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Dedicated to the memory of David Gale.

Abstract. House allocation market is a special type of exchange economy where each agent is endowed with one unit of an indivisible good (house) and wants to end up again with one unit, possibly the best one according to his preferences. If the endowments of all agents are pairwise different, an equilibrium as well as a core allocation always exist. However, for markets in which some agents' houses are equivalent, the existence problem for the economic equilibrium is NP-complete. In this paper we show that the hardness result is not valid if the preferences of all agents are strict, but it remains true in markets with trichotomous preferences. Further we extend some known results about house allocation markets to the case with duplicate houses using graph-theoretical methods.

Keywords: House allocation market, Core, Economic equilibrium, Pareto optimality, NP-completeness

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1 Introduction

The study of markets with indivisible goods started by the seminal paper of Shapley and Scarf [12] where a *house allocation market* was defined. In a house

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allocation market there is a finite set of agents, each one owns one unit of a unique indivisible good (house) and wants to exchange it for another, more preferred one; the preference relation of an agent is a linearly ordered list (possibly with ties) of a subset of goods. In such a market, the set of economic equilibria and the core is always nonempty, which was proved constructively by the Top Trading Cycles (TTC for short) algorithm due to Gale (see [12]).

Roth and Postlewaite [11] made a careful distinction between house allocation markets where preferences of agents contain ties and those where ties are not allowed. Their findings can be summarized as follows: The core is always nonempty, it always contains the set of equilibrium allocations, but may be strictly larger than the latter. The strong core may be empty, but if the agents' preferences do not contain ties, then strong core is nonempty and is equal to a unique equilibrium allocation. Later Wako [13] provided a polynomial algorithm for deciding the nonemptyness of the strong core and showed that each strong core allocation is an equilibrium allocation.

All the above results use the assumption that each agent's house is unique. If the houses of several agents are equivalent, the situation may change. Fekete, Skutella and Woeginger [5] proved that it is NP-complete to decide whether a *house allocation market with duplicate houses* admits an economic equilibrium. Duplicate houses mean that in the preference lists of agents some 'compulsory' ties appear (naturally, equivalent houses must be tied) and there is less freedom in assigning prices to houses (equivalent houses must have equal price). However, in the market constructed in the NP-completeness reduction in [5], some additional ties were used. If ties containing nonequivalent houses were not allowed, would the hardness result still be valid? Such a question was motivated by other allocation markets where the dividing line between efficiently solvable and hard cases is the presence or absence of ties. Let us mention the *stable roommates problem*, where if ties are not present, a polynomial algorithm to decide the existence of a stable matching exists; in case with ties the existence problem is NP-complete (see [7]). Another example is the *stable marriage problem*. Here, a stable matching always exists and in the case without ties all the stable matchings in one instance have the same cardinality. However, in the case with ties, cardinalities of different stable matchings may be different, moreover, the problem of finding a maximum cardinality stable matching is NP-hard [9]. A similar situation occurs also for a modification of the classical house allocation market, in which agents have preferences also over the lengths of trading cycles [4]. In such markets, the notion of economic equilibrium is not applicable, but the TTC algorithm in these settings always finds a (strong) core allocation, if ties are not allowed. In the presence of ties, it is NP-complete to decide whether the core as well as the strong core are nonempty [3].

The aim of the present paper is to study some algorithmic problems for house allocation markets. First, we derive some properties that are common in all house allocation markets, in particular we show how to test the nonemptyness of the strong core and prove that each strong core allocation is an equilibrium allocation. Our proofs, unlike those in [13] do not assume the uniqueness of agents' endowments.

Then we deal with house allocation markets with duplicate houses. We show that in the case of strict preferences the hardness result of [5] is not valid and we propose a simple polynomial time algorithm for deciding the existence of an economic equilibrium. The other end of the spectrum is the case with trichotomous preferences (i.e. all agents consider all acceptable houses equivalent, strictly preferred to their own house). We show that here the existence problem for the economic equilibrium remains NP-complete.

2 Description of the model

Let A be the *set of n agents*, H the *set of m house types*. The *endowment function* $\omega : A \to H$ assigns to each agent the type of house he originally owns. (Notice that in the classical model of Shapley and Scarf [12], $m = n$ and ω is a bijection.) Further, each agent $a \in A$ wishes to have in his possession just one house and it is supposed that his preferences are given as a linear ordering $P(a)$ on a set $H(a) \subseteq H$, the set of *acceptable* house types. The notation $i \succeq_a j$ means that agent a prefers house type i to house type j. If $i \succeq_a j$ and simultaneously $j \succeq_a i$, we say that house types i and j are *tied* in a's preference list and write $i \sim_a j$; if $i \succeq_a j$ and not $j \succeq_a i$, we write $i \succ_a j$ and say that agent a *strictly* prefers house type i to house type j. The n-tuple of preferences $(P(a), a \in A)$ will be denoted by $\mathcal{P}$ and called the *preference profile*.

