Induced Representations of the Affine Group and Intertwining Operators I: Analytical Approach

Abdelhamid S. Elmabrok$^1$ and Ondrej Hutník$^2$

$^1$ School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
E-mail: a.elmabrok@maths.leeds.ac.uk

$^2$ Institute of Mathematics, Faculty of Science, P. J. Šafárik University in Košice, Jesenná 5, SK 040 01 Košice, Slovakia
E-mail: ondrej.hutnik@upjs.sk

Abstract. We analyze the construction and origin of unitary operators describing the structure of the space of continuous wavelet transforms inside the space $L_2(G, d\nu_L)$ of all square-integrable functions on the affine group $G$ with respect to the left-invariant Haar measure from the viewpoint of induced representations of $G$. We show that these operators are, in fact, intertwining operators among pairs of induced representations of the affine group $G$. A characterization of the space of wavelet transforms using the Cauchy-Riemann-type equations is given.


1. Introduction

Let $L_2(X)$ be the Hilbert space of square-integrable functions on a measure set $X$. For example, $X$ can be the real line or the open unit disk. Let $H_2(X)$ be a proper closed subspace of $L_2(X)$, e.g. the Hardy space on the real line or the Bergman space on the unit disk in the above cases, respectively. Then there is the orthogonal projection $P : L_2(X) \to H_2(X)$. Consequently, any function $\phi \in L_\infty(X)$ defines the respective Toeplitz operator $T_\phi$ on $H_2(X)$ by the identity $T_\phi f = P(\phi f)$ for $f \in H_2(X)$.

This association of a function to an operator is the core of the Berezin quantization [3], see the final section of this paper for more details. The principal question for the both operator theory and quantum mechanics is a description of the spectral properties of operator $T_\phi$ in terms of function $\phi$. One of approaches is based on a unitary transformation $U$ which map $L_2(X)$ to a certain function space in such a manner that $T_\phi$ becomes either a multiplication operator [8, 16], or differs from a such by a compact operator. From this point of view it is important to understand better the origin and properties of this unitary transformation $U$.

Thus, in this paper we review numerous connections among the unitary transformations constructed in paper [8], the structure of the underlying affine (or,
"ax + b") group \( G := \{ \zeta = (a, b); a > 0, b \in \mathbb{R} \} \) and its associated (affine) coherent states. Recall that affine coherent states (wavelets) were introduced in [2] to describe the movement of a particle on the real line, see also [1] as a standard reference book for their further development and generalizations. A natural action of \( G \) on \( L^2(\mathbb{R}) \) is then given by

\[
\pi(a,b)f(x) := \frac{1}{\sqrt{a}} f \left( \frac{x - b}{a} \right), \quad (a, b) \in G, \ x \in \mathbb{R},
\]
called also the quasi-regular representation of \( G \) on \( L^2(\mathbb{R}) \). The identity

\[
[W_\psi f](a,b) := \langle f, \pi(a,b)\psi \rangle, \ f \in L^2(\mathbb{R}),
\]
provides a starting point for the so-called wavelet analysis on \( \mathbb{R} \). It defines a map from the initial space of the representation \( \pi \) to a space of functions on the group \( G \). It always has the algebraic intertwining property between \( \pi \) and the left-regular representation of \( G \). In analysis it is desirable to make this map unitary as well, this is expressed by the following resolution of identity

\[
\langle f, g \rangle = \int_G \langle f, \pi(a,b)\psi \rangle \langle \pi(a,b)\psi, g \rangle \frac{da\,db}{a^2},
\]
also known as the Calderón reproducing formula. To achieve this the mother wavelet \( \psi \) shall be admissible: for the affine group this is equivalent to

\[
\int_{\mathbb{R}_+} |\hat{\psi}(a)|^2 \frac{da}{a} = 1, \tag{1}
\]
where \( \hat{\psi} \) stands for the Fourier transform \( F : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) given by \( F\{g\}(\xi) := \hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} \, dx \) (ordinary-frequency convention). Its inverse is usually denoted by \( F^{-1}\{g\} \), or simply \( \hat{g} \).

It is well-known that the Hardy space \( H^2(\mathbb{R}) \) is an irreducible invariant subspace of \( L^2(\mathbb{R}) \) under the quasi-regular representation \( \pi \) of \( G \). On the other hand, a co-adjoint representation of \( G \) spatially splits to irreducible components supported on the orbits, which turn to be positive and negative half-lines. Moreover, the intertwining operator between these two representations of \( G \) built from a group character is the Fourier transform. Indeed, this provides a background for wavelets technique in the theory of Toeplitz operators from [8].

As it follows from the above observations, the whole construction of operators from [8] is closely related to the structure of affine group \( G \), namely to its induced representations. In general, suppose that \( G \) is a locally compact topological group, and that \( H \) is its closed subgroup. Induction is a process that attaches a representation of \( G \) to a representation of \( H \), cf. [9, §13.2]: consider a smooth section \( s : G/H \to G \) which is a left inverse of the natural projection \( p : G \to G/H \). Thus, any element \( g \in G \) can be uniquely decomposed as \( g = s(p(g)) r(g) \), where the map \( r : G \to H \) is defined by the previous identity, i.e., \( r(g) = s(p(g))^{-1} g \). For a character \( \chi \) of \( H \) introduce the lifting map \( L_\chi : L^2(G/H) \to L^2(\chi) \) as follows

\[
[L_\chi f](g) := \chi(r(g)) f(p(g)), \quad f \in L^2(G/H).
\]
The image space of the lifting \( L_\chi \) is invariant under left shifts. We also define the pulling map \( \mathcal{P} : L^2_2(G) \to L_2(G/H) \), which is a left inverse of the lifting and explicitly can be given, for example, as \( \{\mathcal{P}F\}(x) := F(s(x)) \). Then the induced representation \( \rho_\chi \) on \( L_2(G/H) \) is generated by the formula \( \rho_\chi(g) = \mathcal{P}\Lambda(g) L_\chi \) with \( \Lambda \) being the left-regular representation on \( L_2(G) \).

Since the affine group \( G \) may be decomposed as a semi-direct product \( G = N \ltimes A \), where \( N = \{(1, b); b \in \mathbb{R}\} \) is the abelian normal closed subgroup and the quotient group \( A \) is isomorphic to the one-parameter closed subgroup \( \{(a, 0); a > 0\} \cong \mathbb{R}_+ \), see e.g. [5], we will induce representations of the affine group \( G \) in the subspaces which depend on the left homogeneous space \( X = G/H \) with \( H = \{e\} \), \( H = A \) and \( H = N \), respectively. Indeed,

(i) for \( X = G/\{e\} = G \) a character of the subgroup \( \{e\} \) induces a left-regular representation of \( G \) on \( L_2(X) = L_2(G,d\nu_L) \);

(ii) for \( X = G/N \cong A = \mathbb{R}_+ \) a character of the subgroup \( N \) induces a co-adjoint representation of \( G \) on \( L_2(G/N) = L_2(\mathbb{R}_+) \);

(iii) for \( X = G/A \cong N = \mathbb{R} \) a character of the subgroup \( A \) induces a quasi-regular representation of \( G \) on \( L_2(G/A) = L_2(\mathbb{R}) \).

