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On uniform correlation structure

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Abstract

We consider the growth curve model of the form $Y = XBZ + e$, $\mathrm{E}\,e = 0$, $\mathrm{var}(e) = \Sigma \otimes I$, with $\Sigma = \sigma^2((1-\rho)I + \rho\mathbf{1}\mathbf{1}')$, where σ^2 and ρ are unknown covariance components (this structure is known as uniform correlation structure). The distribution of the estimator of ρ is difficult to tackle, however, using similar transformation to Fisher Z-transformation the asymptotic normality can be achieved. This asymptotic normality is shown using U-statistics theory.

Keywords: Growth curve model, asymptotic normality, U-statistics

1 INTRODUCTION

Throughout this paper the following notations will be used:

- sign $\otimes$ denotes the Kronecker product of matrices,
- the vec operator makes column vector from any matrix column-wise,
- the vech operator is a generalization of vec operator for symmetric matrices which stacks the on or below diagonal elements of matrix into a column vector,
- $M_X = I - X(X'X)^- X'$ is the matrix of the orthogonal projection onto the orthogonal complement of the column space of matrix X,
- T_p is the matrix which for any symmetric matrix $X_{p\times p}$ transforms $\mathrm{vec}(X)$ into $\mathrm{vech}(X)$, i. e. $\mathrm{vech}(X) = T_p\,\mathrm{vec}(X)$,
- K_p is the commutation matrix of the type $p^2 \times p^2$ (for details see [4]),
- $\mathbf{1}_p$ is the p-variate vector with all the elements equal to 1,
- J_p is the matrix with all the elements equal to 1, i.e. $J_p = \mathbf{1}_p\mathbf{1}_p'$.

Let us consider the common form of the growth curve model

$$Y = XBZ + e\,,\ \mathrm{E}\,e = 0\,,\quad \mathrm{Var}(\mathrm{vec}\,e) = \Sigma \otimes I, \tag{1}$$

where $Y_{n\times p}$ is a matrix of (independent) p-dimensional observations, $X_{n\times m}$ and $Z_{r\times p}$ are known design matrices (X is an ANOVA design matrix and Z is a matrix of regression constants), $B_{m\times r}$ is a matrix of the first order parameters, $e_{n\times p}$ is an error matrix and Σ is a matrix of the unknown second order parameters.

There is no problem estimating Σ when it is completely unknown. Under normality, its uniformly minimum variance unbiased invariant estimator (UMVUE) is

$$\hat{\Sigma}_n = \frac{1}{n - r(X)} Y' M_X Y, \tag{2}$$

Problems arise in situations when the structure is partially known. One of the most common structures is the uniform correlation structure:

$$\Sigma = \sigma^2((1-\rho)I_p + \rho\mathbf{1}_p\mathbf{1}_p').$$

where $\sigma^2 > 0$ and $\rho \in \left\langle -\frac{1}{p-1},\, 1 \right\rangle$ are unknown parameters. Žežula [6] introduced simple estimators of both parameters based on (2):

$$\hat{\sigma}_S^2 = \frac{\mathrm{Tr}(\hat{\Sigma})}{p},$$
$$\hat{\rho}_S = \frac{1}{p-1}\left(\frac{\mathbf{1}'\hat{\Sigma}\mathbf{1}}{\mathrm{Tr}(\hat{\Sigma})} - 1\right).$$

Although both estimators are based on unbiased estimating equations, the estimator $\hat{\rho}_S$ is biased, and the boundaries are $-\frac{1}{p-1} \le \hat{\rho}_S \le 1$. Its distribution is difficult to tackle. However, using the transformation

$$Z_n = \frac{1}{2}\ln\left(\frac{\frac{1}{p-1} + \hat{\rho}_S}{1 - \hat{\rho}_S}\right), \tag{3}$$

asymptotic normality can be achieved.