The *house allocation market* is a quadruple $\mathcal{M} = (A, H, \omega, \mathcal{P})$.

If $S \subseteq A$, let $\omega(S) = \{\omega(a); a \in S\} \subseteq H$ and conversely, for $T \subseteq H$ we set $A_S(T) = \{a \in S; \omega(a) \in T\}$. If $A' \subseteq A$, we say that $\mathcal{M}' = \mathcal{M} \backslash A' = (A\backslash A', \omega(A\backslash A'), \omega', \mathcal{P}')$ is a *submarket* of $\mathcal{M}$ if ω' and $\mathcal{P}'$ are restrictions of ω and $\mathcal{P}$ to $A\backslash A'$ and $\omega(A\backslash A')$, respectively.

We say that $\mathcal{M}$ is a house allocation market with *strict preferences* if there are no ties in $\mathcal{P}$. On the other hand, $\mathcal{M}$ is a house allocation market with *trichotomous preferences* if for each agent $a \in A$, all the house types $h \in H'(a) = H(a)\backslash\{\omega(a)\}$ are tied. In other words, in a house allocation market with trichotomous preferences each agent $a \in A$ partitions the set H into three classes: $H'(a)$ are the house types that are *better* than his own house type, the second class contains just $\omega(a)$ and the third one those house types that are *worse* than $\omega(a)$. For each agent $a \in A$ we denote by $f_S(a)$ the set of the most preferred ($\succeq_a$-maximal) house types from $\omega(S)$ and $F_S(a) = A_S(f_S(a))$. Notice that when preferences of agents are strict then $|f_S(a)| = 1$ for each $a \in A$ and $S \subseteq A$; for trichotomous preferences we have $f_A(a) = H'(a)$.

We say that a function $x : S \to H$ is an *allocation* on S if there exists a bijection π on S such that $x(a) = \omega(\pi(a))$ for each $a \in S$. Throughout this paper we consider only *individually rational* allocations, i.e. such that $x(a) \in H(a)$ for each $a \in A$. (We always assume that $\omega(a) \in H(a)$ and this is the least preferred house in $H(a)$ for each $a \in A$.) Notice that for each allocation x on S the agent set S can be partitioned into directed cycles (*trading cycles*) of the form $K = (a_0, a_1, \ldots, a_\ell)$ in such a way that $x(a_i) = \omega(a_{i+1})$ for each $i = 0, 1, \ldots, \ell$ (here and elsewhere, indices for agents on cycles are taken modulo ℓ). Therefore we may represent an allocation as a collection of trading cycles. We say that agent a is trading in x if $x(a) \neq \omega(a)$.

A coalition $S \subseteq A$ *blocks* an allocation x on A if there exists an allocation y on S such that

$$y(a) \succ_a x(a) \text{ for each agent } a \in S$$

and a coalition $S \subseteq A$ *weakly blocks* an allocation x on A if there exists an allocation y on S fulfilling the condition

$$y(a) \succeq_a x(a) \text{ for each } \ a \in S \text{ and } y(a) \succ_a x(a) \text{ for at least one } \ a \in S.$$

An allocation x on A is in the *core* of market $\mathcal{M}$ if it admits no blocking coalition and it is in the *strong core* of $\mathcal{M}$ if no coalition weakly blocks it. An allocation x on A is *Pareto optimal* for market $\mathcal{M}$ if A does not block x, and it is *strongly Pareto optimal* if A does not weakly block it. A pair (x, p), where x is an allocation on A and $p : H \to \mathbb{R}$ is a price function, is an *economic equilibrium* for market $\mathcal{M}$ if for each agent $a \in A$, $x(a) \in f_S(a)$ for the set S equal to the *budget set* $B_a(p)$ of agent a under prices p, i.e. for

$$S = B_a(p) = \{h \in H; p(h) \leq p(\omega(a))\}.$$

We shall say that allocation x is an *equilibrium allocation* if there exists a price function p such that the pair (x, p) is an *economic equilibrium.*

The following simple property of equilibrium allocations will often be used.

