# On Solutions of Third Order Nonlinear Differential Equations

Ivan Mojsej and Ján Ohriska

**Abstract.** The aim of our paper is to study oscillatory and asymptotic properties of solutions of nonlinear differential equations of third order with deviating argument. In particular, we prove a comparison theorem for properties A and B as well as a comparison result on property A between nonlinear equations without and with deviating argument. Our assumptions on nonlinearity $f$ are related with its behavior only in a neighbourhood of zero and/or of infinity.

## 1 Introduction

We consider the third-order nonlinear differential equations with deviating argument of the form:

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(h(t))) = 0\,, \quad t \geq 0 \tag{N, h}$$

and

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)f(z(h(t))) = 0\,, \quad t \geq 0 \tag{$\mathrm{N}^{\mathcal{A}}$, h}$$

where

$$r, p, q, h \in C(\langle 0, \infty), \mathbb{R}),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } \langle 0, \infty) \tag{H1}$$

$$f \in C(\mathbb{R}, \mathbb{R}), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0 \tag{H2}$$

$$\int_0^\infty r(t)\,dt = \int_0^\infty p(t)\,dt = \infty \tag{H3}$$

$$\lim_{t\to\infty} h(t) = \infty \tag{H4}$$

*K*eywords and phrases: nonlinear differential equation of third order with deviating argument, asymptotic behavior of nonoscillatory solutions, properties A and B, comparison theorems, quasiderivatives.

This is a preprint of an article published in Central European Journal of Mathematics 4(2006), No.1, 46–63, http://www.cesj.com/.

Without mentioning them again, we shall assume the validity of conditions (H1)–(H4) throughout the paper.

The notation $(N^{\mathcal{A}}, h)$ is suggested by the fact that for linear equation without deviating arguments, i.e., for the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0\,, \tag{L}$$

the *adjoint equation* is

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)z(t) = 0\,. \tag{L$^{\mathcal{A}}$}$$

If $x$ is a solution of $(N, h)$, then the functions

$$x^{[0]} = x,\ x^{[1]} = \frac{1}{r}x',\ x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)',\ x^{[3]} = \frac{1}{q}\left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \frac{1}{q}\left(x^{[2]}\right)'$$

are called the *quasiderivatives* of $x$. For $(N^{\mathcal{A}}, h)$ we can proceed in a similar way. The linear case of equations $(N, h)$, $(N^{\mathcal{A}}, h)$ denote by $(L, h)$, $(L^{\mathcal{A}}, h)$, respectively. For simplicity, when $h(t) \equiv t$, we will denote $(N, h)$ and $(N^{\mathcal{A}}, h)$ with $(N)$ and $(N^{\mathcal{A}})$, respectively. In addition to (H1)–(H4), we sometimes assume

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0 \tag{H5}$$

or

$$\limsup_{u\to 0} \frac{f(u)}{u} < \infty\,. \tag{H6}$$

By a *solution* of an equation of the form $(N, h)$ $[(N^{\mathcal{A}}, h)]$ we mean a function $w \in C^1(\langle 0, \infty), \mathbb{R})$ such that $w^{[1]}(t)$, $w^{[2]}(t) \in C^1(\langle 0, \infty), \mathbb{R})$ satisfying equation $(N, h)$ $[(N^{\mathcal{A}}, h)]$ for all $t \geq 0$. Any solution of $(N, h)$ or $(N^{\mathcal{A}}, h)$ is said to be *proper* if it is defined on the interval $\langle 0, \infty)$ and is nontrivial in any neighborhood of infinity. A proper solution is said to be *oscillatory* (*nonoscillatory*) if it has (does not have) a sequence of zeros converging to $\infty$. In addition, $(N, h)$ $[(N^{\mathcal{A}}, h)]$ is called *oscillatory* if it has at least one nontrivial oscillatory solution and *nonoscillatory* if all its solutions are nonoscillatory. The study of asymptotic behavior of solutions, in the ordinary case as well as in the case with deviating argument, is often connected by introducing the concepts of *equation with property A* and *equation with property B*. Equation $(N, h)$ is said to have *property A* if any proper solution $x$ of $(N, h)$ is either oscillatory or satisfies

