① $y z_x - x z_y = (y-x) z$ $\qquad u = x^2 + y^2$

$z(x,y) = e^{w(u,v)+x+y}$ $\qquad v = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$

$$z_x = z \cdot \left(w_u \cdot u_x + w_v \cdot v_x + 1 \right)$$

$$z_y = z \left(w_u \, u_y + w_v \, v_y + 1 \right)$$

$$z_x = z\left(w_u \cdot 2x - \frac{w_v}{x^2} + 1 \right), \quad z_y = z\left(w_u 2y - \frac{w_v}{y^2} + 1 \right)$$

$$\cancel{2xy w_u} - \frac{y}{x^2} w_v + \cancel{y} - \cancel{2xy w_u} + \frac{x}{y^2} w_v \cancel{-x} = \cancel{y} - x$$

$$\left(\frac{x}{y^2} - \frac{y}{x^2} \right) w_v = 0$$

$$\Rightarrow w(u,v) = F(u) \Rightarrow z(x,y) = e^{x+y} \cdot G(x^2+y^2)$$

$$z(x,x) = e^{2x} \cdot G(2x^2) = e^{x} \Rightarrow G(2x^2) = e^{-x}$$

$$0 < p = 2x^2 \qquad G(p) = e^{-\sqrt{\frac{p}{2}}}$$

$$x = \sqrt{\frac{p}{2}}$$

$$\Rightarrow \boxed{z(x,y) = e^{x+y} \cdot e^{-\sqrt{\frac{x^2+y^2}{2}}}}$$

② $4(x^2+y^2)=(x^2+y^2+2x)^2$

$x=r\cos\varphi$

$y=r\sin\varphi \Rightarrow \quad 4r^2=(r^2+2r\cos\varphi)^2=r^2(r+2\cos\varphi)^2$

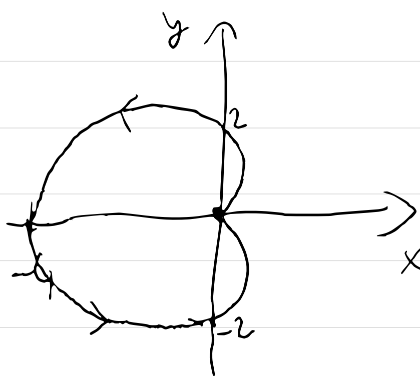
$$4=(r+2\cos\varphi)^2$$

$$\Rightarrow \quad 2=|r+2\cos\varphi| \Rightarrow r=2(1-\cos\varphi)$$

$$\vee \quad r=-2(1+\cos\varphi)$$

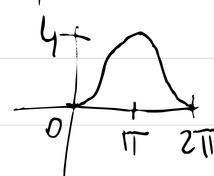
tá istá krivka !

$|r|\leq 4 \Rightarrow$ ohraničená



r rastie pre $\varphi\in(0,\pi)$

klesá pre $\varphi\in(\pi,2\pi)$



$x=0 \Rightarrow 4y^2=y^4 \Rightarrow y=0 \vee y=\pm 2$

$y=0 \Rightarrow 4x^2=x^4+4x^3+4x^2 \Leftrightarrow x^3(x+4)=0$

$x=0,\ x=-4$

Dotyčnica v $[-4,0]$ je priamka $\boxed{x=-4}$

$y'=-\dfrac{x^3+y^2x+3x^2+y^2}{y(x^2+y^2+2x-2)} \Rightarrow y'(0)=-1 \Rightarrow \boxed{y=2-x}$ v bode $[0,2]$