In Section 2 we present a detailed construction of induced representation based on the above mentioned scheme. Since the continuous wavelet transform is the intertwining operator between the quasi-regular and left-regular representation, in Section 3 we analyze the construction and origin of unitary operators describing the structure of the space of continuous wavelet transforms from the viewpoint of induced representations of \( G \). We show that these operators are, in fact, intertwining operators among pairs of induced representations of the affine group \( G \). A characterization of the space of wavelet transforms using the Cauchy-Riemann-type equations is given in Section 4.

2. Affine group and its induced representations

The affine group \( G \), also called the "\( ax + b \)"-group, consists of all transformations \( A_{a,b} \) of the real line \( \mathbb{R} \) of the type \( A_{a,b}(x) := ax + b, \ x \in \mathbb{R} \), where \( a > 0, b \in \mathbb{R} \). Writing \( G := \{\zeta = (a, b); a > 0, b \in \mathbb{R}\} \), one has the multiplication law on \( G \) of the form \((a_1, b_1) \star (a_2, b_2) = (a_1a_2, a_1b_2 + b_1)\). With respect to this introduced group operation \( \star \) the group \( G \) is non-commutative with \( e = (1,0) \) being the identity element, and the inverse of \( \zeta = (a, b) \in G \) given by \( \zeta^{-1} = (a, b)^{-1} = (a^{-1}, -ba^{-1}) \). With respect to the multiplication \( \star \) the affine group \( G \) is a locally compact Lie and Hausdorff group on which the left-invariant Haar measure is given by \( d\nu_L(\zeta) := a^{-2}dadb \) differing from its right-invariant Haar measure \( d\nu_R(\zeta) := a^{-1}dadb \). Thus, \( G \) provides an example of a solvable, but non-unimodular Lie group. Note that the "\( ax + b \)"-group is used not only in mathematics, or signal analysis, but nowadays has its place in quantum physics, see e.g. the paper [17] where the problem of constructing quantum deformation of quantum version of the group \( G \) is studied in detail.
The left-regular representation  The important case of the trivial subgroup \( H = \{ e \} \), for which \( X = \mathbb{G}/H = \mathbb{G} \), generates the left-regular representation arising from the action of the affine group on itself by left translations. Let \( L_2(\mathbb{G}, d\nu_L) \) be the Hilbert space of all square-integrable complex-valued functions on \( \mathbb{G} \) with respect to the measure \( d\nu_L \). The left-regular representation of \( \mathbb{G} \) on \( L_2(\mathbb{G}, d\nu_L) \) is the unitary operator \( \Lambda(a, b) \) given by

\[
[A(a, b)F](u, v) := F \left( ((a, b)^{-1} \ast (u, v)) = F \left( \frac{u}{a}, \frac{v - b}{a} \right) ,
\]

where \((u, v) \in \mathbb{G}\) and \( F \in L_2(\mathbb{G}, d\nu_L) \). Clearly, \( \Lambda \) is a unitary reducible representation. Similarly, we may define the right-regular representation of \( \mathbb{G} \).

The co-adjoint representation  The second case of the left homogeneous space \( X = \mathbb{G}/N \cong A = \mathbb{R}_+ \) generates the co-adjoint representation \( \rho \) of \( \mathbb{G} \) on \( L_2(\mathbb{R}) \). In accordance with the general construction presented in Introduction, let \( s : \mathbb{R}_+ \to \mathbb{G} \) given by \( s(a) = (a, 0) \) be a smooth function which is a left inverse to the natural projection \( p : \mathbb{G} \to \mathbb{R}_+ \) given by \( p(a, b) = a \). Then any \((a, b) \in \mathbb{G}\) has a unique decomposition of the form \((a, b) = s(a) \ast \beta = (a, 0) \ast (1, \frac{b}{a})\), where \( a = p(a, b) \in \mathbb{A} \) and \( \beta = (1, \frac{b}{a}) \in \mathbb{N} \). Note that the subgroup \( \mathbb{A} \) admits a \( \mathbb{G} \)-invariant (Haar) measure \( \frac{da}{a} \). Now, \( X \) is a left homogeneous space with the \( \mathbb{G} \)-action defined in terms of \( p \) and \( s \) as follow

\[
(a, b) : w \mapsto (a, b) \cdot w = p ((a, b) \ast s(w)) = aw ,
\]

where \( \cdot \) is the action of \( \mathbb{G} \) on \( X \) from the left.

Let \( \chi_\omega : \mathbb{N} \to \mathbb{T} \) be a character of \( \mathbb{N} \) defined by \( \chi_\omega (\beta) = e^{2\pi i \frac{b}{a}} \) which induces a linear representation of \( \mathbb{G} \) constructed in the Hilbert space \( L_2^\omega (\mathbb{G}) \) consisting of functions \( F_\omega : \mathbb{G} \to \mathbb{C} \) of the form

\[
F_\omega (a, b) = \chi_\omega \left( 1, \frac{b}{a} \right) F (a, 0) \quad \text{with} \quad ||F_\omega||_A^2 = \int_{\mathbb{R}_+} |F (a, 0)|^2 \frac{da}{a}.
\]

Let \( r : \mathbb{G} \to \mathbb{N} \) be a map given by \( r (a, b) = s(a)^{-1} \ast (a, b) = \beta \). Then we may associate each function \( f \) on the half-line \( \mathbb{R}_+ \) with a function \( F \) on the affine group \( \mathbb{G} \) via the lifting map \( L_{\chi_\omega} : L_2(\mathbb{R}_+) \to L_2^\omega (\mathbb{G}) \) as follows

\[
[L_{\chi_\omega} f] (a, b) := \chi_\omega (r (a, b)) f (p(a, b)) = \chi_\omega \left( 1, \frac{b}{a} \right) f (a) = e^{2\pi i \frac{b}{a}} f (a) ,
\]

where \( f (a) \equiv F (a, 0) \) is a function on the subgroup \( \mathbb{A} \). Then the pulling map \( \mathcal{P} : L_2^\omega (\mathbb{G}) \to L_2(\mathbb{R}_+) \) is defined as follows \( \mathcal{P} F (a) = F (s(a)) := F (a, 0) = f (a) \). It is easy to verify that \( L_{\chi_\omega} \circ \mathcal{P} = I : L_2^\omega (\mathbb{G}) \to L_2^\omega (\mathbb{G}) \) and \( \mathcal{P} \circ L_{\chi_\omega} = I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+) \). Using the commutative diagram on Figure 1, a character \( \chi_\omega \) induces a representation \( \rho_{\chi_\omega}^\pm : L_2 (\mathbb{R}_\pm) \to L_2 (\mathbb{R}_\pm) \) in the sense of Mackey by \( \rho_{\chi_\omega}^\pm (a, b) := \mathcal{P} \Lambda (a, b) L_{\chi_\omega} \). By a simple calculation with \((u, v) \in \mathbb{G}\) we get

