Jozef Kiseľák

Sada úloh na precvičenie VI.
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H. Určte, či je zobrazenia komutujú.

1. $(Af)(x) = 3(x^2+1)f(x)$ a $\frac{1}{x^2+1}f(x)$
2. $(Af)(x) = f'(x)$ a $(Bf)(x) = f(x)^2$
3. $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ a $B_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$ (sú samoadjungované?)
4. operátor hybnosti a kinetickej energie (v kvantovej mechanike)
5. $A = ((\frac{\mathrm{d}}{\mathrm{d}t})^2 + x^3 + a)^2 + 2x$ a $B = ((\frac{\mathrm{d}}{\mathrm{d}t})^2 + x^3 + a)^3 + 3x(\frac{\mathrm{d}}{\mathrm{d}t})^2 + 3\frac{\mathrm{d}}{\mathrm{d}t} + 3x(x^3+a)$ (platí $M^2 = L^3 - a$?)

I. Určte, či je zobrazenie z X do X je prosté, na a ak $X = \ell^2$, či je to izometria.

1. $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$
2. $T(x_1, x_2, x_3 \dots) = (x_2, x_3, \dots)$
3. $A : y_k = (x_k - x_{k+1})/2,\ k \in \mathcal{N}$
4. $T : y_k = x_k/k,\ k \in \mathcal{N}$
5. $M(x_1, x_2, \dots) = (x_2, x_4, \dots)$
6. $T(\mathbf{z}) = \begin{cases} \frac{z_n - 3z_{n+1}}{\sqrt{10}}, & n = 2k+1 \\ \frac{3z_{n-1} + z_n}{\sqrt{10}}, & n = 2k \end{cases}$
7. $\{z_n\} \mapsto \{2^{1-n}(z_1 + z_2 + \cdots + z_n)\}$

J. Overte, že dané zobrazenie (operátor, funkcionál) je lineárne a následne zistite, či je spojité (teda ohraničené ako zobrazenie z X do Y).

1. $F : C([0,1]) \to \mathbb{R},\ \ F(f) = 5f(0) - 7f(1/2) + 3f(1)$
2. $K : C([0,1]) \to \mathbb{R},\ \ K(f) = f(1/2)$, kde $C([0,1]) \subset X = \mathcal{L}^1(0,1)$
3. $I : C([-1,1]) \to \mathbb{R},\ \ I(f) = \int_{-1}^{1} \operatorname{sgn}(x) f(x)\,\mathrm{d}x$
4. $F : \ell^2 \to \mathbb{C},\ \ F(\mathbf{z}) = \sum_{k}^{n} \overline{a}_k z_k,\ \mathbf{a} \in \ell^2$
5. $S : \ell^2 \to \ell^2,\ \ S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$
6. $T : \ell^\infty \to \ell^\infty,\ \ \{z_n\} \mapsto \left\{\frac{z_n}{n^2}\right\}$
7. $M_\lambda : \ell^p \to \ell^p,\ \ M_\lambda(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots),\ \ \lambda \in \ell^\infty$
8. $T : C^1(I) \to C(I),\ \ T(f) = \frac{\mathrm{d}f}{\mathrm{d}t},\ t \in I$
9. $T : X \to C(I),\ \ T(f) = \frac{\mathrm{d}f}{\mathrm{d}t} + f,\ t \in I$, kde $X = (C^1(I), \|\cdot\|),\ \|f\| = \|f\|_\infty + \|f'\|_\infty$
10. $V : C([0,1]) \to C([0,1]),\ \ V(f) = \int_0^x f(t)\,\mathrm{d}t$
11. $T : \mathcal{L}^1(0,1) \to \mathbb{R},\ \ T(f) = \int_0^1 t\,|f(t)|\,\mathrm{d}t$
12. $L : X \to X,\ \ L(f)(t) = f(t)g(t),\ \ g \in C([0,1]),\ \ a)\ X = C([0,1]),\ b) X = \mathcal{L}^2(0,1)$
13. $L : X \to X,\ \ L(f)(t) = f(t^3),\ \ a)\ X = C([0,1]),\ b)\ X = \mathcal{L}^2(0,1)$
14. $T : \mathcal{L}^p(\mathbb{R}) \to \mathcal{L}^1(\mathbb{R}),\ \ T(f) = \frac{f(t)}{1+|x|^\alpha},\ \alpha > 0$

J1. Zistite, či je dané zobrazenie je samoadjungované, pozitívne a projekcia.

