① $I = \int\limits_C (x+y)^{\frac{1}{4}}\, ds$ , $C: \; x^2 - y^2 = \frac{9}{8} z^2$ $\quad A = [0,0,0]$ , $a>0$

$(x-y)^2 = a(x+y)$ $\quad B = [a, 0, \frac{2\sqrt{2}a}{3}]$

param. $C$ : $x(t,r) = \cosh(t)\cdot r$ $\quad \Rightarrow r^2 \cdot 1 = \frac{9}{8} z^2 \Rightarrow z = \frac{2\sqrt{2}}{3} r$

$y(t,r) = \sinh(t)\cdot r$

$r \geq 0$

$r^2\left(2\cosh^2(t) - 1 - 2\cosh(t)\sinh(t)\right) = r^2\left(\cosh(t) - \sinh(t)\right)^2 = r\cdot a\cdot e^t \iff r = a\cdot e^{3t}$

$\Rightarrow \; t = +\frac{1}{3}\ln\left(\frac{r}{a}\right) = \ln\left(\left(\frac{r}{a}\right)^{\frac{1}{3}}\right)$

$\Rightarrow \; x(r) = \frac{1}{2}\left[\left(\frac{r}{a}\right)^{\frac{1}{3}} + \left(\frac{r}{a}\right)^{-\frac{1}{3}}\right]\cdot r$

$y(r) = \frac{1}{2}\left[\left(\frac{r}{a}\right)^{\frac{1}{3}} - \left(\frac{r}{a}\right)^{-\frac{1}{3}}\right]\cdot r$

$z(r) = \frac{2\sqrt{2}}{3} r$ $\quad r \in [0, a]$

$ds = \sqrt{x'^2 + y'^2 + z'^2}\, dr = \sqrt{\left(\frac{2\left(\frac{r}{a}\right)^{\frac{2}{3}} + 1}{3\left(\frac{r}{a}\right)^{\frac{1}{3}}}\right)^2 + \left(\frac{2\left(\frac{r}{a}\right)^{\frac{2}{3}} - 1}{3\left(\frac{r}{a}\right)^{\frac{1}{3}}}\right)^2 + \frac{8}{9}}\; dr$

$= \sqrt{\frac{8}{9}\left(\frac{r}{a}\right)^{\frac{2}{3}} + \frac{8}{9} + \frac{2}{9}\left(\frac{r}{a}\right)^{-\frac{2}{3}}}\; dr = \frac{\sqrt{2}}{3}\left(2\left(\frac{r}{a}\right)^{\frac{1}{3}} + \left(\frac{a}{r}\right)^{\frac{1}{3}}\right) dr$

Teda $I = \frac{\sqrt{2}}{3}\int\limits_0^a \left(2\left(\frac{r}{a}\right)^{\frac{1}{3}} + \left(\frac{a}{r}\right)^{\frac{1}{3}}\right)\cdot\left(r\left(\frac{r}{a}\right)^{\frac{1}{3}}\right)^{\frac{1}{4}} dr = \frac{\sqrt{2}}{3a^{\frac{1}{12}}}\int\limits_0^a \left(2\left(\frac{r}{a}\right)^{\frac{1}{3}} + \left(\frac{a}{r}\right)^{\frac{1}{3}}\right)\cdot r^{\frac{1}{3}} dr$

$= \frac{\sqrt{2}}{3a^{\frac{5}{12}}}\int\limits_0^a \left(2r^{\frac{2}{3}} + a^{\frac{2}{3}}\right) dr = \frac{11}{15} a^{\frac{5}{4}} \sqrt{2}$

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② $z_{xx} - y\, z_{yy} = \frac{1}{2} z_y$ $\quad \phi^{-1}: u = x - 2\sqrt{y}$ , $v = x + 2\sqrt{y}$ ,

$J_{\phi^{-1}} = \begin{bmatrix} 1 & -y^{-\frac{1}{2}} \\ 1 & y^{-\frac{1}{2}} \end{bmatrix}$ , $w(u,v) = z(x,y)$

$\Rightarrow \; z_x = w_u \cdot 1 + w_v \cdot 1$

$z_y = w_u \cdot (-y^{-\frac{1}{2}}) + w_v \cdot y^{-\frac{1}{2}}$

$z_{xx} = w_{uu} + 2w_{uv} + w_{vv}$

$z_{yy} = w_{uu}\cdot y^{-1} - 2\cdot 4\, w_{uv}\left(+\frac{1}{4}\right)\cdot y^{-1} + w_{vv}\cdot y^{-1} + \frac{1}{2} w_u\, y^{-\frac{3}{2}} - \frac{1}{2} w_v\, y^{-\frac{3}{2}}$