Lemma 1 *If (x, p) is an economic equilibrium for market $\mathcal{M}$ then $p(x(a)) = p(\omega(a))$ for each $a \in A$.*

Proof. Let $K = (a_0, a_1, \ldots, a_\ell)$ be any trading cycle of x. According to the definition of equilibrium, $p(\omega(a_i)) \geq p(\omega(a_{i+1}))$ for each $i = 0, 1, \ldots, \ell$, which implies the assertion of the Lemma. ■

In what follows, we shall denote by $Q(\mathcal{M}), C(\mathcal{M}), SC(\mathcal{M}), PO(\mathcal{M})$ and $SPO(\mathcal{M})$ the set of all equilibrium, core, strong core, Pareto optimal and strongly Pareto optimal allocations for market $\mathcal{M}$. The definitions imply that for each house allocation market

$$SC(\mathcal{M}) \subseteq C(\mathcal{M}) \subseteq PO(\mathcal{M}) \text{ and } SC(\mathcal{M}) \subseteq SPO(\mathcal{M}) \subseteq PO(\mathcal{M}),$$

and all the inclusions in the above chain can be strict. Further, in the case without duplicate houses, Wako [13] proved $SC(\mathcal{M}) \subseteq Q(\mathcal{M})$.

a	a_1	a_2	a_3	a_4	a_5
$\omega(a)$	h_1	h_2	h_3	h_4	h_5
$P(a)$	$h_2 \succ h_3 \succ h_1$	$h_1 \sim h_4 \succ h_2$	$h_4 \succ h_2 \succ h_3$	$h_3 \sim h_5 \succ h_4$	$h_2 \succ h_1 \succ h_5$

Figure 1: Endowments and preferences of agents

Example 1 Let us consider the house allocation market $\mathcal{M}$ given by Figure 1.

In this example, either $x(a_1) = h_2$ or $x(a_1) \neq h_2$ for allocation x. In the former case the cycle (a_2, a_4, a_5) is weakly blocking as agent a_5 strictly improves and agents a_2, a_4 are not worse off than in x (they have their most preferred house types on this cycle). In the latter case the set $\{a_1, a_2\}$ is weakly blocking, as a_1 improves and a_2 is not worse off. Hence $SC(\mathcal{M}) = \emptyset$.

Further, $C(\mathcal{M})$ contains six allocations

$$\begin{aligned} x_1 &= (a_1, a_2)(a_3, a_4)(a_5), x_2 = (a_2, a_4, a_5)(a_1)(a_3), x_3 = (a_1, a_3, a_4, a_5, a_2) \\ x_4 &= (a_1, a_3, a_2, a_4, a_5), x_5 = (a_1)(a_2, a_4, a_3)(a_5) \text{ and } x_6 = (a_1, a_2, a_4, a_5)(a_3). \end{aligned}$$

Of them x_3 is strongly Pareto optimal, as the only agent who could strictly improve is a_1 by getting h_2, but since a_5 is not allowed to become worse off than in x_3, he must also be assigned his most preffered house h_2, and so there is no weakly blocking allocation on A for x_3.

Allocation $x_6 = (a_1, a_3, a_2)(a_4)(a_5)$ is Pareto optimal (notice e.g. that agent a_2 cannot strictly improve) but not in the core, as the set $\{a_3, a_4\}$ is blocking.

Finally, allocations x_1 and x_2 form economic equilibria with price vectors p_1 and p_2, where $p_1(a_1) = p_1(a_2) = p_1(a_3) = p_1(a_4) = 1$, $p_1(a_5) = 0$ and $p_2(a_1) = p_2(a_3) = 0$ and $p_2(a_2) = p_2(a_4) = p_2(a_5) = 1$. ■

In the following sections we shall often use digraphs, so let us recall the terminology used; we recommend [8] for further details. A *digraph* is a pair $G = (V, E)$, where V is a set of *vertices*, E is the set of *arcs*, i.e. ordered pairs of vertices. We allow parallel arcs as well as loops, i.e. arcs of the form (i, i) for some $i \in V$. If $V' \subseteq V$, a subdigraph of G *induced* by V' is a digraph $G(V') = (V', E')$ where $E' = \{(i, j) \in E; i, j \in V'\}$.

A *walk* in G is a sequence of vertices $(i_0, i_1, \dots, i_k)$ such that $(i_j, i_{j+1}) \in E$ for each $j = 0, 1, \dots, k-1$. We say that vertex j is a *successor* of a vertex i in G, if G contains a walk from i to j; in this case vertex i is a *predecessor* of vertex j. A vertex j is a *direct successor* of a vertex j and j is a *direct predecessor* of i in G, if $(i, j) \in E$. A vertex $i \in V$ is a *top vertex* in G, if it has no successors in G.

A walk $(i_0, i_1, \dots, i_k)$ is said to be *closed*, if $i_0 = i_k$; if moreover all the vertices $i_1, \dots, i_k$ are pairwise distinct, it is a *cycle*. A collection $\mathcal{K}$ of vertex-disjoint cycles is a *cycle cover* of a digraph G if each vertex of G is contained in some $K \in \mathcal{K}$. A polynomial-time algorithm to decide whether a digraph has a cycle cover is well-known. The approach described in [2] first constructs a bipartite (undirected) graph Γ by duplicating each vertex of G and making the pair $\{i, j'\}$ an edge if and only if $(i, j) \in E$, where j' is the copy of vertex $j \in V$. It is easy to see that G has a cycle cover if and only if Γ has a perfect matching. Moreover, if one creates the so-called persistency partition of the edges of Γ (i.e. labels each edge according to whether it belongs to all, some but not all, or none perfect matchings - a polynomial algorithm is described in [10]), then it can be decided in polynomial time whether a cycle cover containing at least one arc from a given set $E' \subseteq E$ exists.

