is a *probabilistic-valued antimonotone* Π_M*-decomposable supermeasure on* Σ (or, a Π_M-probabilistic submeasure in the terminology of Shen [23]). Under certain condition on the domain of $\Lambda^{\mathscr{H}}$ we will be able to show that it is a τ-decomposable measure w.r.t. an arbitrary triangle function τ (thus, also for Π_M), see Theorem 6.6 below.

Definition 6.4. A set $E \in \Sigma$ is said to be $\Lambda^{\mathscr{H}}$*-measurable w.r.t. a triangle function* τ, if for each $G \in \Sigma$ it holds

$$\Lambda^{\mathscr{H}}_G = \tau\left(\Lambda^{\mathscr{H}}_{G\cap E}, \Lambda^{\mathscr{H}}_{G\cap E^c}\right).$$

Theorem 6.5. *Let* $(\Omega, \mathscr{F}, \tau)$ *be a PM-space. Then each class*

$$\mathbb{S}_\tau = \{E \in \Sigma;\ E \text{ is } \Lambda^{\mathscr{H}}\text{-measurable w.r.t. } \tau\}$$

is an algebra.

Proof. Since $\mathscr{H}_{\emptyset,\Omega} = \mathscr{H}_{\Omega,\Omega} = \varepsilon_0$, then $\emptyset, \Omega \in \mathbb{S}_\tau$. Moreover, from definition it follows that if $E \in \mathbb{S}_\tau$, then $E^c \in \mathbb{S}_\tau$ (and vice versa). Thus, we only have to prove that $\mathbb{S}_\tau$ is closed under the formation of a union.

Let $E, F \in \mathbb{S}_\tau$ and consider an arbitrary $G \in \Sigma$. Since $F \in \mathbb{S}_\tau$, then

$$\Lambda^{\mathscr{H}}_{G\cap(E\cup F)} = \tau\left(\Lambda^{\mathscr{H}}_{G\cap(E\cup F)\cap F}, \Lambda^{\mathscr{H}}_{G\cap(E\cup F)\cap F^c}\right) = \tau\left(\Lambda^{\mathscr{H}}_{G\cap F}, \Lambda^{\mathscr{H}}_{G\cap E\cap F^c}\right),$$

and therefore we get

$$\begin{aligned}
\tau\left(\Lambda^{\mathscr{H}}_{G\cap(E\cup F)}, \Lambda^{\mathscr{H}}_{G\cap(E\cup F)^c}\right) &= \tau\left(\tau\left(\Lambda^{\mathscr{H}}_{G\cap F}, \Lambda^{\mathscr{H}}_{G\cap F^c\cap E}\right), \Lambda^{\mathscr{H}}_{G\cap(E\cup F)^c}\right) \\
&= \tau\left(\tau\left(\Lambda^{\mathscr{H}}_{G\cap F}, \Lambda^{\mathscr{H}}_{G\cap F^c\cap E}\right), \Lambda^{\mathscr{H}}_{G\cap F^c\cap E^c}\right) \\
&= \tau\left(\Lambda^{\mathscr{H}}_{G\cap F}, \tau\left(\Lambda^{\mathscr{H}}_{G\cap F^c\cap E}, \Lambda^{\mathscr{H}}_{G\cap F^c\cap E^c}\right)\right) \\
&= \tau\left(\Lambda^{\mathscr{H}}_{G\cap F}, \Lambda^{\mathscr{H}}_{G\cap F^c}\right) \\
&= \Lambda^{\mathscr{H}}_G.
\end{aligned}$$

Thus, the set $E \cup F$ is $\Lambda^{\mathscr{H}}$-measurable w.r.t. τ, i.e., $E \cup F \in \mathbb{S}_\tau$. □

Theorem 6.6. *Let* $(\Omega, \mathscr{F}, \tau)$ *be a PM-space. The restriction of the set function* $\Lambda^{\mathscr{H}}$ *to* $\mathbb{S}_\tau$ *is a* τ*-decomposable measure.*

Proof. Let $E, F \in \mathbb{S}_\tau$ be disjoint. Since $E \cup F$ is $\Lambda^{\mathscr{H}}$-measurable w.r.t. τ, then

$$\Lambda^{\mathscr{H}}_{E\cup F} = \tau\left(\Lambda^{\mathscr{H}}_{(E\cup F)\cap E}, \Lambda^{\mathscr{H}}_{(E\cup F)\cap E^c}\right) = \tau\left(\Lambda^{\mathscr{H}}_E, \Lambda^{\mathscr{H}}_F\right),$$

i.e., $\Lambda^{\mathscr{H}}$ is a τ-decomposable measure on $\mathbb{S}_\tau$. □

Concluding remarks

In this paper, we have introduced a generalized decomposable (sub)measure notion related to triangle functions in probabilistic metric spaces, which covers our previously introduced notions of (sub)measures as well as certain recent notions from literature. We have given a characterization of decomposable measures and discussed construction methods providing new decomposable (sub)measures from the given ones. In connection with this "aggregation" the following problem arises:

Problem 6.7. *Characterize all the mappings which preserve the class of* τ*-decomposable measures.*

Further, we have extended Shen's results related to probabilistic-valued decomposable measures and the probabilistic Hausdorff distance for an arbitrary triangle function. We have proved that the class of all measurable sets forms an algebra on its power set. Thus, the second problem is a restatement of the Shen's question in [23]:

Problem 6.8. *Is the class $\mathbb{S}_\tau$ of all $\Lambda^{\mathscr{H}}$-measurable sets w.r.t. a triangle function τ a σ-algebra? Does this property depend on a choice of triangle function τ?*

As it is well-known, triangle functions have been used and discussed almost exclusively in the theory of probabilistic metric and probabilistic normed spaces. Particular examples of triangle functions also appear in, e.g., the treatment of fuzzy numbers or in information theory (compare also [17]). As it was described in the introduction, the probabilistic-valued (sub)measures w.r.t. the triangle function τ_M may be naturally interpreted in the context of fuzzy sets and fuzzy numbers. Thus, one may be interested in the following:

Problem 6.9. *Is there room in these theories (both PM-spaces as well as fuzzy sets) for τ-decomposable (sub)measures w.r.t. triangle functions of other type? Which interpretation do these objects have?*

Furthermore, the idea of the probabilistic integral introduced in [2] motivates to develop an integral w.r.t. probabilistic-valued decomposable (sub)measures for an arbitrary triangle function. This topic is already discussed in our recent paper [7]. This approach may be understood as a modification of the Aumann integral, or, of the Choquet-type integral based on interval-valued measures as discussed e.g. in [12].

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