\[
\left[ \rho_{\chi_\omega}^\pm (a, b) f \right] (u) = e^{-2\pi i \frac{b}{a}} f \left( \frac{u}{a} \right) , \quad f \in L_2 (\mathbb{R}_\pm) .
\]
The interesting feature of these representations consists in fact that their direct sum
\( \rho_{\chi} (a, b) = \rho_{\chi^+} (a, b) \oplus \rho_{\chi^-} (a, b) \) is given by
\[
[\rho_{\chi} (a, b) f] (u) = e^{-2\pi i \frac{\chi}{\omega}} f \left( \frac{u}{a} \right), \quad f \in L_2 (\mathbb{R}).
\]
Changing the variables \( t = u^{-1} \), \( g (t) = t^{-\frac{1}{2}} f (t^{-1}) \) we finally have
\[
[\rho_{\chi} (a, b) g] (t) = \sqrt{a} e^{-2\pi i \frac{\chi}{\omega}} g (at), \quad g \in L_2 (\mathbb{R}),
\]
which is the co-adjoint representation of the affine group \( \mathbb{G} \) on \( L_2 (\mathbb{R}) \). For the following observation recall that if \( \phi_1, \phi_2 \) are two representations of a group \( G \) on Hilbert spaces \( H_{\phi_1}, H_{\phi_2} \), respectively, then the operator \( T : H_{\phi_1} \to H_{\phi_2} \) intertwines representations \( \phi_1 \) and \( \phi_2 \), iif \( T \phi_1 (g) = \phi_2 (g) T \) for each \( g \in G \).

**Proposition 2.1** The lifting map \( \mathcal{L}_{\chi} : L_2 (\mathbb{R}_+) \to L_2^\chi (\mathbb{G}) \) intertwines the co-adjoint representation \( \rho_{\chi^+}^\pm \) and the left-regular representation \( \Lambda \) of \( \mathbb{G} \). Therefore, the image \( \mathcal{L}_{\chi} (L_2 (\mathbb{R}_+)) \) is invariant under the left-regular representation \( \Lambda \) of \( \mathbb{G} \).

In what follows we will consider only the most simple case \( \omega = 1 \) and therefore the subscript \( \chi \) will be usually omitted.

The quasi-regular representation Now we come up to the third case of the left homogeneous space \( X = \mathbb{G} / \mathbb{A} \cong \mathbb{N} = \mathbb{R} \), which generates the quasi-regular representation \( \pi^+ \) of \( \mathbb{G} \) on the Hardy space \( H_2 (\mathbb{R}) \). In fact, the Hardy space \( H_2 (\mathbb{R}) \) will be an irreducible component of the quasi-regular representation on the real line.

Let \( s : \mathbb{R} \to \mathbb{G} \) given by \( s (b) = (1, b) \) be a smooth function, which is a left inverse to the natural projection \( p : \mathbb{G} \to \mathbb{R} \) in the form \( p (a, b) = b \). Then any \( (a, b) \in \mathbb{G} \) has a unique decomposition of the form \( (a, b) = s (b) \ast \alpha = (1, b) \ast (a, 0) \), where \( b = p (a, b) \in \mathbb{R} \) and \( \alpha = (a, 0) \in \mathbb{A} \). Note that the subgroup \( \mathbb{N} \) admits a \( \mathbb{G} \)-invariant (Haar) measure \( db \). Now, \( X \) is a left homogeneous space with the \( \mathbb{G} \)-action defined in terms of \( p \) and \( s \) as follows
\[
(a, b) : x \mapsto (a, b) \cdot x = p ((a, b) \ast s (x)) = ax + b,
\]
where \( \cdot \) denotes the action of \( \mathbb{G} \) on \( X \) from the left. Also, this explains the name ”\( ax + b \)”-group for the group \( \mathbb{G} \).

Let \( \sigma_\tau : \mathbb{A} \to \mathbb{T} \) be a character of \( \mathbb{A} \) defined by \( \sigma_\tau (a, 0) = a^\tau, \tau \in \mathbb{R} \), which induces a linear representation of \( \mathbb{G} \) constructed in the Hilbert space \( L_2^\omega (\mathbb{G}) \) consisting of functions \( F_\tau : \mathbb{G} \to \mathbb{C} \) such that
\[
F_\tau (a, b) = \sigma_\tau (a, 0) F_0 (1, b) \quad \text{with the norm} \quad \| F_\tau \|_N^2 = \int_{\mathbb{R}} \| F (1, b) \|^2 \, db.
\]
Introduce the map \( r : \mathbb{G} \to \mathbb{A} \) given by \( r (a, b) = s (b)^{-1} \ast (a, b) = \alpha \). Now, we associate each function \( f \) on the real line \( \mathbb{R} \) with a function \( F \) on the affine group \( \mathbb{G} \) via the lifting map \( \mathcal{L}_{\sigma_\tau} : H_2 (\mathbb{R}) \to L_2^{\sigma_\tau} (\mathbb{G}) \) by
\[
[\mathcal{L}_{\sigma_\tau} f] (a, b) := \sigma_\tau (r (a, b)) f (p (a, b)) = a^\tau f (b),
\]
Induced Representations of the Affine Group and Intertwining Operators I

\[ L_{\chi_{\omega}} \] \[ \xrightarrow{\Lambda} \] \[ L_{\chi_{\omega}}(G) \] \[ \xrightarrow{\mathcal{L}_{\chi_{\omega}}} \] \[ L_{2}(\mathbb{R}_{+}) \] \[ \xrightarrow{\rho_{\chi_{\omega}}} \] \[ L_{2}(\mathbb{R}_{+}) \] \[ L_{\pi}(\mathbb{G}) \] \[ \xrightarrow{\Lambda} \] \[ L_{\pi}(\mathbb{G}) \] \[ \xrightarrow{\mathcal{L}_{\pi}} \] \[ L_{2}(\mathbb{R}) \] \[ \xrightarrow{\pi^{+}_{\sigma_{\tau}}} \] \[ L_{2}(\mathbb{R}) \] \[ \xrightarrow{\mathcal{P}'} \] \[ L_{\pi^{+}}(\mathbb{G}) \]

**Figure 1.** Relationship among the induced representations of the affine group \( G \) via lifting and pulling map

where \( f(b) \equiv F(1, b) \) is a function on the subgroup \( N \). Then the pulling map \( \mathcal{P}': L_{\pi}^{+}(G) \to H_{2}(\mathbb{R}) \) is given by \((\mathcal{P}'F)(b) := F(s(b)) = F(1, b) = f(b)\), and we have \( \mathcal{L}_{\sigma_{\tau}} \circ \mathcal{P}' = I : L_{\pi}^{+}(G) \to L_{\pi}^{+}(G) \) and \( \mathcal{P}' \circ \mathcal{L}_{\sigma_{\tau}} = I : H_{2}(\mathbb{R}) \to H_{2}(\mathbb{R}) \). Using the commutative diagram on Figure 1, a character \( \sigma_{\tau}(a, 0) = a^{ir+\frac{1}{2}} \) induces a representation \( \pi^{+}_{\sigma_{\tau}} : H_{2}(\mathbb{R}) \to H_{2}(\mathbb{R}) \) by \( \pi^{+}_{\sigma_{\tau}}(a, b) := \mathcal{P}' \Lambda (a, b) \mathcal{L}_{\sigma_{\tau}} \). Easily, for \( (u, v) \in G \) we have the formula

\[
[\pi^{+}_{\sigma_{\tau}}(a, b) f](v) = \left( \frac{1}{a} \right)^{ir+\frac{1}{2}} f \left( \frac{v - b}{a} \right), \quad f \in H_{2}(\mathbb{R}), \ (a, b) \in G.
\]

The obtained representation \( \pi^{+}_{\sigma_{\tau}} \) is called the quasi-regular representation of the affine group \( G \) on \( H_{2}(\mathbb{R}) \). It is an irreducible unitary and square-integrable representation of \( G \) on \( H_{2}(\mathbb{R}) \).