$$|x^{[i]}(t)| \downarrow 0 \ \mathit{as}\ t \to \infty,\ \ i = 0, 1, 2,$$

and equation $(N^{\mathcal{A}}, h)$ is said to have *property B* if any proper solution $z$ of $(N^{\mathcal{A}}, h)$ is either oscillatory or satisfies

$$|z^{[i]}(t)| \uparrow \infty \text{ as } t \to \infty \ i = 0, 1, 2.$$

The notations $u(t) \downarrow 0$ and $u(t) \uparrow \infty$ mean that function $u$ monotonically decreases to zero as $t \to \infty$ or monotonically increases to infinity as $t \to \infty$, respectively.

Denote by $\mathcal{N}[(N, h)]$, $\mathcal{N}[(N^{\mathcal{A}}, h)]$, $\mathcal{N}[(L, h)]$, $\mathcal{N}[(L^{\mathcal{A}}, h)]$ the sets of all proper solutions of $(N, h)$, $(N^{\mathcal{A}}, h)$, $(L, h)$ ,$(L^{\mathcal{A}}, h)$, respectively. From slight modification of the well-known lemma of Kiguradze (see, e.g., [6]) it follows that nonoscillatory solutions $x$ of $(N, h)$ and $(L, h)$ can be divided into the following two classes in the same way as in [3]:

$$\mathcal{N}_0 = \{x \text{ solution}, \exists\, T_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T_x\}$$

$$\mathcal{N}_2 = \{x \text{ solution}, \exists\, T_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T_x\}$$

Similarly nonoscillatory solutions $z$ of $(N^{\mathcal{A}}, h)$ and $(L^{\mathcal{A}}, h)$ can be divided into the following two classes:

$$\mathcal{M}_1 = \{z \text{ solution}, \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) < 0 \text{ for } t \geq T_z\}$$

$$\mathcal{M}_3 = \{z \text{ solution}, \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) > 0 \text{ for } t \geq T_z\}$$

It is clear that $(N, h)$ $[(L, h)]$ has property A if and only if all nonoscillatory solutions $x$ of $(N, h)$ $[(L, h)]$ belong to the class $\mathcal{N}_0$ and $\lim_{t\to\infty} x^{[i]}(t) = 0,\ i = 0, 1, 2$. Similarly $(N^{\mathcal{A}}, h)$ $[(L^{\mathcal{A}}, h)]$ has property B if and only if all nonoscillatory solutions $z$ of $(N^{\mathcal{A}}, h)$ $[(L^{\mathcal{A}}, h)]$ belong to the class $\mathcal{M}_3$ and $\lim_{t\to\infty} |z^{[i]}(t)| = \infty,\ i = 0, 1, 2$. In addition, if $x \in \mathcal{N}_0$, then its quasiderivatives satisfy the inequality $x^{[i]}(t)x^{[i+1]}(t) < 0$ for $i = 0, 1, 2$, for all sufficiently large $t$ and in the literature they are called *Kneser solutions.* If $z \in \mathcal{M}_3$, then its quasiderivatives satisfy the inequality $z^{[i]}(t)z^{[i+1]}(t) > 0$ for $i = 0, 1, 2$, for all sufficiently large $t$ and are called *strongly monotone solutions.*

The oscillatory and asymptotic properties of solutions of differential equations of the third order with quasiderivatives (linear and nonlinear, and with delay) have been largely investigated in [1–5].

The aim of this paper is to continue in study of such equations with deviating argument and with advanced argument. Our research is based on a study of asymptotic behavior of nonoscillatory solutions of $(N, h)$ and $(N^{\mathcal{A}}, h)$, on a linearization device as well as on a comparison result between equations with different deviating arguments. Such a comparison criterion, in the form here used, is quoted in section 2. The paper is organized as follows: Section 2 summarizes results which will be useful in the sequel. In the section 3 we give a comparison theorem for properties A and B, which is more suitable for application than others existing in the literature. This theorem extends Theorem 4 in [5]. As consequence we obtain sufficient conditions ensuring property A for $(N, h)$ and property B for $(N^{\mathcal{A}}, h)$ as well as a comparison result on property A between nonlinear equations without and with deviating argument. Some results on the asymptotic behavior of nonoscillatory solutions of $(N, h)$ $[(N^{\mathcal{A}}, h)]$ which belong to the class $\mathcal{N}_0$ $[\mathcal{M}_3]$ will be considered in the section 4. Section 5 gives new integral criteria in order for $(N, h)$ $[(N^{\mathcal{A}}, h)]$ to have property A [B].