2 U-STATISTICS FOR VECH(Σ)

Let $Y_1^1, \dots, Y_{n_1}^1, Y_1^2, \dots, Y_{n_2}^2, \dots, Y_1^m, \dots, Y_{n_m}^m$ be the p-variate independent and normally distributed random vectors with $\mathrm{E}(Y_i^j) = \mu^j$ and $\mathrm{Var}(Y_i^j) = \Sigma$, $j = 1, \dots, m$, $i = 1, \dots, n_j$. Let $n = \sum_{j=1}^m n_j$. Then the matrix Y from the model (1) can be written as

$$Y = \left(Y_1^1, \quad \cdots, \quad Y_{n_1}^1, \quad \cdots, \quad Y_1^m, \quad \cdots, \quad Y_{n_m}^m\right)'. \tag{4}$$

According to theory of generalized U-statistics [3] consider the kernel of degree $(2, \dots, 2)$

$$h(Y_1^1, Y_2^1, \dots, Y_1^m, Y_2^m) = \frac{1}{2m}\left(\mathrm{vech}\left[(Y_1^1 - Y_2^1)(Y_1^1 - Y_2^1)'\right] + \cdots \right.$$
$$\left. \cdots + \mathrm{vech}\left[(Y_1^m - Y_2^m)(Y_1^m - Y_2^m)'\right]\right).$$

Then $\mathrm{E}\, h(Y_1^1, Y_2^1, \dots, Y_1^m, Y_2^m) = \mathrm{vech}(\Sigma)$ and the U-statistics for $\mathrm{vech}(\Sigma)$ is

$$U_n = \frac{1}{\binom{n_1}{2}\cdots\binom{n_m}{2}} \sum_{\substack{1 \le \beta_1^1 < \beta_2^1 \le n_1 \\ \cdots \\ 1 \le \beta_1^m < \beta_2^m \le n_m}} h(Y_{\beta_1^1}^1, Y_{\beta_2^1}^1, \dots, Y_{\beta_1^m}^m, Y_{\beta_2^m}^m).$$

After short calculation this statistics can be written in the equivalent form

$$\begin{aligned} U_n = & \frac{1}{m(n_1 - 1)} \sum_{i=1}^{n_1} \mathrm{vech}\left[(Y_i^1 - \bar{Y}^1)(Y_i^1 - \bar{Y}^1)'\right] + \cdots + \\ & + \frac{1}{m(n_m - 1)} \sum_{i=1}^{n_m} \mathrm{vech}\left[(Y_i^m - \bar{Y}^m)(Y_i^m - \bar{Y}^m)'\right], \end{aligned} \tag{5}$$

where $\bar{Y}^j = \frac{1}{n_j}\sum_{i=1}^{n_j} Y_i^j$.

Define for $0 \le j_1 \le 2, \dots, 0 \le j_m \le 2$

$$h_{j_1 \dots j_m}(Y_1^1, \dots, Y_{j_1}^1, \dots, Y_1^m, \dots, Y_{j_m}^m) = \\ = \mathrm{E}\left(h(Y_1^1, Y_2^1, \dots, Y_1^m, Y_2^m) | Y_1^1, \dots, Y_{j_1}^1, \dots, Y_1^m, \dots, Y_{j_m}^m\right),$$

and

$$\Psi_{j_1 \dots j_m} = \mathrm{Var}\left[h_{j_1 \dots j_m}(Y_1^1, \dots, Y_{j_1}^1, \dots, Y_1^m, \dots, Y_{j_m}^m)\right].$$

The following theorem describes the asymptotic distribution of the statistics U_n.

Theorem 1 *Suppose that there exist constants $p_1, \dots, p_m$ in the interval $(0,1)$ such that $n_j/n \to p_j$ for all j and that $\Psi_{2 \dots 2} < \infty$. Then*

$$\sqrt{n}\,(U_n - \mathrm{vech}(\Sigma)) \xrightarrow{\mathcal{L}} N(0, \Gamma),$$

where

$$\Gamma = \frac{4}{p_1} \Psi_{10 \dots 00} + \dots + \frac{4}{p_m} \Psi_{00 \dots 01}$$

Proof. See [3].■

We can compute $\Psi_{0 \dots 1 \dots 0}$ (the index 1 is on the k-th place). First we need

$$h_{0 \dots 1 \dots 0}(Y_1^k) = \mathrm{E}\left(h(Y_1^1, Y_2^1, \dots, Y_1^m, Y_2^m) | Y_1^k\right) = \\ = \frac{1}{2m}\left[\sum_{\substack{j=1 \\ j \ne k}}^{m} \mathrm{vech}\left(\mathrm{E}\left((Y_1^j - Y_2^j)(Y_1^j - Y_2^j)'\right)\right) + \right. \\ \left. + \mathrm{vech}\left(\mathrm{E}\left((Y_1^k - Y_2^k)(Y_1^k - Y_2^k)' | Y_1^k\right)\right)\right] = \\ = \frac{m-1}{m}\,\mathrm{vech}\,(\Sigma) + \frac{1}{2m}\,\mathrm{vech}\left(\mathrm{E}\left(((Y_1^k - \mu^k) - (Y_2^k - \mu^k)) \times \right.\right.$$