$\Rightarrow$ PDR $\quad 4 w_{uv} = 0 \iff w(u,v) = F(u) + G(v)$

$$\Rightarrow z(x,y) = F(x-2\sqrt{y}) + G(x+2\sqrt{y}), \quad F, G \in C^2$$

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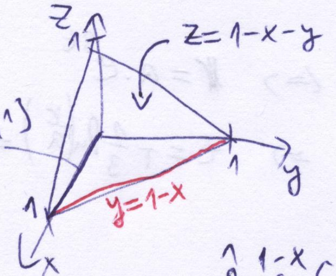
(3) $I = \iiint_\Omega (1+x+y+z)^{-3}\, d\lambda_3$, $\quad \Omega: \ x+y+z=1, \ x=0, \ y=0, \ z=0$

integrál $\exists$, lebo integrand je spojitý a $\Omega$ je kompakt




$$\Rightarrow I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-3}\, d\lambda_3 = \int_0^1\int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y}$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{8} + \frac{1}{2(1+x+y)^2}\right] dy\, dx = -\int_0^1 \left[\frac{y}{8} + \frac{1}{2(1+x+y)}\right]_0^{1-x} dx =$$

$$= -\int_0^1 \left(\frac{3}{8} - \frac{x}{8} - \frac{1}{2(1+x)}\right) dx = \left[-\frac{3}{8}x + \frac{x^2}{16} + \ln\sqrt{1+x}\right]_0^1 = -\frac{5}{16} + \ln\sqrt{2}$$

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(4) $I = \int_0^\infty \ln(1-e^{-x})\, dx$

$$\ln(1-z) = -\sum_{n=1}^\infty \frac{z^n}{n}, \quad |z|<1, \quad z = e^{-x} < 1 \ \forall x>0 \ ✓$$

$$I = -\int_0^\infty \sum_{n=1}^\infty \frac{e^{-xn}}{n}\, dx \overset{?}{=} -\underbrace{\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty e^{-xn}\, dx}_{J} = +\sum_{n=1}^\infty \frac{1}{n}\left[\frac{e^{-nx}}{n}\right]_0^\infty =$$

$$= -\sum_{n=1}^\infty \frac{1}{n^2} = -\frac{\pi^2}{6}$$

na (?) použijeme Leviho vetu – stačí ukázať, že $v_n = \frac{e^{-xn}}{n}$ sú nez. → zrejmé

a že $J$ konverguje (čo sme už ukázali)...

(5) $I(a,b) = \int_0^\infty \frac{\ln(b^2+x^2)}{a^2+x^2}\,dx$

predovšetkým, ak pre $a>0, b>0$ $I$ $\exists$ $\Rightarrow I(-a,-b) = I(-a,b) =$
$= I(a,-b) = I(a,b)$

pre $a>0, b>0$ je $\lim_{x\to 0^+} \frac{\ln(b^2+x^2)}{a^2+x^2} = \frac{\ln(b^2)}{a^2}$ ✓

$\lim_{x\to\infty}$ —||— $= 0$ (NP splnená)

porovn. krit.: $g(x) = \frac{1}{x^{3/2}}$ $\in L(1,\infty)$ $\lim_{x\to\infty} \frac{\ln(b^2+x^2)}{(a^2+x^2)} x^{3/2} = 0$ ✓

pre $a>0, b=0$ na okolí „$\infty$" je to obdobné ...

na okolí „0" $\lim_{x\to 0^+} \frac{\ln(x^2)}{a^2+x^2} = -\infty$

ale $\int_0^1 \ln x^2\,dx = -2 \Rightarrow$ aj $I(a,0)$ konv.

pre $a=0, b>0$ int zrejme diverguje $\frac{1}{x^2} \notin L(0,1)$ ...

pre $a>0, b\geq 0$

$$\frac{\partial I}{\partial b} = 2b\int_0^\infty \frac{dx}{(b^2+x^2)(a^2+x^2)} = \begin{cases} \frac{2b}{a^2-b^2}\left(\int_0^\infty \frac{dx}{b^2+x^2} - \int_0^\infty \frac{dx}{a^2+x^2}\right) = \left[\frac{2\operatorname{arctg}\left(\frac{x}{b}\right)}{a^2-b^2} - \frac{2b\operatorname{arctg}\left(\frac{x}{a}\right)}{a(a^2-b^2)}\right]_0^\infty \\ 2b\int_0^\infty \frac{dx}{(b^2+x^2)^2} = \left[\frac{x}{b(b^2+x^2)} + \frac{\operatorname{arctg}\left(\frac{x}{b}\right)}{b^2}\right]_0^\infty = \end{cases}$$

$= \frac{\pi}{a(a+b)}$, $a \neq b$

$\Rightarrow \frac{\partial I}{\partial b} = \frac{\pi}{a(a+b)} \Rightarrow I = \int \frac{\pi}{a(a+b)}\,db = \frac{\pi}{a}\ln(a+b) + c$

$= \frac{\pi}{2b^2}$, ak $a=b$

$\lim_{a\to\infty} I(a,b) = 0 \Rightarrow c = 0$

rozšírenie pre $a\neq 0, b\in\mathbb{R}$ $\quad I(a,b) = \frac{\pi}{|a|}\ln(|a|+|b|)$