(3) $M = x^2 y - y\sin^2 y$

$N = xy\sin(2y)$

$$\mu = h(x^\alpha y^\beta)$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\frac{\partial M}{\partial y} = x^2 - \sin^2 y - 2y\sin y\cos y$$

$$\frac{\partial N}{\partial x} = y\sin(2y)$$

$$h'\beta x^\alpha y^{\beta-1}(x^2 y - y\sin^2 y) + h(x^2 - \sin^2 y - 2y\sin y\cos y) =$$

$$h'\cdot \alpha x^{\alpha-1} y^\beta xy\sin(2y) + h\cdot y\cdot \sin(2y)$$

$$h' x^\alpha y^\beta\left[\beta x^2 - \beta\sin^2 y - \alpha y\sin(2y)\right] = h\left[2y\sin(2y) - x^2 + \sin^2 y + \cancel{y\sin(2y)}\right]$$

$$\Rightarrow \boxed{\alpha = -2} \quad \boxed{\beta = -1}$$

$$h'\cdot z = h \Rightarrow h(z) = z \Rightarrow \boxed{\mu = \frac{1}{x^2 y}}$$

$\Downarrow$

$$\left(1 - \frac{\sin^2 y}{x^2}\right)dx + \frac{\sin(2y)}{x}dy = 0$$

$$U = \int\left(1 - \frac{\sin^2 y}{x^2}\right)dx = x + \frac{\sin^2 y}{x} + c(y)$$

$$\frac{\partial v}{\partial y} = \frac{2\sin y \cos y}{x} + c'(y) = \frac{\sin(2y)}{x}$$

$c(y) = \text{konšt.}$

$$U = x + \frac{\sin^2(y)}{x} = C \;, C \in \mathbb{R} \qquad y(1) = \frac{\pi}{3}$$

$$1 + \sin^2\left(\frac{\pi}{3}\right) = C \Rightarrow 1 + \frac{\sqrt{3}}{2} = C$$

$$\Rightarrow y = \arcsin\sqrt{x\left(1 + \frac{\sqrt{3}}{2} - x\right)} \qquad x \in \left(0, 1 + \frac{\sqrt{3}}{2}\right)$$

---

4 i) h je harm. $\Rightarrow \Delta h = 0$ na Ω

chceme $\Delta f(h) = 0 \Leftrightarrow (\nabla \cdot \nabla) f(h) = \operatorname{div}(f' \cdot \nabla h) =$

$= f' \operatorname{div}(\nabla h) + \nabla f' \cdot \nabla h = f' \Delta h + f''(\nabla h \cdot \nabla h) =$

$= f' \underbrace{\Delta h}_{= 0} + f'' \cdot \underbrace{\|\nabla h\|^2}_{\neq 0} = 0 \Rightarrow \boxed{f'' = 0}$

inak $h = \text{konšt.}$

$\boxed{f(u) = au + b}$

ii) $\operatorname{div}\left(g(r)\frac{\vec{r}}{r}\right) = 0$

$\Leftrightarrow \frac{g(r)}{r}\operatorname{div}(\vec{r}) + \nabla\left(\frac{g(r)}{r}\right)\cdot\vec{r} = 0$

$\Leftrightarrow \frac{2g(r)}{r} + \left(\frac{1}{r}\nabla g(r) + g(r)\nabla\frac{1}{r}\right)\cdot\vec{r} = 0$

$\frac{2g(r)}{r} + \left(\frac{g'(r)}{r}\nabla r - \frac{g(r)}{r^2}\nabla r\right)\cdot\vec{r} = 0$

$\frac{2g(r)}{r} + \left(\frac{g'(r)}{r} - \frac{g(r)}{r^2}\right)\frac{\vec{r}\cdot\vec{r}}{r} = 0$

$\frac{\cancel{2}g(r)}{r} + g'(r) - \cancel{\frac{g(r)}{r}} = 0$

$\frac{dg}{g} = -\frac{dr}{r} \Rightarrow g(r) = \frac{c}{r}, \quad c \in \mathbb{R}$

iii) $\operatorname{rot}(\vec{a} \operatorname{arctg} r) = \operatorname{arctg} r \cdot \underbrace{\operatorname{rot}(\vec{a})}_{=0} + \nabla \operatorname{arctg} r \times \vec{a} =$

$= \frac{1}{1+r^2} \nabla r \times \vec{a} = \frac{1}{r(1+r^2)} \vec{r} \times \vec{a} =$

$= \frac{1}{r(1+r^2)} \left(a_3 y - a_2 z \,,\, a_1 z - a_3 x \,,\, a_2 x - a_1 y \right)$