$$\left.\left. \times\ ((Y_1^k - \mu^k) - (Y_2^k - \mu^k))' | Y_1^k\right)\right) = \frac{m-1}{m}\,\mathrm{vech}\,(\Sigma) + \\ + \frac{1}{2m}\,\mathrm{vech}\left((Y_1^k - \mu^k)(Y_1^k - \mu^k)'\right) + \frac{1}{2m}\,\mathrm{vech}(\Sigma) = \\ = \left(1 - \frac{1}{2m}\right)\mathrm{vech}(\Sigma) + \frac{1}{2m}\,\mathrm{vech}\left((Y_1^k - \mu^k)(Y_1^k - \mu^k)'\right).$$

Then

$$\Psi_{0 \dots 1 \dots 0} = \mathrm{Var}(h_{0 \dots 1 \dots 0}(Y_1^k)) = \\ = \mathrm{Var}\left(\left(1 - \frac{1}{2m}\right)\mathrm{vech}(\Sigma) + \frac{1}{2m}\,\mathrm{vech}((Y_1^k - \mu^k)(Y_1^k - \mu^k)')\right) = \\ = \frac{1}{4m^2}\, T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma) T_p'.$$

This implies that

$$\Gamma = \frac{1}{m^2} \sum_{i=1}^{m} \frac{1}{p_i} T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'.$$

3 ASYMPTOTIC EQUIVALENCE

The estimator $\hat{\Sigma}_n$ from (2) using (4) can be equivalently written in the form

$$\begin{aligned}\hat{\Sigma}_n =& \frac{1}{n-m} \left(\sum_{i=1}^{n_1} (Y_i^1 - \bar{Y}^1)(Y_i^1 - \bar{Y}^1)' + \cdots + \right. \\ &\left. + \sum_{i=1}^{n_m} (Y_i^m - \bar{Y}^m)(Y_i^m - \bar{Y}^m)' \right).\end{aligned} \tag{6}$$

Both estimators, U_n and $\mathrm{vech}(\hat{\Sigma}_n)$, are unbiased estimators of $\mathrm{vech}(\Sigma)$. The only difference is the constant by which each argument in the sum is multiplied.

Lemma 1 *Let p_j be defined as in the Theorem 1. Then $\sqrt{n}(U_n - \mathrm{vech}(\Sigma))$ and $\sqrt{n}(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma))$ for U_n and $\hat{\Sigma}_n$ defined in (5) and (6) are asymptotically equivalent iff $p_j = 1/m$ for all j.*

Proof. It suffices to show that

$$\mathrm{E}\left(\left\| \sqrt{n}(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma)) - \sqrt{n}(U_n - \mathrm{vech}(\Sigma)) \right\|^2 \right) \to 0.$$

Because both estimators are unbiased, it must hold $\mathrm{E}(\mathrm{vech}(\hat{\Sigma}_n) - U_n) = 0$. Therefore

$$\begin{aligned}
&\mathrm{E}\left(\left\| \sqrt{n}(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma)) - \sqrt{n}(U_n - \mathrm{vech}(\Sigma)) \right\|^2 \right) = \\
&= n\,\mathrm{E}\left(\left\| \mathrm{vech}(\hat{\Sigma}_n) - U_n \right\|^2 \right) = n\,\mathrm{E}\left[(\mathrm{vech}(\hat{\Sigma}_n) - U_n)'(\mathrm{vech}(\hat{\Sigma}_n) - U_n) \right] = \\
&= n\,\mathrm{Tr}\left[\mathrm{E}((\mathrm{vech}(\hat{\Sigma}_n) - U_n)(\mathrm{vech}(\hat{\Sigma}_n) - U_n)') \right] = \\
&= n\,\mathrm{Tr}\left[\mathrm{Var}(\mathrm{vech}(\hat{\Sigma}_n) - U_n) + \mathrm{E}(\mathrm{vech}(\hat{\Sigma}_n) - U_n)\,\mathrm{E}(\mathrm{vech}(\hat{\Sigma}_n) - U_n)' \right] = \\
&= n\,\mathrm{Tr}\left[\mathrm{Var}(\mathrm{vech}(\hat{\Sigma}_n) - U_n) \right] = \\
&= n\,\mathrm{Tr}\left[\mathrm{Var}(\mathrm{vech}(\hat{\Sigma}_n) + \mathrm{Var}(U_n) - 2\,\mathrm{Cov}(\mathrm{vech}(\hat{\Sigma}_n), U_n) \right].
\end{aligned}$$