**Proposition 2.2** The lifting map \( \mathcal{L}_{\sigma_{\tau}} : H_{2}(\mathbb{R}) \to L_{\pi}^{+}(G) \) intertwines the quasi-regular representation \( \pi^{+}_{\sigma_{\tau}} \) and the left-regular representation \( \Lambda \) of \( G \). Thus, the image \( \mathcal{L}_{\sigma_{\tau}}(H_{2}(\mathbb{R})) \) is invariant under the left-regular representation \( \Lambda \) of \( G \).

Similarly as above we get \( \pi^{-}_{\sigma_{\tau}}(a, b) f \) for \( f \in H_{2}(\mathbb{R})^{\perp} \), and hence,

\[
[\pi^{-}_{\sigma_{\tau}}(a, b) f](v) = \left( \frac{1}{a} \right)^{ir+\frac{1}{2}} f \left( \frac{v - b}{a} \right), \quad f \in L_{2}(\mathbb{R}).
\]

In what follows we will consider the case \( \tau = 0 \), therefore its index will be omitted, i.e.,

\[
[\pi(a, b) f](v) = \frac{1}{\sqrt{a}} f \left( \frac{v - b}{a} \right), \quad f \in L_{2}(\mathbb{R}). \tag{3}
\]

From this action on signals (we identify a signal with an element \( f \in L_{2}(\mathbb{R}) \)) we observe that \( G \) consists precisely of the transformations we apply to a signal: *translation* (time-shift) by an amount \( b \), and *zooming in or out* by the factor \( a \). Hence, the group \( G \) naturally relates to the geometry of signals.

The Hilbert space \( L_{2}(\mathbb{R}) \) under the action \( \pi \) contains precisely two closed proper invariant subspaces \( H_{2}(\mathbb{R}) \) and \( H_{2}(\mathbb{R})^{\perp} \), such that \( L_{2}(\mathbb{R}) = H_{2}(\mathbb{R}) \oplus H_{2}(\mathbb{R})^{\perp} \). Thus, \( \pi \) is a reducible representation on \( L_{2}(\mathbb{R}) \), and we can decompose the reducible quasi-regular representation \( \pi \) into two irreducible representations, such that \( \pi(a, b) = \pi^{+}(a, b) \oplus \pi^{-}(a, b) \). From it follows that only the Hardy space \( H_{2}(\mathbb{R}) \) is considered,
although the discussion is equally valid for the conjugate Hardy space $H_2(\mathbb{R})^\perp$. Let us mention for later purposes that the orthogonal projections $P_R$ and $P_R^\perp$ of $L_2(\mathbb{R})$ onto $H_2(\mathbb{R})$ and $H_2(\mathbb{R})^\perp$ are called the Szegö projections, respectively.

**Induced representations of the affine group and its coherent states** Without any doubt, the unitary (quasi-regular) representation $\pi$ of $G$ given by (3) is not new. As we have stated in Introduction it is used quite a long time in wavelet analysis on $L_2(\mathbb{R})$. In what follows a (mother) wavelet is a function $\psi \in L_2(\mathbb{R})$ satisfying the admissibility condition (1). The group-theoretical methods providing constructions generalizing admissible wavelets are given in papers [10] and [12]. However, many interesting types of wavelets arise from group representations which are not square integrable or vacuum vectors which are not admissible, cf. [11].

For a fixed admissible mother wavelet $\psi \in L_2(\mathbb{R})$, the functions $W\psi f$ on $G$ of the form $[W\psi f](\zeta) := \langle f, \pi(\zeta)\psi \rangle$ with $f \in L_2(\mathbb{R})$, $\zeta \in G$, form a reproducing kernel Hilbert space $W\psi (L_2(\mathbb{R}))$ called the space of wavelet transforms. Then the integral operator $P_\psi : L_2(G, d\nu_L) \to L_2(G, d\nu_L)$ given by

$$[P_\psi F](\eta) := \int_G F(\zeta) \langle \pi(\zeta)\psi, \pi(\eta)\psi \rangle \, d\nu_L(\zeta), \quad F \in L_2(G, d\nu_L),$$

(4)

is the orthogonal projection onto $W\psi (L_2(\mathbb{R}))$.

Summarizing results and constructions of previous sections we have the relationship among the induced representations $\Lambda$, $\rho$ and $\pi^+$ of the affine group $G$ schematically described on Figure 2. Observe that on the front size of this cuboid are described relations between the induced representations using unitary operators of continuous wavelet transform $W\psi$ and Fourier transform $\mathcal{F}$, whereas on the lateral face these operators describe relations between orthogonal projections of considered spaces.

From these results it follows that the wavelet transform $W\psi$ built on the quasi-regular representation $\pi$ is an intertwining operator to the left-regular representation $\Lambda$. Thus, the image $W\psi (L_2(\mathbb{R}))$ is a left-invariant subspace. Moreover, it follows that there is an intertwining operator between the co-adjoint representation $\rho$ and quasi-regular representation $\pi$. This is the Fourier transform $\mathcal{F}$ in the essence, since it uses the characters which induce those representations. Intertwining operators between various wavelet transforms can be constructed by Proposition 2.16 in [10].

### 3. Intertwining property of unitary operators

In [7] the second author presented a method based on the general decomposition scheme of Vasilevski, see [16], which enables to obtain the information “how much room occupies the space $W\psi (L_2(\mathbb{R}))$ inside $L_2(G, d\nu_L)$”. Then these results were used in application to Toeplitz-type operators related to wavelets in [8]. In fact, the basic idea is based on the construction of unitary operators providing decomposition of wavelet transform. Since $W\psi$ is an intertwining operator between $\pi$ and $\Lambda$, our aim is to provide a detailed
description and construction of intertwining operators related to $L_2$-type spaces in terms of representations of the affine group $\mathbb{G}$.

Let $\psi \in L_2(\mathbb{R})$ be a fixed admissible mother wavelet. Represent the Hilbert space $L_2(\mathbb{G}, d\nu_L)$ as a tensor product in the form $L_2(\mathbb{G}, d\nu_L(\eta)) = L_2(\mathbb{R}^+, u^{-2} \, du) \otimes L_2(\mathbb{R}, dv)$, where $\eta = (u, v) \in \mathbb{G}$, and introduce the unitary operator

$$U_1 = (I \otimes \mathcal{F}) : L_2(\mathbb{G}, d\nu_L(\eta)) \to L_2(\mathbb{R}^+, u^{-2} \, du) \otimes L_2(\mathbb{R}, dv)$$

with $\mathcal{F}$ being the Fourier transform. The image $\Delta_1$ of the space $W_\psi(L_2(\mathbb{R}))$ in the mapping $U_1$ consists of all functions

$$F(u, \omega) = \sqrt{u} f(\omega) \hat{\psi}(\omega u), \quad f \in L_2(\mathbb{R}), \ (u, \omega) \in \mathbb{G}. \quad (5)$$

Obviously, $\|F(u, \omega)\|_{\Delta_1} = \|f(\omega)\|_{L_2(\mathbb{R}, d\omega)}$. Then the operator $\lambda_1 : L_2(\mathbb{G}, d\nu_L) \to \Delta_1$ given by $\lambda_1 = U_1 P_\psi U_1^* = (I \otimes \mathcal{F}) P_\psi (I \otimes \mathcal{F}^{-1})$ has the explicit form

$$[\lambda_1 F](u, \omega) = \sqrt{u} \psi(\omega u) \int_{\mathbb{R}^+} F(s, \omega) \hat{\psi}(\omega s) \frac{ds}{s^{1/2}}.$$

Thus, $\text{Im} \lambda_1 = \Delta_1$. Moreover, $\lambda_1^2 = \lambda_1$ and $\lambda_1$ is obviously self-adjoint.