A digraph G is said to be Eulerian, if it contains a closed walk containing all arcs of G exactly once – this can also be decided in polynomial time.

A digraph G with vertex set V is *strongly connected*, if for each pair i, j of distinct vertices of V there is a walk from i to j as well as a walk from j to i in G. A *strongly connected component* (SCC for short) of a digraph G is a maximal strongly connected subdigraph of G. We shall call a SCC *trivial*, if it contains just one vertex and no arcs. The *condensation* $G^* = (V^*, E^*)$ of a directed graph G is obtained by merging the vertices of each SCC of G (and perhaps deleting eventual parallel arcs). For $i \in V$ we shall denote by i^* the vertex of G^* corresponding to the SCC of G containing i. As G^* is acyclic, the vertices of G^* are partially ordered, $i^* \ll j^*$ if and only if i^* is a predecessor of j^* in G^* and this ordering can easily by extended to a linear ordering, sometimes called a *topological ordering*. A polynomial algorithm to construct a condensation of a digraph is described e.g. in [8].

3 Common properties of house allocation markets

If $\mathcal{M}$ is any house allocation market, for finding a core allocation the famous TTC algorithm [12] can be used. TTC starts with the whole set of agents A. An arbitrary agent $a_0 \in A$ is chosen and he proposes to one (arbitrary) agent in $F_A(a_0)$, say agent a_1. Then a_1 proposes to one of the agents in $F_A(a_1)$, say agent a_2, etc. After a finite number of proposals, a cycle K arises, which will be the first trading cycle. Agents corresponding to K are deleted from the market and the whole procedure is repeated for the submarket $\mathcal{M} \backslash K$ until all agents are assigned to some trading cycle.

Theorem 1 *[12] The core of each house allocation market is nonempty.*

An implementation of the TTC algorithm for the no-ties, no-duplicate-houses case in $O(r)$ time, where $r = \sum_{a \in A} |H(a)|$ is described in [1]. Obviously, this implementation can be used in any market, if at the beginning the ties are broken arbitrarily.

When TTC is applied to a house allocation market without duplicate houses and ties, its outcome is unique. This need not be the case when either duplicate houses or ties are present. Moreover, in the classical model each equilibrium allocation is an output of some realization of the TTC algorithm irrespective of ties, but it may happen that some core allocations cannot be obtained in this way. In Example 1, this applies e.g. to allocation x_5, since as soon as agent a_3 receives the proposal from a_4, he proposes back to a_4. Let us remark here, that a complete description of the structure of the core of a house allocation market is not known.

Strong core of a house allocation market may be empty [11] and its nonemptyness can be decided in polynomial time by an algorithm proposed in [13]. Here we give a simpler algorithm for this task. Let $G_F(\mathcal{M}) = (A, E)$ be a digraph defined by $(i, j) \in E$ if $j \in F_A(i)$.

Lemma 2 *Let $\mathcal{M} = (A, H, \omega, \mathcal{P})$ be any house allocation market and let D be a top SCC in $G_F(\mathcal{M})$. Then $SC(\mathcal{M}) \neq \emptyset$ if and only if D admits a cycle cover and $SC(\mathcal{M}') \neq \emptyset$ for the submarket $\mathcal{M}\backslash V(D)$.*

Proof. Let $x \in SC(\mathcal{M})$. Suppose that D is a top SCC in G_F that does not have a cycle cover. Then there exists $a \in V(D)$ such that $x(a) \notin f_A(a)$, but as $a \in V(D)$, there exists a cycle K in D containing a. Assigning to each agent on K the endowment of his direct successor on K, it is easy to see that the vertices of K form a weakly blocking set for x. So D must have a cycle cover and $x(a) \in \omega(V(D)$ for each agent $a \in V(D)$. Hence the restriction of x to agents of $A\backslash V(D)$ is a strong core allocation for $\mathcal{M}'$.

Conversely, let $x' \in SC(\mathcal{M}')$ and let D have a cycle cover $\mathcal{K}$. It is easy to see that x' augmented by the trading cycles defined by the cycles of $\mathcal{K}$ gives a strong core allocation for $\mathcal{M}$. ■

Based on Lemma 2, it is sufficient to find any top SCC D of the graph $G_F(\mathcal{M})$ and to test whether it admits a cycle cover. If the answer is negative, $SC(\mathcal{M}) = \emptyset$, otherwise the same procedure is invoked with the submarket $\mathcal{M}\backslash V(D)$. All this can easily be done in polynomial time.

