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Sada úloh na precvičenie VII.
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A. Riešte PDR prvého rádu s danými podmienkami.

1. $u_t - 3u_x = 0,\ \ u(x,0) = \mathrm{e}^{-x^2}$ $\qquad u(x,t) = \mathrm{e}^{-(x+3t)^2}$
2. $u_t + 3u_x = 0,\ \ u(x,0) = \sin x$ $\qquad u(x,t) = \sin(x-3t)$
3. $\sqrt{1-x^2}u_x + u_y = 0,\ \ u(0,y) = y$ $\qquad u(x,y) = y - \arcsin x$
4. $u_x + u_y + u = \mathrm{e}^{x+2y},\ \ u(x,0) = 0$ $\qquad u(x,y) = \frac{\mathrm{e}^{x+2y} - \mathrm{e}^{x-2y}}{4}$
5. $u_t + au_x = x^2t + 1,\ a \in \mathbb{R},\ \ u(x,0) = x+2$ $\qquad u(x,t) = x - at + 2 + t + \frac{t^2x^2}{2} - \frac{axt^3}{3} + \frac{a^2t^4}{12}$
6. $u_t + t^{\alpha}u_x = 0,\ \alpha > -1,\ \ u(x,0) = \phi(x)$ $\qquad u(x,t) = \phi\left(x - \frac{t^{\alpha+1}}{\alpha+1}\right)$
7. $u_t + xtu_x = x^2,\ \ u(x,0) = \phi(x)$ $\qquad u(x,t) = \phi\left(x\mathrm{e}^{-t^2/2}\right) + x^2\mathrm{e}^{-t^2}\int_0^t \mathrm{e}^{s^2}\,\mathrm{d}s$
8. $u_t + 2u_x - 3u = 0,\ \ u(x,0) = \frac{1}{1+x^2}$ $\qquad u(x,t) = \frac{\mathrm{e}^{-3t}}{1+4t^2-4tx+x^2}$
9. $u_t + u_x - 3u = t,\ \ u(x,0) = x^2$ $\qquad u(x,t) = -\frac{t}{3} - \frac{1}{9} + \mathrm{e}^{3t}\left((x-t)^2 + \frac{1}{9}\right)$

B. Riešte PDR prvého rádu a načrtnite niektoré z charakteristík.

$(1+x^2)u_x + u_y = 0$ $\qquad u(x,y) = F(y - \arctan x)$

C. Riešte PDR pomocou substitúcie $v = u_y$.

$u_{xy} + 3u_y = 0$ $\qquad u(x,y) = \mathrm{e}^{-3x}G(y) + F(x)$

D. Nájdite všeobecné riešeniev PDR prvého rádu.

1. $x\,u_x + y\,u_y = \sqrt{x^2+y^2}$ $\qquad u(x,y) = \sqrt{x^2+y^2} + F\left(\frac{y}{x}\right)$
2. $y^2\,u_x + ax^2\,u_y = (bx^2+cy^2)u,\ \ a \neq 0$ $\qquad u(x,y) = \mathrm{e}^{cx+by/a}F(ax^3 - y^3)$
3. $u_x + u_y = u + \sin(x+y)$ $\qquad u(x,y) = \mathrm{e}^{x}F(x-y) - \cos(x+y) - \sin(x+y)/2$
4. $a\,u_x + by\,u_y + cz\,u_z = 0$ $\qquad u(x,y,z) = F\left(|y|^a\mathrm{e}^{-bx}, |z|^a\mathrm{e}^{-cx}\right)$
5. $u_x + u_y + u_z = xyz$ $\qquad u(x,y,z) = x^2yz/2 - x^3(z+y)/6 + x^4/12 + F(y-x, z-x)$

E. Riešte vlnovú rovnicu s danými podmienkami.

1. $u_{tt} = c^2 u_{xx},\ \ u(x,0) = \mathrm{e}^x,\ u_t(x,0) = \sin x$ $\qquad u(x,t) = \frac{\mathrm{e}^{x+ct}+\mathrm{e}^{x-ct}}{2} - \frac{\cos(x+ct)-\cos(x-ct)}{2c}$

2. $u_{tt} = c^2 u_{xx},\ \ u(x,0) = \ln(x^2+1),\ u_t(x,0) = 4+x$ $\qquad u(x,t) = \ln\sqrt{(1+(x+ct)^2)(1+(x-ct)^2)} + t(4+x)$

3. $u_{tt} = u_{xx},\ \ u(x,0) = 0,\ u_t(x,0) = -2x\mathrm{e}^{-x^2}$ $\qquad u(x,t) = \frac{\mathrm{e}^{-(x+t)^2}+\mathrm{e}^{-(x-t)^2}}{2}$

4. $u_{tt} = u_{xx},\ \ u(x,0) = 0,\ u_t(x,0) = \frac{x}{(1+x^2)^2}$ $\qquad u(x,t) = \frac{1}{4}\left(\frac{1}{1+(x-t)^2} - \frac{1}{1+(x+t)^2}\right)$

5. $u_{tt} = u_{xx},\ \ u(x,0) = 0,\ \ u_t(x,0) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases}$

$$u(x,t) = \frac{g(x+t)-g(x-t)}{2},\ \ g(x) = \begin{cases} -\frac{2}{3} & \text{for } x \le -1 \\ x - \frac{x^3}{3} & \text{for } |x| \le 1 \\ \frac{2}{3} & \text{for } x \ge 1 \end{cases}$$

6. $u_{tt} = c^2 u_{xx} + xt,\ \ u(x,0) = 0,\ u_t(x,0) = 0$ $\qquad u(x,t) = xt^3/6$