**Theorem 3.1** The action $\Lambda_1 = U_1 \Lambda U_1^*$ on $L_2(\mathbb{R}^+, u^{-2} \, du) \otimes L_2(\mathbb{R}, d\omega)$ has the form

$$[\Lambda_1(a, b) F](u, \omega) = a e^{-2\pi i b \omega} F \left( \frac{u}{a}, a \omega \right).$$

**Proof.** For $(a, b) \in \mathbb{G}$ we have

$$[\Lambda_1(a, b) F](u, \omega) = \left( (I \otimes \mathcal{F}) \tilde{F}_2 \left( \frac{u}{a}, \frac{v - b}{a} \right) \right) = \int_{\mathbb{R}} \tilde{F}_2 \left( \frac{u}{a}, \frac{v - b}{a} \right) e^{-2\pi i \omega v} \, dv$$

$$= \int_{\mathbb{R}} \tilde{F}_2 \left( \frac{u}{a}, t \right) e^{-2\pi i (at + b)\omega} \, dt = a e^{-2\pi i b \omega} F \left( \frac{u}{a}, a \omega \right).$$
where $\hat{F}_2(u,v) = [(I \otimes F^{-1})F(u,\omega)]$ is the inverse Fourier transform of the function $F(u,\omega)$ with respect to the second variable.

From (5) we immediately get that the action $\Lambda_1 : \Delta_1 \to \Delta_1$ has the form

$$[\Lambda_1(a,b)F](u,\omega) = \sqrt{a}\ e^{-2\pi iby} f(aw)\psi(u\omega).$$

Furthermore, the image $U_1(W_\psi(L_2(\mathbb{R}))) = \Delta_1$ is invariant under the representation $\Lambda_1$, and $U_\psi^*(\Delta_1)$ is invariant under the representation $\Lambda$ of $G$.

Introduce the isometric imbedding $R_0 : L_2(\mathbb{R}) \to \Delta_1$ by the rule $[R_0f](u,\omega) = \sqrt{u}\ f(\omega)\psi(u\omega)$. The adjoint operator $R_0^* : L_2(G, d\nu_L) \to L_2(\mathbb{R})$ is given by

$$[R_0^*F](y) = \int_{\mathbb{R}_+} F(s,y)\hat{\psi}(sy)\frac{ds}{s^{3/2}},$$

and it is easy to verify that the operators $R_0$ and $R_0^*$ provide the following decomposition of identity on $L_2(\mathbb{R})$ and of orthogonal projection $\Lambda_1$, i.e.,

$$R_0^*R_0 = I : L_2(\mathbb{R}) \to L_2(\mathbb{R}),$$
$$R_0R_0^* = \lambda_1 : L_2(G, d\nu_L) \to \Delta_1.$$

Thus, we immediately have

**Theorem 3.2** The action $\rho$ of $G$ on $L_2(\mathbb{R})$ is unitarily equivalent to $R_0^*\Lambda_1R_0$.

**Proof.** By a direct computation we have

$$[R_0^*\Lambda_1R_0f](y) = \left(R_0^*\left(a\ e^{-2\pi iby}[R_0f]\left(\frac{s}{a},ay\right)\right)\right) = a\ e^{-2\pi iby}\int_{\mathbb{R}_+}[R_0f]\left(\frac{s}{a},ay\right)\hat{\psi}(sy)\frac{ds}{s^{3/2}}$$

$$= \sqrt{a}\ e^{-2\pi iby} f(ay)\int_{\mathbb{R}_+} |\hat{\psi}(sy)|^2\frac{ds}{s} = \sqrt{a}\ e^{-2\pi iby} f(ay) = [\rho(a,b)f](y)$$

with $(a,b) \in G$. \hfill $\Box$

**Corollary 3.3** The image $R_0(L_2(\mathbb{R}))$ is invariant under the representation $\Lambda_1$ of $G$ and $R_0^*(\Delta_1)$ is invariant under the co-adjoint representation $\rho$ of $G$.

Moreover, the operator $R = R_0^*U_1$ maps the space $L_2(G, d\nu_L)$ onto $L_2(\mathbb{R})$, and its restriction $R|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to L_2(\mathbb{R})$ is an isometrical isomorphism. The adjoint $R^* = U_1^*R_0 : L_2(\mathbb{R}) \to W_\psi(L_2(\mathbb{R})) \subset L_2(G, d\nu_L)$ is an isometrical isomorphism of $L_2(\mathbb{R})$ onto the wavelet subspace $W_\psi(L_2(\mathbb{R}))$.

**Theorem 3.4** The action $\rho$ of $G$ on $L_2(\mathbb{R})$ is unitarily equivalent to $R\Lambda R^*$.

**Proof.** Using the above stated results we immediately have the following identities

$$R\Lambda R^* = R_0^*U_1\Lambda U_1^*R_0 = R_0^*\Lambda_1R_0 = \rho.$$
Corollary 3.5 The unitary operator \( R \) gives the isometrical isomorphism of \( L_2(G, d\nu_L) \) onto \( L_2(\mathbb{R}) \) and its restriction \( R|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to L_2(\mathbb{R}) \) intertwines the left-regular representation \( \Lambda \) of \( G \) on \( W_\psi(L_2(\mathbb{R})) \) and the co-adjoint representation \( \rho \) of \( G \) on \( L_2(\mathbb{R}) \). The adjoint operator \( R^* : L_2(\mathbb{R}) \to W_\psi(L_2(\mathbb{R})) \) intertwines \( \rho \) of \( G \) on \( L_2(\mathbb{R}) \) and \( \Lambda \) of \( G \) on \( W_\psi(L_2(\mathbb{R})) \).

Corollary 3.6 The image \( R(W_\psi(L_2(\mathbb{R}))) \) is invariant under the co-adjoint representation \( \rho \) of \( G \) and \( R^* (L_2(\mathbb{R})) \) is invariant under the left-regular representation \( \Lambda \) of \( G \).

Introducing the operator \( \tilde{R} : L_2(G, d\nu_L) \to L_2(\mathbb{R}) \) as follows \( \tilde{R} = \mathcal{F}^{-1}R_0^*(I \otimes \mathcal{F}) \) we immediately have \( \tilde{R}\Lambda\tilde{R}^* = \mathcal{F}^{-1}R_0^*U_1\Lambda U_1^* R_0 \mathcal{F} = \mathcal{F}^{-1}R_0^*\Lambda R_0 \mathcal{F} = \mathcal{F}^{-1}\rho \mathcal{F} = \pi^+ \), which shows that the action \( \pi^+ \) of \( G \) on \( H_2(\mathbb{R}) \) is unitarily equivalent to \( \tilde{R}\Lambda\tilde{R}^* \).