Now take the sum $\sum_{i=1}^{n_j} (Y_i^j - \bar{Y}^j)(Y_i^j - \bar{Y}^j)'$ for fixed j (such a sum is a part of both $\mathrm{vech}(\hat{\Sigma}_n)$ and U_n). This sum has Wishart distribution $W_p(n_j - 1, \Sigma)$, so (see e.g. [5])

$$\begin{aligned}
&\mathrm{Var}\left(\mathrm{vech}\left(\sum_{i=1}^{n_j} (Y_i^j - \bar{Y}^j)(Y_i^j - \bar{Y}^j)' \right) \right) = \\
&= \mathrm{Var}\left(T_p\,\mathrm{vec}\left(\sum_{i=1}^{n_j} (Y_i^j - \bar{Y}^j)(Y_i^j - \bar{Y}^j)' \right) \right) = \\
&= 2(n_j - 1)T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'.
\end{aligned}$$

Making use of independence of $\sum_{i=1}^{n_j}(Y_i^j - \bar{Y}^j)(Y_i^j - \bar{Y}^j)'$ for all j it is easily seen that

$$\mathrm{Var}(\mathrm{vech}(\hat{\Sigma}_n)) = \frac{2}{n-m}\, T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'\,,$$
$$\mathrm{Var}(U_n) = \frac{2}{m^2}\sum_{j=1}^{m}\frac{1}{n_j - 1}\, T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'\,,$$

and

$$\mathrm{Cov}(\mathrm{vech}(\hat{\Sigma}_n), U_n) = \frac{2}{n-m}\, T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'\,.$$

Then

$$\mathrm{E}\left(\left\|\sqrt{n}(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma)) - \sqrt{n}(U_n - \mathrm{vech}(\Sigma))\right\|^2\right) =$$
$$= \left(\frac{2n}{m^2}\sum_{j=1}^{m}\frac{1}{n_j - 1} - \frac{2n}{n-m}\right)\mathrm{Tr}\left[T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'\right],$$

and

$$\lim_{n\to\infty}\mathrm{E}\left(\left\|\sqrt{n}(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma)) - \sqrt{n}(U_n - \mathrm{vech}(\Sigma))\right\|^2\right) =$$
$$= \left(\frac{2}{m^2}\sum_{j=1}^{m}\frac{1}{p_j} - 2\right)\mathrm{Tr}\left[T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'\right].$$

The last term is equal to 0 iff $p_j = 1/m$ for all j. This concludes the proof. ■

The following corollary follows from Theorem 1 and Lemma 1.

Corollary 1 *Suppose that $n_j/n \to 1/m$ for all j and that $\Psi_{2\ldots 2} < \infty$. Then*

$$\sqrt{n}\left(\mathrm{vech}(\hat{\Sigma}_n) - \mathrm{vech}(\Sigma)\right) \xrightarrow{\mathcal{L}} N(0, \Gamma),$$

where

$$\Gamma = T_p(I_{p^2} + K_p)(\Sigma \otimes \Sigma)T_p'.$$

4 ASYMPTOTIC NORMALITY OF $\hat{\rho}_S$

The statistics Z_n from (3) can be written as $Z_n = \varphi(\mathrm{vech}(\hat{\Sigma}_n))$, where for any symmetric matrix $A_{p\times p}$ is $\varphi(\mathrm{vech}(A)) = f(g(\mathrm{vech}(A)))$,

$$g(\mathrm{vech}(A)) = \frac{2}{p-1}\left[\frac{(\mathrm{vech}(J_p))'\,\mathrm{vech}(A)}{(\mathrm{vech}(I_p))'\,\mathrm{vech}(A)} - 1\right],$$

and

$$f(x) = \frac{1}{2}\ln\left(\frac{\frac{1}{p-1} + x}{1-x}\right).$$

Theorem 2 *Suppose that $n_j/n \to 1/m$ for all j and that $\Psi_{2\ldots2} < \infty$. Then*