7. $u_{tt} = c^2 u_{xx} + \mathrm{e}^{at},\ \ u(x,0) = 0,\ u_t(x,0) = 0$ $\qquad u(x,t) = \frac{\mathrm{e}^{at}-at-1}{a^2}$

8. $u_{tt} = c^2 u_{xx} + \cos x,\ \ u(x,0) = \sin x,\ u_t(x,0) = 1+x$ $\qquad u(x,t) = \cos ct \sin x + (1+x)t + \frac{\cos x}{c^2} - \frac{\cos x \cos ct}{c^2}$

9. $u_{tt} = u_{xx} + \mathrm{e}^{x-t},\ \ u(x,0) = 0,\ u_t(x,0) = 0$ $\qquad u(x,t) = \frac{\mathrm{e}^{x+t}-\mathrm{e}^{x-t}}{4} - \frac{t\mathrm{e}^{x-t}}{2}$

F. Riešte úlohu $u_{tt} - 3u_{xt} - 4u_{xx} = 0,\ \ u(x,0) = x^2,\ u_t(x,0) = \mathrm{e}^x$ **podobne ako pri riešení vlnovej rovnice.**

$u(x,t) = 4t^2 + x^2 + \frac{\mathrm{e}^x}{5}(\mathrm{e}^{4t} - \mathrm{e}^{-t})$

G. Nájdite všeobecné riešenie rovníc využitím faktorizovateľnosti diferenciálneho operátora.

1. $u_{xx} + u_x = u_{yy} + u_y$ $\qquad u(x,y) = \mathrm{e}^{-x}F(y-x) + G(y+x)$

2. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$ $\qquad u(x,y) = F(y-3x) + G(3y-x)$

H. Riešte rovnicu s danými podmienkami.

1. $u_t = ku_{xx}, \;\; u(x,0) = \mathrm{e}^{3x}$ $\qquad u(x,t) = \mathrm{e}^{3x+9kt}$

2. $u_t = ku_{xx}, \;\; u(x,0) = \begin{cases} 1, & x > 0, \\ 3, & x \leq 1. \end{cases}$ $\qquad u(x,t) = 2 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$

3. $u_t = u_{xx}, \;\; u(x,0) = \begin{cases} 1-x, & x \in [0,1], \\ 1+x, & x \in [-1,0], \\ 0, & |x| > 1. \end{cases}$

   $u(x,t) = \sqrt{4kt}\mathrm{e}^{-\frac{-(x+1)^2}{4tk}} + \sqrt{4kt}\mathrm{e}^{-\frac{-(x-1)^2}{4tk}} - 4\sqrt{kt}\mathrm{e}^{-\frac{-x^2}{4tk}} + \frac{(x-1)}{2}\operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right) + \frac{(x+1)}{2}\operatorname{erf}\left(\frac{x+1}{\sqrt{4kt}}\right) - x\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$

4. $u_t = ku_{xx} + \sin x, \;\; u(x,0) = 0$ $\qquad u(x,t) = \sin x(1-\mathrm{e}^{-kt})/k$

I. Pomocou substitúcie $u(x,t) = \mathrm{e}^{bt}v(x,t)$ riešte rovnicu $u_t = ku_{xx} + bu$ s podmienkou $u(x,0) = \phi(x)$.

$u(x,t) = \mathrm{e}^{bt}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{4kt}}}{2\sqrt{\pi kt}}\phi(y)\,\mathrm{d}y$

J. Zistite, či dané rovnice riešia Laplaceovu rovnicu (či sú harmonické).

1. $u = x^2 - y^2$
2. $u = e^y \cos x$
3. $u = \frac{1}{x^2+y^2}$
4. $u = \arctan\left(\frac{y}{x}\right)$
5. $u = \arctan\left(\frac{y}{x}\right)\frac{y}{x^2+y^2}$
6. $u = z^2$, $z = x + \mathrm{i}y$
7. $u = z^3$, $z = x + \mathrm{i}y$
8. $u = \mathrm{e}^z$, $z = x + \mathrm{i}y$

K. Nájdite dve harmonické funkcie také, že ich súčin nebude harmonická funkcia.

L. Ukážte, že ak u aj u^2 sú harmonické, potom u je konštanta.

M. Uvažujte úlohu

$$\begin{cases} u_{xx} + u_{yy} = 0 & \textbf{na } \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = 0, & u_y(x,0) = \frac{\cos nx}{n^2}. \end{cases}$$

Ukážte, že $u_n(x,y) = \frac{\sinh ny \cos nx}{n^3}$ **je riešením, ale** $\lim_{n\to\infty} ||u_n(x,y)||_\infty \neq 0$**.**

N. Nájdite radiálne symetrické riešenie rovnice $u_{xx} + u_{yy} = 1$ **v kruhu** $x^2 + y^2 < a^2$ **s podmienkou** $u(x,y) = 0$ **na hranici kruhu.**

$u(x,y) = \frac{x^2+y^2-a^2}{4}$

O. Nájdite radiálne symetrické riešenie rovnice $u_{xx} + u_{yy} = 1$ **v medzikruží** $a^2 < x^2 + y^2 < b^2$ **s podmienkou** $u(x,y) = 0$ **na oboch hraniciach kruhov.**

$u(x,y) = \frac{1}{4}(x^2 + y^2 + \frac{(b^2-a^2)\ln(x^2+y^2)}{2(\ln a - \ln b)}) + \frac{a^2 \ln b - b^2 \ln a}{\ln a - \ln b}$