1. $A : \mathbb{R}^2 \to \mathbb{R}^2,\ \ A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$
2. $P : \mathbb{R}^n \to \mathbb{R}^n,\ \ P(x_1, \dots, x_n) = (0, x_2, \dots, x_n)$
3. $A : l^2 \to l^2,\ \ A(x_1, x_2, \dots) = (0, 0, x_3, x_4, \dots)$
4. $M_c : l^2 \to l^2,\ \ M_c(x_1, x_2, \dots) = (c_1 x_1, c_2 x_2, \dots)$, kde $(c_n)_{n\in\mathcal{N}} \subset \mathbb{C} : \sup_{n\in\mathcal{N}} |c_n| < \infty$
5. $A : D_A \to C([a,b]),\ \ Au = u''$, kde $D_A := \{u \in C^2([a,b]) : u(a) = u(b) = 0\} \subset \mathcal{L}^2(a,b)$
6. $A : C^2([a,b]) \to C([a,b]),\ \ Au = u'',\ \ C^2([a,b]) \subset \mathcal{L}^2(a,b)$
7. $L : C^2([a,b]) \to C([a,b]),\ \ Lu = p_0 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + p_1 \frac{\mathrm{d}u}{\mathrm{d}x} + p_2 u,\ \ p_i \in C^{2-i}([a,b]), p_0 > 0$
   $C^2([a,b]) \subset \mathcal{L}^2(a,b)$
8. $A : D_A \to C(\Omega),\ \ Au = -\Delta u$, kde $D_A := \{u \in C^2(\overline{\Omega}) : u(\partial\Omega) = 0\} \subset \mathcal{L}^2(a,b)$
9. $M_\phi : \mathcal{L}^2(0,1) \to \mathcal{L}^2(0,1),\ \ (M_\phi f)(x) = f(x)\phi(x)$, kde $\phi \in \mathcal{L}^\infty([0,1], \mathbb{C})$
10. $A : \mathcal{L}^2(0,1) \to \mathcal{L}^2(0,1),\ \ (Af)(x) = \phi(x) \int_0^1 \phi(t) f(t)\, \mathrm{d}t$, kde $\phi \in C([0,1])$
11. $U : \mathcal{L}^2(a,b) \to \mathcal{L}^2(a,b),\ \ (Uf)(x) = f(\phi^{-1}(x)),\ \ \forall x \in [a,b]$, kde ϕ je C^1-difeomorfizmus

J2. Zistite, či je dané zobrazenie je unitárne.

1. rotácia na $\mathbb{R}^2$
2. $(U_A f)(x) = f(A^{-1}x)|\det(A)|^{-1/2}$ na $L^2(\mathbb{R}^n)$, ak $\exists\ A^{-1}$
3. T^n, ak T je unitárne
4. $S : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R}), S(x_1, x_2, \dots) = (x_2, x_3, \dots)$
5. $Kz = \overline{z}$ na $\mathbb{C}$
6. $(Tx)(t) = \begin{cases} x(t), & t \geq 0 \\ -x(t), & t < 0 \end{cases}$ na $\mathcal{L}^2(\mathbb{R})$

J3. Zistite, či je dané zobrazenie je normálne.

1. $A = \begin{bmatrix} -i & 2+i \\ -2+i & 0 \end{bmatrix}$
2. $(Fx)(t) = f(t)x(t)$ na $\mathcal{L}^2(I)$, kde f je ohraničená komplexná funkcia
3. T^*, ak $T : H \to H$ je normálne, spojité, lineárne
4. $T - \lambda I$, ak $T : H \to H$ je normálne
5. $(Tf)(\theta) = \mathrm{e}^{\mathrm{i}\theta} f(\theta)$ na $\mathcal{L}^2(-\pi, \pi)$

J4. Zistite, či je dané zobrazenie je disipatívne.

