Séria úloh č. 8

1. Vyšetrite konvergenciu nasledujúcich radov:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}$;

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{3^n}}$;

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \cdot 4^n$;

d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n+1}$;

e) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 5 \ldots (6n-7)(6n-4)}{1 \cdot 5 \ldots (8n-11)(8n-7)}$;

f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{[5+(-1)^n]^n}$;

g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(\sqrt{2}+(-1)^n)^n}{3^n}$;

h) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\ldots(a+n)}$, $a > 0$;

i) $\displaystyle\sum_{n=1}^{\infty} n^4 \operatorname{arctg}^{2n} \frac{\pi}{4n}$;

j) $\displaystyle\sum_{n=1}^{\infty} \frac{2n!}{\sqrt{2^n+3}}$;

k) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{6^n}$;

l) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{6n-5}$;

m) $\displaystyle\sum_{n=1}^{\infty} \frac{3+(-1)^n}{(-3)^n}$;

n) $\displaystyle\sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{1}{4^n}$;

o) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^{100} \cdot 99^n}{100^n}$;

p) $\displaystyle\sum_{n=2}^{\infty} \frac{4^{n-1}\sqrt{n^2+5}}{(n-1)!}$;

r) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{1000n^2+3}$;

s) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)!}$;

t) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{(-1)^n\sqrt{n}}{n-1} - \frac{1}{n-1} \right)$;

u) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt[3]{n+1}}$;

v) $\displaystyle\sum_{n=0}^{\infty} \frac{1000^n}{n!}$;

w) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$.

2. Vyšetrite konvergenciu nasledujúcich radov:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{3}{8}\right)^n$;

b) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{\frac{1}{n^{n+1}}}$;

c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$;

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$;

e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2^n(2^n-1)}$;

f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-4)^2}$;

g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin \frac{1}{n}$;

h) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n} \operatorname{tg} \frac{1}{n}$;

i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+3)^{\frac{5}{3}}}$;

j) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2+\cdots+n}{n^3}$;

k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\left(\sqrt{n^2+n\sqrt{n}}-n\right)}$;

l) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{3^n}\right)$;

m) $\displaystyle\sum_{n=1}^{\infty} 2^n \sin\frac{\pi}{3^n}$;

n) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{3+(-1)^n}{n^2}\right)$;

o) $\displaystyle\sum_{n=5}^{\infty} \frac{\binom{n}{2}+\binom{n}{3}}{\binom{n}{4}+\binom{n}{5}}$;

p) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1+2^n}{3^n}+\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt[4]{n}}\right)$;

q) $\displaystyle\sum_{n=1}^{\infty} \frac{n!+1}{(n+2)!+2}$;

r) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}-\sqrt{2n-1}}{\sqrt{n}} \sin\frac{1}{\sqrt[3]{n}}$;

s) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^{n-1}+\sqrt{n}}$;

t) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n \left(1+\frac{2}{n}\right)^{n^2}$;

u) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n+1}}{2^{3n-1}}$;

v) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$;

w) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$;

x) $\displaystyle\sum_{n=1}^{\infty} \frac{\operatorname{arctg}(2+(-1)^n)}{\ln(1+n)}$;

y) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+3n-4}{2n^2+5}\right)^3$;

z) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot 5^n}$.

3. Vypočítajte nasledujúce limity:

a) $\displaystyle\lim_{n\to\infty} \frac{n^n}{(n!)^2}$;

b) $\displaystyle\lim_{n\to\infty} \frac{\mathrm{e}^n}{n!}$;

c) $\displaystyle\lim_{n\to\infty} \frac{n+1}{2^n(n-1)!}$;

d) $\displaystyle\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{n^2}}{2^{2n}}$;

e) $\displaystyle\lim_{n\to\infty} \frac{n!}{(2n-1)^n}$;

f) $\displaystyle\lim_{n\to\infty} \frac{(n+1)^{n^2}}{3^n\cdot n^{n^2}}$;

g) $\displaystyle\lim_{n\to\infty} \frac{n^2}{(n+2)!}$;

h) $\displaystyle\lim_{n\to\infty} \frac{n!(2n+1)!}{(3n)!}$;

i) $\displaystyle\lim_{n\to\infty} \frac{\binom{2n}{n}}{2^{2n}}$.

4. Uveďte príklad spojitej funkcie $f:\langle 1,+\infty)\to\langle 0,+\infty)$ takej, že

(i) rad $\displaystyle\sum_{n=1}^{\infty} f(n)$ konverguje, ale $(\mathcal{R})\displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x=+\infty$;

(ii) rad $\displaystyle\sum_{n=1}^{\infty} f(n)$ diverguje, ale $(\mathcal{R})\displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x<+\infty$.

5.* Nech f je nezáporná nerastúca funkcia na intervale $\langle 1,+\infty)$. Potom rad $\sum\limits_{n=1}^{\infty} f(n)$ konverguje práve vtedy, keď postupnosť $(I_n)_1^{\infty}$ je ohraničená, kde

$$I_n=(\mathcal{R})\int_1^{n} f(x)\,\mathrm{d}x,\quad n\in\mathbb{N}.$$

6*. Nech $(F_n)_1^{\infty}$ je Fibonacciho postupnosť. Dokážte, že rad $\sum\limits_{n=1}^{\infty} \frac{1}{F_n}$ konverguje.