We now give a simpler proof of another theorem from [13], which also shows that the next property holds in house allocation markets with duplicate houses too.

Theorem 2 *In any house allocation market $\mathcal{M}$, $SC(\mathcal{M}) \subseteq Q(\mathcal{M})$.*

Proof. Let $x \in SC(\mathcal{M})$ be arbitrary. We will show that for a suitably defined price function p, the pair (x, p) is an economic equilibrium of $\mathcal{M}$.

Let us take the digraph $G_x = (H, E_x^{\sim} \cup E_x^{\succ})$ where

$$(h_i, h_j) \in E_x^{\sim} \quad \text{if there exist } i, j \text{ such that } \omega(i) = h_i, \omega(j) = h_j \text{ and } h_j \sim_i x(i)$$
$$(h_i, h_j) \in E_x^{\succ} \quad \text{if there exist } i, j \text{ such that } \omega(i) = h_i, \omega(j) = h_j \text{ and } h_j \succ_i x(i).$$

Arcs from $E_x^{\sim}$ will be called *weak* arcs, those from $E_x^{\succ}$ are *strong* arcs. As x is a strong core allocation, any cycle K in G_x consists exclusively of weak arcs, otherwise the agents corresponding to the arcs of K would form a weakly blocking set for x. Let us now take the condensation $G^* = (V^*, E^*)$ of $G_x^{\sim} = (H, E_x^{\sim})$. It is easy to see that strong arcs (now defined accordingly) connect different vertices of G^*, moreover digraph $G_x^{\succ} = (V^*, E^* \cup E_x^{\succ})$ is acyclic. So taking any topological ordering $\gg$ of V^* in $G_x^{\succ}$, the prices of house types can be defined by

$$p(h_i) > p(h_j) \text{ if and only if } \ h_i^* \gg h_j^*.$$

It is easy to see that for the pair (x, p) we have:

(i) $p(\omega(i)) = p(x(i))$ for each agent, as any trading cycle of x corresponds to a cycle in $G_x^{\sim}$;

(ii) if agent i prefers a house type $\omega(j)$ to $x(i)$, then $p(\omega(j)) > p(\omega(i))$

and so (x, p) is an economic equilibrium for $\mathcal{M}$. ■

A detailed study of Pareto optimality for house allocation markets can be found in [1]. A polynomial algorithm for finding a strongly Pareto optimal matching and some results concerning the structure of strongly Pareto optimal matchings are described, however, all is done under the assumption of strict preferences. If ties and/or duplicate houses are present, one can use a different approach. Let x be any allocation in $\mathcal{M}$. Let us take the digraph $G_x^{\succ} = (A, E_x^{\succ})$ where $(i, j) \in E_x^{\succ}$ if $\omega(j) \succ_i x(i)$ (this digraph is similar to the one used in the proof of Theorem 2, but now its vertices are agents, not house types). Allocation x is not Pareto optimal if and only if $G_x^{\succ}$ admits a cycle cover, and along this cycle cover each agent in A can get a strictly preferred house to the one he is assigned in x. Similarly, x is not strongly Pareto optimal if and only if in the digraph $G_x^{\succsim} = (A, E^{\sim} \cup E_x^{\succ})$ with $(i, j) \in E_x^{\sim}$ if $\omega(j) \sim_i x(i)$ a cycle cover exists that contains at least one arc from $E_x^{\succ}$. Using at most n of upgrades, a Pareto optimal or a strongly Pareto optimal allocation, respectively, will be obtained. Summarizing:

Theorem 3 *In any house allocation market $\mathcal{M}$, Pareto optimal as well as strongly Pareto optimal allocations exist and they can be found in polynomial time.*

4 House allocation markets with strict preferences

Let $\mathcal{M}$ be a house allocation market with strict preferences. We define a digraph $G_H = (H, E)$, called the *house-type digraph*, with arcs corresponding to agents in such a way that $e(a) = (h_i, h_j) \in E$ if $\omega(a) = h_i$ and $h_j \in f_A(a)$. Notice that since preferences are strict, the correspondence between agents and arcs is one-to-one.

Lemma 3 *Let $\mathcal{M}$ be a market with strict preferences where an economic equilibrium with price function p exists and let D be a top SCC in G_H. Then*

(i) $p(h_i) = p(h_j)$ *for each* $h_i, h_j \in V(D)$*;*

(ii) *D is Eulerian and*

(iii) *if* $(h_i, h_j) \in E$ *for some* $h_i \notin V(D)$, $h_j \in V(D)$*, then* $p(h_i) < p(h_j)$*.*

Proof. Let the economic equilibrium in $\mathcal{M}$ be (x, p).

