(1) $x z_x + y z_y = z$

transf. $x = u$
$y = u \cdot v \rightarrow v = \frac{y}{x}$
$z(x,y) = w(u,v)$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = -\frac{y}{x^2} \quad \frac{\partial v}{\partial y} = \frac{1}{x}$

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = w_u \cdot 1 + w_v \left(-\frac{y}{x^2}\right)$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = w_u \cdot 0 + w_v \cdot \frac{1}{x}$$

$$\Rightarrow \quad x w_u - \cancel{\frac{y}{x} w_v} + \cancel{\frac{y}{x} w_v} = z$$

$$\Rightarrow \quad u \cdot w_u = w$$

$$\Rightarrow \quad \frac{\frac{\partial w}{\partial u}}{w} = \frac{1}{u} \quad \Rightarrow \quad \ln|w| = \ln|u| + c(v)$$

$$w = k(v) u \ , \ k \in C^1$$

$$\Rightarrow z(x,y) = k\left(\frac{y}{x}\right) \cdot x \ , \quad \text{ale } z(x,x) = k(1) \cdot x = x$$

$$\Rightarrow \boxed{k(1) = 1}$$

$$\Rightarrow \boxed{z(x,y) = k\left(\frac{y}{x}\right) \cdot x \ , \ k(1) = 1 \ , \ k \in C^1}$$

② $4(x^2+y^2) = (x^2+y^2+2x)^2 \quad (*)$

$x = r\cos\varphi \quad r \geq 0$
$y = r\sin\varphi \quad \varphi \in [0, 2\pi)$ $\quad \rightarrow \quad 4r^2 = (r^2 + 2r\cos\varphi)^2$

$\Leftrightarrow \quad 4r^2 = r^2(r + 2\cos\varphi)^2 \quad / \frac{1}{\cancel{r^2}} \quad [r=0 \ [0,0]]$

$4 = (r + 2\cos\varphi)^2$

$-2 = r + 2\cos\varphi \Leftrightarrow -2(1+\cos\varphi) = r$
$2 = -r - 2\cos\varphi$

$\Leftrightarrow \quad |2| = |r + 2\cos\varphi|$

$\Leftrightarrow \quad 2 = r + 2\cos\varphi \quad \Rightarrow r = 2(1-\cos\varphi)$

<u>ohr.</u> $0 \leq r \leq 2(1+|\cos\varphi|) \leq 4 \quad \rightarrow$ leží v kruhu s polomerom 4

<u>singul. body</u> $\quad \nabla(*) \Leftrightarrow 8(x,y) = 4\left((x^2+y^2+2x)(x+1), (x^2+y^2+2x)y\right)$

[alebo cez polárne] $\Rightarrow \boxed{y=0} \ \wedge \ 8x = 4(x^2+2x)(x+1) = 4x(x+2)(x+1)$

$2 = (x+2)(x+1)$

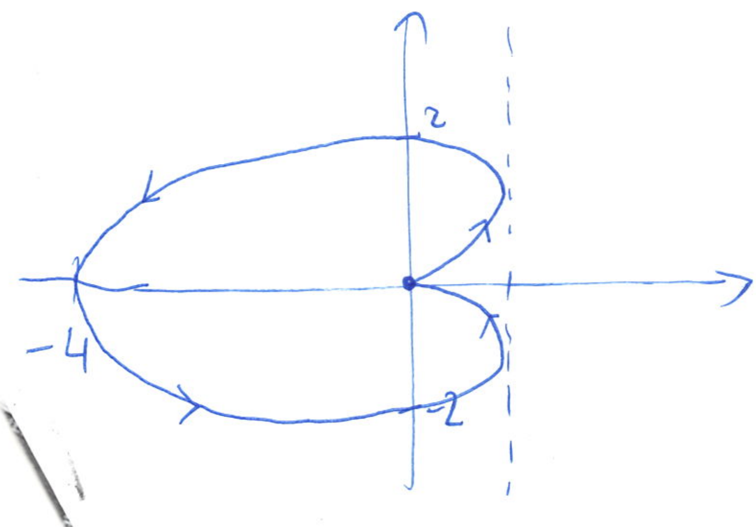
$\boxed{x=0}$ $\qquad \cancel{\boxed{x=-3}}$ lebo $[-3,0]$ nerieši $(*)$

$[0,0]$ leží na $*$ ✓

$r = 2(1-\cos\varphi), \quad \varphi \in [0, 2\pi) = D_\varphi$

rastie na $[0, \pi]$
klesá na $[\pi, 2\pi]$

$r(0) = 0$
$r(\pi/2) = 2$
$r(\pi) = 4$
$r(3/2\pi) = 2$
$r(2\pi) = 0$



(3) $\underbrace{(x^2-\sin^2(y))}_{M}dx+\underbrace{x\sin(2y)}_{N}dy=0 \quad , \quad y(1)=\frac{\pi}{3}$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-2\sin(y)\cos(y)-\sin(2y)=-2\sin(2y)\neq 0$$

$$0=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)+\frac{\partial \mu}{\partial y}\cdot M-\frac{\partial \mu}{\partial x}\cdot N$$

(1) $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=-\frac{2\sin(2y)}{x^2-\sin^2 y} \Rightarrow \mu(y) \text{ nepomôže}$$