$$\sqrt{n}\,(Z_n - \varphi(\mathrm{vech}(\Sigma))) \xrightarrow{\mathcal{L}} N(0,\gamma),$$

where

$$\gamma = d^2\,(\mathrm{vech}(C_p))'\,T_p(I_{p^2}+K_p)(\Sigma\otimes\Sigma)T_p'\,\mathrm{vech}(C_p),$$

and

$$d = \frac{1}{2\sigma^2}\cdot\frac{1}{[1+(p-1)\rho\,](1-\rho)}\,,$$

$$C_p = \frac{2}{p-1}J_p - \left(\frac{2}{p-1}+\rho\right)I_p\,.$$

Proof. Asymptotic normality is direct consequence of the Corollary 1 and the delta method. As for γ we have

$$\gamma = [\nabla\varphi(\mathrm{vech}(\Sigma))]'\,\Gamma\,\nabla\varphi(\mathrm{vech}(\Sigma)),$$

where symbol ∇ denotes the gradient of a function. For a symmetric matrix A is

$$\nabla\varphi(\mathrm{vech}(A)) = \left(\frac{\partial\varphi(\mathrm{vech}(A))}{\partial(\mathrm{vech}(A))'}\right)' = \left(\frac{\partial f(g(\mathrm{vech}(A)))}{\partial(\mathrm{vech}(A))'}\right)' =$$
$$= \left(\frac{\partial f(g(\mathrm{vech}(A)))}{\partial g(\mathrm{vech}(A))}\cdot\frac{\partial g(\mathrm{vech}(A))}{\partial(\mathrm{vech}(A))'}\right)' =$$
$$= \frac{p}{2\,[1-g(\mathrm{vech}(A))]\,[1+(p-1)g(\mathrm{vech}(A))]}\left(\frac{\partial g(\mathrm{vech}(A))}{\partial(\mathrm{vech}(A))'}\right)'.$$

Let us now compute the last term (the fact that $(\mathrm{vech}(I_p))'\,\mathrm{vech}(A) = \mathrm{Tr}(A)$ will be used)

$$\frac{\partial g(\mathrm{vech}(A))}{\partial(\mathrm{vech}(A))'} = \frac{\partial}{\partial(\mathrm{vech}(A))'}\left[\frac{2}{p-1}\left(\frac{(\mathrm{vech}(J_p))'\,\mathrm{vech}(A)}{(\mathrm{vech}(I_p))'\,\mathrm{vech}(A)}-1\right)\right] =$$
$$= \frac{2}{p-1}\cdot\frac{[\mathrm{vech}(I_p)]'\,\mathrm{vech}(A)[\mathrm{vech}(J_p)]' - [\mathrm{vech}(J_p)]'\,\mathrm{vech}(A)[\mathrm{vech}(I_p)]'}{[(\mathrm{vech}(I_p))'\,\mathrm{vech}(A)]^2} =$$
$$= \frac{2}{(p-1)\,\mathrm{Tr}(A)}\big((\mathrm{vech}(J_p))' - (\mathrm{vech}(I_p))'\big) - \frac{g(\mathrm{vech}(A))}{\mathrm{Tr}(A)}(\mathrm{vech}(I_p))' =$$
$$= \frac{1}{\mathrm{Tr}(A)}(\mathrm{vech}(D_p(A)))',$$

where

$$D_p(A) = \frac{2}{p-1}J_p - \left(\frac{2}{p-1}+g(\mathrm{vech}(A))\right)I_p\,.$$

Then

$$\nabla\varphi(\mathrm{vech}(A)) = \frac{1}{2\,\mathrm{Tr}(A)[1-g(\mathrm{vech}(A))][1+(p-1)g(\mathrm{vech}(A))]}\,\mathrm{vech}(D_p(A))\,.$$

Because $g(\mathrm{vech}(\Sigma)) = \rho$, $\mathrm{Tr}(\Sigma) = p\sigma^2$, and $D_p(\Sigma) = C_p$, we have

$$\nabla\varphi(\mathrm{vech}(\Sigma)) = \frac{p}{2\sigma^2[1+(p-1)\rho\,](1-\rho)}\,\mathrm{vech}(C_p)\,.$$

Then the result for γ follows. ■

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