All these observations may be formulated in the following theorem. The presented relationships among all the above constructed unitary (intertwining) operators among induced representations of \( G \) are visualized on Figure 3.

Theorem 3.7 The operator \( \tilde{R}|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to H_2(\mathbb{R}) \) gives an isometrical isomorphism and intertwines the quasi-regular representation \( \pi^+ \) of \( G \) on \( H_2(\mathbb{R}) \) and the left-regular representation \( \Lambda \) of \( G \) on \( W_\psi(L_2(\mathbb{R})) \). The adjoint operator \( \tilde{R}^*|_{H_2(\mathbb{R})} : H_2(\mathbb{R}) \to W_\psi(L_2(\mathbb{R})) \) intertwines \( \Lambda \) of \( G \) on \( W_\psi(L_2(\mathbb{R})) \) and \( \pi^+ \) of \( G \) on \( H_2(\mathbb{R}) \). The operators \( \tilde{R} \) and \( \tilde{R}^* \) provide the decomposition of the wavelet \( P_\psi \) and Szegö \( P_\mathbb{R} \) projections as follows
\[
\tilde{R}^* \tilde{R} = P_\psi : L_2(G, d\nu_L) \to W_\psi(L_2(\mathbb{R})),
\]
\[
\tilde{R}\tilde{R}^* = P_\mathbb{R} : L_2(\mathbb{R}) \to H_2(\mathbb{R}).
\]

Corollary 3.8 The image \( \tilde{R}(W_\psi(L_2(\mathbb{R}))) \) is invariant under the quasi-regular representation \( \pi \) of \( G \) and \( \tilde{R}^*(H_2(\mathbb{R})) \) is invariant under the left-regular representation \( \Lambda \) of \( G \).

Following [8] introduce now the unitary operator \( U_2 : L_2(\mathbb{R}^+, u^{-2} du) \otimes L_2(\mathbb{R}, d\nu) \to L_2(\mathbb{R}^+, dx) \otimes L_2(\mathbb{R}, dy) \) by the rule
\[
U_2 : F(u, \omega) \mapsto \sqrt{|y|} \frac{F(x/|y|, y)}{x}.
\]
Then the inverse operator \( U_2^{-1} = U_2^* : L_2(\mathbb{R}^+, dx) \otimes L_2(\mathbb{R}, dy) \to L_2(\mathbb{R}^+, u^{-2} du) \otimes L_2(\mathbb{R}, d\omega) \) is given by
\[
U_2^{-1} : F(x, y) \mapsto \sqrt{|\omega|} u F(u|\omega|, \omega).
\]
Put \( \Delta_2 = U_2(\Delta_1) \). Then the operator \( \lambda_2 = U_2 \lambda_1 U_2^{-1} \) is obviously the orthogonal projection of \( L_2(G, d\nu_L) \) onto \( \Delta_2 \) of the form \( [\lambda_2 F](x, y) = \ell_+(x) \int_{\mathbb{R}^+} F(s, y) \ell_+(s) \, ds \), where \( \ell_\pm(x) = x^{-1/2} \psi(\pm x) \) are functions such that \( \ell_\pm(x) \in L_2(\mathbb{R}^+, dx) \) and
Theorem 3.9 The action \( \Lambda_2 = U_2 \Lambda_1 U_2^* \) on \( L_2(\mathbb{R}_+, dx) \otimes L_2(\mathbb{R}, dy) \) has the form
\[
[\Lambda_2(a, b) F](x, y) = \sqrt{a} e^{-2\pi iby} F(x, ay).
\]

Proof. By a direct computation we have
\[
[\Lambda_2(a, b) F](x, y) = U_2 \left( a e^{-2\pi ib\omega} \left[ U_2^* F \left( \frac{u}{a}, a\omega \right) \right] \right) = \left( U_2 \left( a e^{-2\pi ib\omega} \sqrt{|a\omega|} F \left( |a\omega| \frac{u}{a}, a\omega \right) \right) \right) = \sqrt{a} e^{-2\pi iby} F(x, ay).
\]
with \((a, b) \in \mathbb{G}\.\)

For the co-adjoint representation \( \rho \) of \( \mathbb{G} \) on \( L_2(\mathbb{R}, dy) \) let \([\rho^+(a, b) f](y) = \chi_+(y)|\rho(a, b) f|(y)\) be its restriction onto positive and negative half-lines, respectively. Then for \( F \in \Delta_1 \) the function \( U_2 F \) from the space \( \Delta_2 \) may be represented by \([U_2 F](x, y) = [\rho^+(a, b) f](y) \ell_+(x) + [\rho^-(a, b) f](y) \ell_-(x)\), and we immediately get that the action \( \Lambda_2 : \Delta_2 \to \Delta_2 \) has the form \([\Lambda_2(a, b) F](x, y) = [\rho(a, b) f](y) \ell_\pm(x)\).

Corollary 3.10 The image \( U_2(\Delta_1) = \Delta_2 \) is invariant under the representation \( \Lambda_2 \) of \( \mathbb{G} \) and \( U_2^*(\Delta_2) \) is invariant under the representation \( \Lambda_1 \) of \( \mathbb{G} \).
Denote by $L_\pm$ the one-dimensional subspaces of $L_2(\mathbb{R}_+, dx)$ generated by functions $\ell_\pm(x)$. Then the space $\Delta_2$ may be written as $\Delta_2 = L_+ \otimes L_2(\mathbb{R}_+) \oplus L_- \otimes L_2(\mathbb{R}_-)$, and the one-dimensional projections $P_\pm$ of $L_2(\mathbb{R}_+, dx)$ onto $L_\pm$ have the form

$$[P_\pm h](x) = (\langle h, \ell_\pm \rangle) \ell_\pm = \frac{1}{\sqrt{x}} \hat{\psi}(\pm x) \int_{\mathbb{R}_+} h(s) \hat{\psi}(\pm s) \frac{ds}{\sqrt{s}}.$$ 

In this light the projection $\lambda_2$ has the form $\lambda_2 = P_+ \otimes \chi_+ I \oplus P_- \otimes \chi_- I$. Introducing the operator $U = U_2 U_1 : L_2(\mathbb{G}, d\nu_L) \to L_2(\mathbb{R}_+, dx) \otimes L_2(\mathbb{R}, dy)$ we have that $U \Delta U^* = U_2 U_1 \Lambda U_1^* U_2^* = U_2 \Lambda_2 U_2^* = \Lambda_2$, i.e., the action $\Lambda_2$ is unitarily equivalent to $U \Delta U^*$. Summarizing the above statements we get the following result.

**Theorem 3.11** The unitary operator $U$ gives an isometrical isomorphism of the space $L_2(\mathbb{G}, d\nu_L)$ onto $L_2(\mathbb{R}_+, dx) \otimes L_2(\mathbb{R}, dy)$, and its restriction $U|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to \Delta_2$ intertwines $\Lambda$ on $W_\psi(L_2(\mathbb{R}))$ and $\Lambda_2 = \mathrm{Id} \otimes \rho$ on $\Delta_2$. Thus, the image $U(W_\psi(L_2(\mathbb{R}))) = \Delta_2$ is invariant under the representation $\Lambda_2$, and $U^*(\Delta_2)$ is invariant under the representation $\Lambda$ of $\mathbb{G}$.