(2) $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=-\frac{2\sin(2y)}{x\sin(2y)}=-\frac{2}{x} \quad \checkmark \quad \boxed{\mu(x)}$$

$$\Rightarrow \mu'=\frac{\partial \mu}{\partial x}=\mu\cdot\left(-\frac{2}{x}\right) \qquad \frac{d\mu}{dx}=-\frac{2\mu}{x} \Rightarrow \boxed{\mu(x)=\frac{1}{x^2}}$$

$$\hookrightarrow \left(1-\frac{\sin^2(y)}{x^2}\right)dx+\frac{\sin(2y)}{x}dy=0$$

$$U=\int N\,dy=\frac{1}{x}\int\sin(2y)\,dy=-\frac{\cos(2y)}{2x}+h(x)=\frac{\sin^2(y)-\cos^2(y)}{2x}+h(x)$$

$$\frac{\partial U}{\partial x}=\cancel{1}-\frac{\sin^2(y)}{x^2}=\frac{-\sin^2(y)+\cos^2(y)}{2x^2}+h'(x)$$

$$h'(x)=1-\frac{2\sin^2(y)}{2x^2}+\frac{\sin^2(y)}{2x^2}-\frac{1-\sin^2(y)}{2x^2}=1-\frac{1}{2x^2}$$

$$h(x)=x+\frac{1}{2x}$$

$$U(x,y)=x+\frac{1}{2x}-\frac{\cos(2y)}{2x}=x+\frac{1-\cos^2(y)+\sin^2(y)}{2x}=\boxed{x+\frac{\sin^2(y)}{x}=\text{konšt.}}$$

$$y(1)=\frac{\pi}{3} \to O\left(1,\frac{\pi}{3}\right) \Rightarrow 1+\sin^2(\pi/3)=\text{konšt.}$$

$$\Rightarrow \frac{7}{4}=1+\frac{3}{4}=\text{konšť}$$

$$\sin^2(y)=\frac{7}{4}x-x^2 \Rightarrow \boxed{y=\arcsin\sqrt{x\left(\frac{7}{4}-x\right)}} \quad x\in\left[0,\frac{7}{4}\right]$$

4 iii) nech $u \neq 0$ a $u\vec{F} = \nabla v$ / rot

$$\mathrm{rot}(u\vec{F}) = \mathrm{rot}(\nabla v) \Leftrightarrow \nabla u \times \vec{F} + u\,\mathrm{rot}\,\vec{F} = \vec{0} \quad / \cdot \vec{F}$$

$$\underbrace{\vec{F} \cdot (\nabla u \times \vec{F})}_{=0} + u\vec{F} \cdot \mathrm{rot}\,\vec{F} = \vec{0} \cdot \vec{F} = 0$$

$=0$, lebo vekt. súčin je normála k pôvodným vektorom

$$\Rightarrow u\vec{F} \cdot \mathrm{rot}\,\vec{F} = 0 \quad / \frac{1}{u} \quad \Rightarrow \vec{F} \cdot \mathrm{rot}\vec{F} = 0 \Rightarrow \underline{\vec{F} \perp \mathrm{rot}\,\vec{F}}$$

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ii) $$\mathrm{div}\left(\frac{g(r)}{r} \cdot \vec{r}\right) = \frac{g(r)}{r}\,\mathrm{div}(\vec{r}) + \nabla \frac{g(r)}{r} \cdot \vec{r} =$$

$\boxed{\nabla h(r) = h'(r)\frac{\vec{r}}{r}}$

$\boxed{\nabla r = \frac{\vec{r}}{r}}$

$\boxed{\vec{r} = (x, y)}$

$$= \frac{g(r)}{r} \cdot 2 + \left(\frac{g(r)}{r}\right)' \frac{\vec{r}}{r} \cdot \vec{r} =$$

$$= \frac{2g(r)}{r} + r^2 \frac{(g'(r)r - g(r))}{r^3} =$$

$$= \boxed{\frac{g(r)}{r} + g'(r) = 0}$$

pre n-rozmer

$$\frac{(n-1)\,g(r)}{r} + g'(r)$$

$$\Downarrow \frac{dg}{g} = -\frac{dr}{r} \Leftrightarrow \ln|g| = c - \ln|r|$$

$$\Leftrightarrow g(r) = \frac{k}{r}, k \in \mathbb{R}$$

$$\underline{g(x,y) = \frac{k}{\sqrt{x^2+y^2}}}$$

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i) $\Delta h = 0 \overset{②}{\Rightarrow} \Delta f(h) = 0$

$$\Delta f(h) = \nabla \cdot \nabla f(h) = \nabla \cdot (f'(h) \cdot \nabla h) = f'(h)\,\nabla \cdot \nabla h + \nabla(f'(h)) \cdot \nabla h =$$

$$= f'(h) \cdot \underbrace{\Delta h}_{=0} + f''(h)\,\nabla h \cdot \nabla h = f''(h) \cdot \underbrace{\|\nabla h\|^2}_{\neq 0} = 0$$

pre $h = 0 \Rightarrow f$ je ľub.

pre $h \neq 0$ $\quad f''(h) = 0 \Rightarrow f(h) = c_1 h + c_2$, $c_i \in \mathbb{R}$