(i) Suppose that $V(D)$ is partitioned into two nonempty disjoint sets V^1 and V^2 in such a way that $p(h_i) > p(h_j)$ for each $h_i \in V^1$ and each $h_j \in V^2$. As D is strongly connected, there exists an agent a such that $e(a) = (h_i, h_j)$ for some $h_i \in V^1$, $h_j \in V^2$. Then, since h_j is the only most preferred house type for a in A and agent a can afford it, we have $x(a) = h_j$ – a contradiction with Lemma 1. Therefore the prices of all the house types in $V(D)$ are equal.

(ii) It follows from (i) and the definition of D that all the agents correspondig to arcs of D trade only among themselves, i.e. the arc set of D can be partitioned into several arc-disjoint directed cycles (possibly loops) and so D is Eulerian.

(iii) If $e(a) = (h_i, h_j) \in E$ for $h_i \notin V(D)$, $h_j \in V(D)$ and $p(h_i) \geq p(h_j)$, then agent a can afford house type h_j and since this is his only most preferred house type in $\mathcal{M}$, he must be in x on a trading cycle containing house types from $V(D)$, but this is a contradiction with the proof of (ii). ■

Notice that if a top SCC of the house-type digraph consists of a single vertex, then it contains a loop, so it is trivially Eulerian.

Lemma 4 *Let* $\mathcal{M} = (A, H, \omega, \mathcal{P})$ *be a market with strict preferences and let D be a top SCC in* G_H*. Then an economic equilibrium exists in* $\mathcal{M}$ *if and only if D is Eulerian and in the reduced market* $\mathcal{M}' = \mathcal{M} \backslash E(D)$ *an economic equilibrium exists.*

Proof. Let (x, p) be an equilibrium for $\mathcal{M}$. Lemma 3 implies that $x(a) \in V(D)$ for each agent $a \in E(D)$, that D is Eulerian and no agent a with $\omega(a) \notin V(D)$ can afford a house type in $V(D)$. Then the restriction of (x, p) to $\mathcal{M}'$ is an economic equilibrium for $\mathcal{M}'$.

Conversely, let $(x', p') \in Q(\mathcal{M}')$. Then we construct an economic equilibrium (x, p) for $\mathcal{M}$ in the following way: For each $a \in A \backslash E(D)$ set $x(a) = x'(a)$, for each $h \in H \backslash V(D)$ let $p(h) = p'(h)$. Now take an arbitrary constant $\xi > max\{p(h); h \in H \backslash V(D)\}$ and set $p(h) = \xi$ for all $h \in V(D)$. Clearly, houses in $V(D)$ are now outside the budget set for all agents in $A \backslash E(D)$, so these agents still have their most preferred affordable houses in (x, p). Each agent $a \in E(D)$ corresponds to an arc $e(a) = (h_i, h_j)$, so we set $x(a) = h_j$ for this agent. As D is Eulerian, the supply equals demand for the houses in D, thus (x, p) is an economic equilibrium for $\mathcal{M}$. ■

Now Lemmas 3 and 4 directly imply the following simple algorithm: for a given market $\mathcal{M}$ create the house-type digraph G_H and take any top SCC D in

G_H. If D is not Eulerian, $\mathcal{M}$ does not admit any economic equilibrium. If D is Eulerian, the agents and house types corresponding to D are deleted from $\mathcal{M}$ and the whole procedure continues for the obtained submarket. Summarizing:

Theorem 4 *If the preferences of all agents in a house allocation market $\mathcal{M}$ are strict, then the existence of an economic equilibrium for $\mathcal{M}$ can be decided in polynomial time.*

Notice that if a market admits an economic equilibrium, then the equilibrium allocation is unique, although it may be supported by several different price functions and not even the linear order of the prices is uniquelly determined.

Now we turn our attention to the strong core. We derived in Section 3 a condition that enables to decide the nonemptyness of the strong core for any house allocation market in polynomial time, but here we give an alternative characterization for markets with strict preferences using the house-type digraph.

Lemma 5 *Let $\mathcal{M}$ be a market with strict preferences and D a top SCC in G_H.*

(i) If $SC(\mathcal{M}) \neq \emptyset$, then D is Eulerian.

(ii) $SC(\mathcal{M}) \neq \emptyset$ if and only if $SC(\mathcal{M} \backslash E(D)) \neq \emptyset$ and D is Eulerian.

Proof. (i) Let us suppose that for a strong core allocation x an agent $a \in E(D)$ exists such that $x(a) \notin V(D)$. This means that a did not receive in x his most preferred house. Further, since D is nontrivial, there exists a cycle K in D, containing arc $e(a)$. Let us assign to each agent b associated with an arc $e(b) \in K$ the house type corresponding to the endvertex of $e(b)$. It is easy to see that $E(K)$ is a weakly blocking coalition – a contradiction with x being a strong core allocation. Now we use the same argument as in the proof of Lemma 3 (ii).