1. $A = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$
2. $Ax = -x$ na $\mathbb{R}^n$
3. $Au = \frac{\mathrm{d}}{\mathrm{d}t} u$ na $C^1([0,1]) \subset L^2(0,1)$, ak $u(1) = 0$
4. laplacián na $\mathcal{L}^2(\Omega)$, kde $\Omega \subset \mathbb{R}^n$, kde Ω je oblasť a funkcie sú nulové na $\partial\Omega$

K. Nájdite izometriu z M do M, ktorá nie je surjektívna.

K1. Nájdite spektrum operátora.

1. $A = \frac{\mathrm{d}}{\mathrm{d}t}, X = C[0,1], D_A = \{x \in C^1[0,1] : x(0) = 0\}$
2. $A = \frac{\mathrm{d}}{\mathrm{d}t}, X = C[0,1], D_A = \{x \in C^1[0,1]\}$
3. $A = \frac{\mathrm{d}}{\mathrm{d}t}, X = C[0,1], D_A = \{x \in C^1[0,1], x(0) = x(1)\}$
4. $(Tf)(x) = f(x+a), a > 0, X = \mathcal{L}^2(\mathbb{R})$
5. $(Kx)(t) = \int_0^t k(t,s)x(s)\,\mathrm{d}s, k \in C([0,1] \times [t,1]), X = \mathcal{L}^2(0,1)$
6. $Sx = y, x = (\dots, x_{-1}, x_0, x_1, \dots), y_k = x_{k-1}, X = \ell^2(\mathbb{R})$
7. $Sx = y, x = (\dots, x_{-1}, x_0, x_1, \dots), y_k = x_{k-1}, X = \ell^2[0,\infty)$
8. $(Af)(x) = \int_{-\infty}^t \mathrm{e}^{\alpha(t-s)} f(s)\,\mathrm{d}s, Re\alpha < 0, X = \mathcal{L}^2(\mathbb{R})$
9. L^{-1}, ak $L : X \to X$ a aj L^{-1} je spojité lineárne

L. Vypočítajte $\delta\Phi(y_0;h)$, $\delta^2\Phi(y_0;h,k)$ na X.

1. $\Phi(y) = f(x, y(x))$, f je diferencovateľná na $\mathbb{R}^2$ a $X = C([0,1])$
2. $\Phi(y) = \int_a^b \left(y^2 + (y')^2\right)\, \mathrm{d}x - \int_a^b fy\,\mathrm{d}x$, kde $X = C^1([a,b])$
3. $\Phi(y) = \int_{-1}^1 x^2(y')^2, \mathrm{d}x$ kde $X = C^1([-1,1])$
4. $\Phi(y) = \int_1^4 \frac{(y')^4-(y')^2}{2x}, \mathrm{d}x$ kde $X = C^1([-1,1])$
5. $\Phi(u) = \int_\Omega ||\nabla u(x,y)||^2\,\mathrm{d}(x,y)$, kde Ω je ohraničená oblasť, $X = C^1(\overline{\Omega})$
6. $\Phi(\mathbf{x},\mathbf{y},\mathbf{z}) = \int_a^b \sum_{i=1}^{n} \frac{m_i}{2}((x_i')^2 + (y_i')^2 + (z_i')^2)\,\mathrm{d}t$, kde $X = [C^1([a,b])]^{3n}$

M. Nájdite stacionárne body nasledujúcich funkcionálov a určte ich hladkosť.

1. $\Phi(y) = \int_2^3 \frac{x^3}{(y')^2}\,\mathrm{d}x$, pričom $y(2) = 4$, $y(3) = 9$
2. $\Phi(y) = \int_{-1}^1 x^{\frac{2}{3}}(y')^2\,\mathrm{d}x$, pričom $y(-1) = -1$, $y(1) = 1$
3. $\Phi(y) = \int_0^T [(y')^2 - y^2]\,\mathrm{d}x$, pričom $y(0) = 0$, $y(T) = \sin T$
4. $\Phi(y) = \int_1^2 \frac{\sqrt{1+(y')^2}}{x}\,\mathrm{d}x$, pričom $y(1) = 0$, $y(2) = 1$
5. $\Phi(y) = \int_{\frac{1}{2}}^1 \frac{\sqrt{1+(y')^2}}{x}\,\mathrm{d}x$, pričom $y(1/2) = -\sqrt{3}/2$, $y(1) = 0$
6. $\Phi(y) = \int_{-1}^1 y^2(1-y')^2\,\mathrm{d}x$, pričom $y(-1) = 0$, $y(1) = 1$
7. $\Phi(y) = \int_0^1 (y')^2 - 6x^2y + y^3y'\,\mathrm{d}x$, pričom $y(0) = 1$, $y(1) = 2$
8. $\Phi(y) = \int_0^1 (y')^5\,\mathrm{d}x$, pričom $y(0) = 0$, $y(1) = B$