Recall that the Hardy spaces $H_2(\mathbb{R})$ and $H_2(\mathbb{R})^\perp$ are the proper closed invariant subspaces of $L_2(\mathbb{R})$ under the action $\pi$ of $\mathbb{G}$. Therefore, put $U_3 = (I \otimes \mathcal{F}^{-1}) U_2 (I \otimes \mathcal{F})$, and denote by $\Delta_3$ the image of $W_\psi(L_2(\mathbb{R}))$ under the mapping $U_3$. Then $\Delta_3 = L_+ \otimes H_2(\mathbb{R}) \oplus L_- \otimes H_2(\mathbb{R})^\perp$.

**Theorem 3.12** The action $\Lambda_3 = U_1^* \Lambda_2 U_1$ on $L_2(\mathbb{R}_+, dx) \otimes L_2(\mathbb{R}, dy)$ has the form

$$[\Lambda_3(a, b)F](x, y) = \frac{1}{\sqrt{a}} F\left(x, \frac{y-b}{a}\right).$$

**Proof.** Obviously,

$$[\Lambda_3(a, b)F](x, y) = \left[U_1^*[\Lambda_2(a, b) \hat{F}_2(x, \omega)]\right] = \sqrt{a} \int_{\mathbb{R}} \hat{F}_2(x, a\omega) e^{2\pi i \omega (y-b)} d\omega = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \hat{F}_2(x, t) e^{2\pi i (y-b)/a} \frac{dt}{\sqrt{a}} F\left(x, \frac{y-b}{a}\right),$$

where $(a, b) \in \mathbb{G}$, and $\hat{F}_2(x, \omega) = [(I \otimes \mathcal{F}) F(x, y)]$ is the Fourier transform of the function $F(x, y)$ with respect to the second variable. $\Box$

Easily, the action $\Lambda_3$ is unitarily equivalent to $U_3 \Delta U_3^*$. Clearly, $\lambda_3 = U_1^* \lambda_2 U_1$ is the orthogonal projection of $L_2(\mathbb{G}, d\nu_L)$ onto $\Delta_3$, and has the form $\lambda_3 = P_+ \otimes P_\mathbb{R} \oplus P_- \otimes P_\mathbb{R}^\perp$.

**Theorem 3.13** The unitary operator $U_3$ gives an isometrical isomorphism of the space $L_2(\mathbb{G}, d\nu_L)$ onto $L_2(\mathbb{R}_+, dx) \otimes L_2(\mathbb{R}, dy)$, and its restriction $U_3|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to \Delta_3$ intertwines $\Lambda$ on $W_\psi(L_2(\mathbb{R}))$ and $\Lambda_3 = \mathrm{Id} \otimes \pi$ on $\Delta_3$.

**Corollary 3.14** The image $U_3(W_\psi(L_2(\mathbb{R}))) = \Delta_3$ is invariant under the representation $\Lambda_3$ of $\mathbb{G}$ and $U_3^*(\Delta_3)$ is invariant under the representation $\Lambda$ of $\mathbb{G}$.

The whole situation with the above constructed operators is described schematically on Figure 3.
4. Characterization of the image of wavelet transform by Cauchy-Riemann-type equation

In this section we provide a characterization of the image of the space of wavelet (Calderón) transforms $W_\psi (L^2_\mathbb{R})$ within the entire $L^2$-type spaces using the differential equation of Cauchy-Riemann type. In what follows let

$$X_A = (1, 0) \text{ and } X_N = (0, 1),$$

be the basis of the corresponding Lie algebra $\mathfrak{g}$ generating one-parameter subgroups $A$ and $N$ of $G$,

$$\exp (tX_A) = (e^t, 0) \text{ and } \exp (tX_N) = (1, t),$$

respectively. Recall, see [13], that the derived representation of a representation $\phi$ of the Lie algebra $\mathfrak{g}$ is a linear map $d\phi^{X_H} : \mathcal{H}_{\phi}^\infty \to \mathcal{H}_{\phi}^\infty$, depending linearly on $X_H \in \mathfrak{g}$, such that

$$d\phi^{X_H} f = \left. \frac{d}{dt} \phi (\exp (tX_H)) f \right|_{t=0}$$

for $f \in \mathcal{H}_{\phi}^\infty$, where $\mathcal{H}_{\phi}^\infty$ is the space of all $C^\infty$-functions on $G$.

The derived representations of co-adjoint representation $\rho$ defined by (2) acts on the basis as follows

$$[d\rho^{X_A} f] (u) = \left. \frac{d}{dt} \rho (\exp (tX_A)) f (u) \right|_{t=0} = uf'(u) + \frac{1}{2} f (u); \quad (7)$$

$$[d\rho^{X_N} f] (u) = \left. \frac{d}{dt} \rho (\exp (tX_N)) f (u) \right|_{t=0} = -2\pi iuf (u). \quad (8)$$

By equations (7) and (8), we solve the following equation

$$[d\rho^{X_A} + i d\rho^{X_N}] f (u) = 0 \iff uf'(u) + \left(1 + \frac{4\pi u}{2}\right) f(u) = 0.$$  \quad (9)

Its solution is $f(u) = \frac{c}{\sqrt{\pi}} e^{-2\pi u}$, where $c$ is a constant.

Considering the quasi-regular representation $\pi$ its derived representations acts on the basis as follows

$$[d\pi^{X_A} f] (v) = \left. \frac{d}{dt} \pi (\exp (tX_A)) f (v) \right|_{t=0} = -vf'(v) - \frac{1}{2} f(v); \quad (10)$$

$$[d\pi^{X_N} f] (v) = \left. \frac{d}{dt} \pi (\exp (tX_N)) f (v) \right|_{t=0} = -f'(v). \quad (11)$$

By equations (10) and (11) we again look for a solution of the following equation

$$[d\pi^{X_A} + i d\pi^{X_N}] f (v) = 0 \iff f'(v)(v + i) + \frac{1}{2} f(v) = 0,$$  \quad (12)

which has the form $f(v) = c(v + i)^{-1/2}$, where $c$ is a constant.
For each $X_H$ (a basis of the Lie algebra $\mathfrak{g}$) the Lie derivative is defined as a linear map $\mathfrak{L}^{X_H} : C^\infty(\mathbb{G}) \to C^\infty(\mathbb{G})$ by the formula

$$\left[ \mathfrak{L}^{X_H} F \right] (\eta) = \left. \frac{d}{dt} F (\eta \exp (tX_H)) \right|_{t=0},$$

see [13]. It is clear that $\mathfrak{L}^{X_H}$ is left $\mathbb{G}$-invariant. Then any function $F$ on the group $\mathbb{G}$ may be written in terms of $\mathbb{R}^2$-coordinates $(u,v)$, such that

$$F (\eta_\ast(t)) = F (u(t), v(t)), \quad \eta_\ast(t) = \eta \exp (tX_H),$$

and

$$\left. \frac{d}{dt} F (\eta_\ast(t)) \right|_{t=0} = \left. \frac{\partial F}{\partial u} \frac{du}{dt} \right|_{t=0} + \left. \frac{\partial F}{\partial v} \frac{dv}{dt} \right|_{t=0}.$$