(ii) If $x \in SC(\mathcal{M})$ then clearly its restriction belongs to $SC(\mathcal{M} \backslash E(D))$ as any weakly blocking set in $\mathcal{M} \backslash E(D)$ is weakly blocking in $\mathcal{M}$ too. Conversely, if $x' \in SC(\mathcal{M} \backslash E(D))$, we extend allocation x' to an allocation x of $\mathcal{M}$ in such a way that an agent a corresponding to the arc $e(a) = (h_i, h_j) \in E(D)$ will get $x(a) = h_j$, for other agents $x(a) = x'(a)$. Suppose now that x is weakly blocked by a set Z with allocation y. Then neither $Z \subseteq E \backslash E(D)$ nor $Z \subseteq E(D)$ is possible (the former because x' is a strong core allocation for the submarket, the latter because no agent from $E(D)$ can strictly improve), but then necessarily Z contains an agent a, for whom $\omega(a) \in V(D)$ and $y(a) \notin V(D)$, hence a is worse off in allocation y, a contradiction. ∎

Now it is easy to see that the following assertion holds also for markets with duplicate houses:

Corollary 1 *A house allocation market with strict preferences admits a strong core allocation if and only if it admits an economic equilibrium.*

5 Trichotomous markets

First we derive a simple condition for the nonemptyness of the strong core for any house allocation market with trichotomous preferences. Let $G_T(\mathcal{M}) = (A, E_T)$ where $(i,j) \in E_T$ if $\omega(j) \succ_i \omega(i)$, i.e. if $\omega(j) \in H'(i)$. An allocation x in $\mathcal{M}$ corresponds to a collection of vertex-disjoint trading cycles and some single vertices in $G_T(\mathcal{M})$. It is easy to see that allocation x admits no weakly blocking set, if each agent contained in a cycle in $G_T(\mathcal{M})$ is trading in x. This implies

Theorem 5 *In a house allocation market $\mathcal{M}$ with trichotomous preferences, $SC(\mathcal{M}) \neq \emptyset$ if and only if each nontrivial SCC of $G_T(\mathcal{M})$ admits a cycle cover.*

In a sharp contrast with the above result is the following assertion.

Theorem 6 *In a house allocation market with trichotomous preferences it is NP-complete to decide whether an economic equilibrium exists.*

Proof. In the polynomial reduction we shall use the problem ONE-IN-THREE-SAT, see [6, Problem LOG4]. Here a Boolean formula B in conjunctive normal form with variables $v_1, v_2, \ldots, v_n$ and clauses $C_1, C_2, \ldots, C_m$ is given. Each clause contains exactly three literals, no variable is negated in B and the question is whether there exists a truth assignment f such that there is exactly one true literal in each clause.

Let us construct a market $\mathcal{M}(B)$ for each formula B. The set of agents contains one agent c_i for each clause C_i and for each variable v_j there are agents $\phi_j^1, \phi_j^2, \ldots \phi_j^{n_j}$ where n_j is the number of occurences of variable v_j in B. The set of houses is $\{C_1, \ldots, C_m, v_1, \ldots, v_n\}$ and the endowments are defined by

$$\begin{aligned} \omega(c_i) &= C_i \text{ for each } i, \\ \omega(\phi_j^k) &= v_j \text{ for each } j, k. \end{aligned}$$

Further, $H'(c_i) = \{v_{j_1}, v_{j_2}, v_{j_3}\}$ where $v_{j_1}, v_{j_2}, v_{j_3}$ are the literals present in C_i and $H'(\phi_j^k)$ is equal to the set of all houses C_i such that variable v_j occurs in clause C_i.

Now suppose that the truth assignment f with the required properties exists. We shall define the pair (x, p) in the following way:

$$\begin{aligned} p(c_i) &= 1 \text{ for each } i \\ p(v_j) &= \begin{cases} 1 & \text{if } f(v_j) = \text{true and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Further, for each clause agent we set $x(c_i) = v_j$ if the only true literal in C_i is variable v_j and

$$x(\phi_j^k) = \begin{cases} C_i & \text{if } \phi_j^k \text{ corresponds to a true literal in clause } C_i \text{ and} \\ v_j & \text{otherwise.} \end{cases}$$

It is easy to see that (x, p) is an economic equilibrium for $\mathcal{M}(B)$.