N. Nájdite extremály, ktoré minimalizujú nasledujúce funkcionály na $C^1([a,b])$.

1. $\Phi(y) = \int_0^1 \left[(y')^4 - 6(y')^2\right]\, \mathrm{d}x$, pričom $y(0) = 0,\ y(1) = A$
2. $\Phi(y) = \int_0^1 \left[1 + (1+x^2)(y')^2\right]\, \mathrm{d}x$, pričom $y(0) = 0,\ y(1) = \frac{\pi}{4}$
3. $\Phi(y) = \int_{-1}^1 (y' - 2|x|)^2\, \mathrm{d}x$, pričom $y(\pm 1) = \pm 1$
4. $\Phi(y) = \int_0^1 ((y')^2 - 1)((y')^2 - M)\, \mathrm{d}x$, pričom $y(0) = 0,\ y(1) = 1, M \geq 1$
5. $\Phi(y) = \int_0^{\frac{\pi}{2}} \sqrt{(y')^2 + y^2}\, \mathrm{d}x$, pričom $y(0) = y(\pi/2) = 1$
6. $\Phi(y) = \int_1^2 (y')(1 + x^2 y')\, \mathrm{d}x$, pričom $y(1) = -1,\ y(2) = 1$
7. $\Phi(y) = \int_1^2 \left[x(y')^4 - 2y\,(y')^3\right]\, \mathrm{d}x$, pričom $y(1) = 0,\ y(2) = 1$
8. $\Phi(y) = \int_0^1 \left[(y')^2 + x^2\right]\, \mathrm{d}x$, pričom $y(0) = -1,\ y(1) = 1$
9. $\Phi(y) = \int_0^a \left[1 - \mathrm{e}^{-(y')^2}\right]\, \mathrm{d}x$, pričom $y(0) = 0,\ y(a) = b, a > 0$
10. $\Phi(y) = \int_0^1 y(y')^2\, \mathrm{d}x$, pričom $y(0) = p > 0,\ y(1) = q > 0$
11. $\Phi(y) = \int_1^2 \left[x^3(y'')^2 - 12xy\right]\, \mathrm{d}x$, pričom $y(1) = \frac{1}{2},\ y(2) = 2$ (na $C^2([a,b])$)

O. Bez overovania nájdite extremály, ktoré minimalizujú nasledujúce funkcionály na $C^1([a,b])$.

1. $\Phi(y) = \int_0^\pi (y')^2\, \mathrm{d}x$ s podmienkami $y(0) = 0,\ y(\pi) = 0$ a s väzbou $\int_0^\pi y^2\, \mathrm{d}x = 1$
2. $\Phi(y) = \int_0^b y\, \mathrm{d}x$ s podmienkami $y(0) = 0,\ y(b) = 0$ a s väzbou $\int_0^b \sqrt{1 + (y')^2}\, \mathrm{d}x = L \in (b, \pi b/2)$
3. $\Phi(y) = \int_0^\pi [2y \sin x + (y')^2]\, \mathrm{d}x$ s podmienkami $y(0) = 0,\ y(\pi) = 0$ a s väzbou $\int_0^\pi y\, \mathrm{d}x = 1$
4. $\Phi(y) = \int_0^2 [y + (y')^2]\, \mathrm{d}x$ s podmienkami $y(0) = 1,\ y(3) = 48$ a s väzbou $\int_0^2 xy\, \mathrm{d}x = 43$
5. $\Phi(y) = \int_0^b \sqrt{1 + y^2}\, \mathrm{d}x$ s väzbou $\int_0^b y\, \mathrm{d}x = C$
6. $\Phi(y) = \int_0^1 y\, \mathrm{d}x$ s väzbou $\int_0^1 y^2\, \mathrm{d}x \leq 9$
7. $\Phi(y) = \int_0^1 (y^2 - 2y)\, \mathrm{d}x$ s väzbou $\int_0^1 x|y|\, \mathrm{d}x \leq \frac{1}{6}$