Let $X_A$ and $X_N$ be the basis of the affine Lie algebra $\mathfrak{g}$. Then the Lie derivatives of a function $F$ on the affine group $\mathbb{G}$ have the form

$$\left[ \mathfrak{L}^{X_A} F \right] (\eta) = \left. \frac{d}{dt} F (\eta \exp (tX_A)) \right|_{t=0} = \left. \frac{\partial F}{\partial u} (ue^t, v) \right|_{t=0} = u \frac{\partial F}{\partial u}; \quad (13)$$

$$\left[ \mathfrak{L}^{X_N} F \right] (\eta) = \left. \frac{d}{dt} F (\eta \exp (tX_N)) \right|_{t=0} = \left. \frac{d}{dt} F (u, ut + v) \right|_{t=0} = u \frac{\partial F}{\partial v}. \quad (14)$$

Then the (Cauchy-Riemann-) Dirac operator $\mathfrak{L}^X$ of a function $F$ is given by

$$\left[ \mathfrak{L}^X F \right] (u,v) = -iu \left( \frac{\partial F}{\partial v} + i \frac{\partial F}{\partial u} \right). \quad (15)$$

In general, the space of wavelet transforms $W_{\psi}(L_2(\mathbb{R}))$ can be characterized by left-invariant differential operators as follows, see [12, Corollary 24].

**Proposition 4.1** Let $G$ be a Lie group with a Lie algebra $\mathfrak{g}$ and $\phi$ be a unitary representation of $G$ with the derived representation $d\phi$ of $\mathfrak{g}$. Let a mother wavelet $\psi_0$ be a null-solution (i.e., $A\psi_0 = 0$) for the operator $A = \sum_j a_j d\phi^{X_j}$ with $X_j \in \mathfrak{g}$. Then the image of the wavelet transform $[W_{\psi_0} f](\zeta) = \langle f, \phi(\zeta) \psi_0 \rangle$ consists of all functions $F$, such that

$$DF = 0 \quad \text{with} \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j},$$

where $\mathfrak{L}^{X_j}$ are the left-invariant fields (Lie derivatives) on $\mathbb{G}$ corresponding to $X_j$.

In what follows we will give a practical effect to the Proposition 4.1 for two cases of unitary representations of the affine group $\mathbb{G}$.

**Example 4.2** Let the derived representations of the co-adjoint representation $\rho$ of $\mathbb{G}$ be given as in equations (7) and (8). The corresponding left-invariant vector fields on $\mathbb{G}$ are $\mathfrak{L}^{X_A} = u \partial_u$ and $\mathfrak{L}^{X_N} = u \partial_v$, respectively. Since the mother wavelet $\psi_0 (u) = \frac{1}{\sqrt{u}} e^{-2\pi u}$ is a null-solution of the operator

$$d\rho^{X_A} + i d\rho^{X_N} = \left( \frac{1 + 4\pi u}{2} \right) I + u \frac{d}{du};$$

...
then by Proposition 4.1 the image \([W_\psi f](\zeta)\) consists of all the null-solutions of the Cauchy-Riemann operator (15).

**Example 4.3** Let \(\pi\) be the quasi-regular representation of \(G\) on \(L_2(\mathbb{R})\) given by (3), and let (10-11) be its derived representations. The mother wavelet \(\psi_0(v) = c(v + i)^{-1/2}\) is a null-solution of the operator

\[
d\pi^X A + i d\pi^X N = -\mathbf{I}_2 - (v + i) \frac{d}{dv}.
\]

Therefore by Proposition 4.1 the image of wavelet transform \([W_\psi f](\zeta)\) consists of all the null solutions of the operator (15).

Since the unitary operators \(U_1^*\) and \(R_0\) provide the decomposition of wavelet transform \(W_\psi\), then by Proposition 4.1 we immediately get the following result

**Proposition 4.4** Let \(\psi \in L_2(\mathbb{R})\) be an admissible wavelet. Then \([\mathcal{L}^X W_\psi f](\zeta) = 0\) if and only if \([\mathcal{L}^X U_1^* R_0 f](\zeta) = 0\).

5. Concluding remarks

For a quantum particle with one degree of freedom we can naturally associate the collection of its states with \(L_2(\mathbb{R})\). For many reasons, it is more convenient to use the Fock-Segal-Bargmann (FSB) space of analytic functions on the plane [6]. The advantage of FSB model is the availability of methods from complex analysis. In particular, the quantization of observables can be achieved by the transition from a function to the respective Toeplitz operator, see the first paragraph of this paper. Of course, the two descriptions – \(L_2(\mathbb{R})\) and FSB space – are isomorphic through the wavelet transform generated by the Heisenberg group and a Gaussian as a mother wavelet [6, 10].

We can also consider the unitary wavelet transform on \(L_2(\mathbb{R})\) generated by the affine group \(G\) and an admissible mother wavelet. The image of wavelet transform will consist of functions on the upper half-plane. Then Proposition 4.4 provides the necessary and sufficient condition for analyticity of the images of such a wavelet transform. If the condition is satisfied, we have another analytic model for quantum mechanics. Again, Toeplitz operators provide a natural framework for quantization of observables. This is conceptually similar to the Berezin quantization [3], moreover an explicit connection can be established through the Cayley map from the unit disk to the upper half-plane of the complex plane.

It was observed in several cases [8, 16] that Toeplitz operators can be transformed into multiplication operators by means of certain unitary maps. Such maps were previously constructed case-by-case by try-and-error methods. In this paper we have shown that unitary maps from paper [8] has the following property related to group representations:

(i) they intertwine respective representations of the affine group \(G\);
(ii) they provide a spatial separation of the irreducible components of the affine group’s representations.

Furthermore, we have shown in Section 3 that these properties makes these unitary maps useful for characterization of the image $W_\psi(L_2(\mathbb{R}))$ of the wavelet transform inside the space $L_2(G, d\nu_L)$. Therefore, we got a key for a description of the orthogonal projection $P_\psi : L_2(G, d\nu_L) \rightarrow W_\psi(L_2(\mathbb{R}))$ and the associated Toeplitz operators.

Induced representations now occupy a central place, often implicit, in much of representation theory, automorphic forms, but also in quantum mechanics as can be seen in [14]. Once the group-representation origin of the unitary maps is analyzed, it is possible to use this knowledge for an effective synthesis of such maps for various Toeplitz operators on the Hardy, Bergman or FSB spaces. However, this task is scheduled for further works.

Acknowledgments

Some results of paper are a part of the first author’s thesis [4] written under the supervision of Dr. Vladimir V. Kisil to whom both authors would like to thank for his idea behind the paper, useful suggestions and improvements of the previous versions of manuscript, and for his kind help with finalizing the paper. We are also indebted to anonymous referees for their criticism. The work was partially supported by the London Mathematical Society (Scheme 4 Grant N41038) and VVGS 45/10-11.

References

[13] Lang S 1975 $SL_2(\mathbb{R})$ (Amsterdam: Addison-Wesley)