Conversely, let (x, p) be an economic equilibrium for $\mathcal{M}(B)$. First let us realize that any trading cycle in x contains alternately players c_i and ϕ_j^k, so if all the ϕ-agents endowed with the same house v_j are trading, then they use up all the houses C_i corresponding to clauses containing variable v_j. This implies that if some agent c_i is not trading, then there exists also some agent ϕ_j^k with variable v_j contained in clause C_i who also is not trading. This is however impossible, since if $p(C_i) \geq p(v_j)$ then x does not assign to agent c_i the best house he can afford; and if $p(C_i) < p(v_j)$ then agent ϕ_j^k is not assigned the best house in his budget set – both is in a contradiction with the definition of an economic equilibrium. Further, if any of the agents ϕ_j^k, $k = 1, 2, \ldots, n_j$ is trading, then so are all of them. For, suppose that e.g. $x(\phi_j^1) = C_i$ and $x(\phi_j^2) = v_j$. Then $p(C_i) = p(v_j)$ and agent ϕ_j^2 can afford house C_i, but is assigned a worse house – again a contradiction. So if we set v_j to be true if all the agents ϕ_j^k, $k = 1, 2, \ldots, n_j$ are trading in x and set v_j to be false if none of them is trading, this will be a consistent truth assignment in which each clause contains exactly one true literal. ∎

6 Conclusion

In this paper we studied house allocation markets with duplicate houses. We extended several results known for the classical case and narrowed down the border line between easy and hard cases in the house allocation markets with duplicate houses: the existence of an economic equilibrium can be decided in polynomial time if the preferences of all agents are strict (in contrast with [5]), but it remains NP-complete if all the acceptable house-types are tied in the preference list of each agent. Because of this intractability result, we propose

Definition 1 *A pair (x, p) is an* α-defficient equilibrium *for house allocation market $\mathcal{M}$ with trichotomous preferences, if x is an allocation on A, $p : H \to \mathbb{R}^+$ is a price function and*

$$|\{a \in A; B_a(p) \cap H'(a) \neq \emptyset \ \&\ x(a) = \omega(a)\}| \leq \alpha.$$

Defficiency *of a house allocation market with trichotomous preferences is the minimum such α for which an α-defficient equilibrium exists.*

For a further research we propose the following problem:

PROBLEM **1** *Given a house allocation market $\mathcal{M}$ with trichotomus preferences, compute its defficiency.*

Although we know that the core of each house allocation market is nonempty, still little is known about its structure. As the number of agents that receive in a core allocation their first choice house (let us say that such an agent is *happy* in the allocation) may be different in different core allocations, one may consider the following problem:

PROBLEM **2** *For a given house allocation market $\mathcal{M}$ find a core allocation that maximizes the number of happy agents.*

Notice that a core allocation making all the agents happy exists if and only if the digraph whose vertex set is the agents set and $(i, j) \in E$ if $\omega(j) \in f_A(i)$ admits a cycle cover. If this is not the case, nothing is known for PROBLEM 2.

References

[1] D. Abraham, K. Cechlárová, D. Manlove and K. Mehlhorn, *Pareto optimality in house allocation problems*, Lecure Notes in Comp. Sci. **3341**, Algorithms and Computation, ISAAC 2004, Hong Kong, December 2004, Eds. R. Fleischer, G. Trippen, 3-15 (2004).

[2] D. Abraham, A. Blum, T. Sandholm, *Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges*, EC'07, June 11–15, 2007, San Diego, California.

[3] K. Cechlárová and J. Hajduková, *Computational complexity of stable partitions with $\mathcal{B}-$preferences*, Int. J. Game Theory 31 (2002) 3, 353–364.

[4] K. Cechlárová and A. Romero Medina, *Stability in coalition formation games*, International Journal of Game Theory 29, 2001, 487-494.

[5] S. Fekete, M. Skutella, G. Woeginger, *The complexity of economic equilibria for house allocation markets*, Inform. Process. Lett. 88/5 (2003), 219–223.

[6] M.R. Garey and D.S. Johnson, *Computers and Intractability*, Freeman, San Francisco (1979).

[7] D. Gusfield, R. W. Irving, *The Stable Marriage Problem: Structure and Algorithms*, Foundations of Computing, MIT Press, Cambridge (1989).

[8] G. Chartrand, O.R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill, New York (1993).

[9] K. Iwama, D. Manlove, S. Miyazaki, Y. Morita, *Stable marriage with incomplete lists and ties*, Proceedings of ICALP '99 (J. Wiedermann, P. Van Emde Boas, M. Nielsen Eds.), Lecture Notes in Computer Science Vol 1644, 443–452 (1999).

[10] L. Lovász, M.D. Plummer, *Matching theory*, Akad. Kiadó, Budapest, 1986.

[11] A. E. Roth, A. Postlewaite, *Weak versus strong domination in a market with indivisible goods*, J. Math. Econ. **4**:131–137 (1977).

[12] L. Shapley, H. Scarf, *On cores and indivisibility*, J. Math. Econ. **1**:23–37 (1974).

[13] J. Wako, *A note on the strong core of a market with indivisible goods*, J. Math. Econ. **13**:189–194 (1984).

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