We point out that our assumptions on nonlinearity $f$ are related with its behavior only in a neighbourhood of zero and/or of infinity. No monotonicity conditions are required as well as no assumptions involving the behavior of $f$ in the whole $\mathbb{R}$ are supposed.

# 2 Preliminary results

We introduce the following notation:

$$I(u_i) = \int_0^\infty u_i(t)\, dt, \qquad I(u_i, u_j) = \int_0^\infty u_i(t) \int_0^t u_j(s)\, ds\, dt, \quad i, j = 1, 2$$

$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t) \int_0^t u_j(s) \int_0^s u_k(b)\, db\, ds\, dt, \quad i, j, k = 1, 2, 3,$$

where $u_i$, $i = 1, 2, 3$ are continuous positive functions on $\langle 0, \infty)$.
For simplicity, sometimes we will write $u(\infty)$ instead of $lim_{t\to\infty} u(t)$.

In the recent papers [1, 2, 5] authors have studied relationships among properties A and B and both the oscillation and the asymptotic behavior of nonoscillatory solutions for linear equations without deviating argument. We recall some of these results which will be useful in the sequel.

**Theorem 2.1** *([1], Theorem 2.2) The following assertions are equivalent:*

*(i)* *(L) has property A.*

*(i')* *$(L^{\mathcal{A}})$ has property B.*

*(ii)* *(L) is oscillatory and $I(q, p, r) = \infty$.*

*(ii')* *$(L^{\mathcal{A}})$ is oscillatory and $I(q, p, r) = \infty$.*

**Lemma 2.1** *([1], Lemma 2.1) If there exists a Kneser solution $x$ of equation (L) such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$, then $I(q, p, r) = \infty$.*

**Remark 1.** Theorem 2.1 and Lemma 2.1 hold even if $I(r) < \infty$ or $I(p) < \infty$.

The following comparison theorem and a result on Kneser solutions we will use in our consideration.

**Theorem 2.2** *([2], Theorem 1) Let the following condition be satisfied:*

$$\text{either} \quad \limsup_{t\to\infty} \int_0^t p(s)\,ds \int_t^\infty q(s)\,\frac{\int_0^s r(u)\int_0^u p(v)\,dv\,du}{\int_0^s p(u)\,du}\,ds = \infty \qquad \text{or} \quad I(q,r) = \infty \tag{H$^*$}$$

*If for some $K > 0$ the equation*

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + Kq(t)x(t) = 0 \tag{L$_\mathrm{K}$}$$

*has property A, then the equation*

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + kq(t)x(t) = 0 \tag{L$_\mathrm{k}$}$$

*has property A for every $k > 0$.*

**Proposition 2.1** *([2], Proposition 6) Every Kneser solution of (L) tends to zero for $t \to \infty$ if and only if $I(q,p,r) = \infty$.*

**Remark 2.** From Proposition 2.1 it follows the following statements: If $(L)$ is oscillatory and does not have property A, then $(L)$ has Kneser solution tending to nonzero limit and $I(q,p,r) < \infty$.

To extend known results to differential equations with deviating argument we will use the following comparison criterion. It is a particular case of a more general theorem which is stated in [6] for functional differential equations of higher order.

**Theorem 2.3** *([6], Theorem 1) Consider the differential equations* $(i = 1, 2)$

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q_i(t)x(h_i(t)) = 0 \tag{L, h$_\mathrm{i}$)$_\mathrm{i}$ ($}$$

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q_i(t)z(h_i(t)) = 0 \tag{L$^\mathcal{A}$, h$_\mathrm{i}$)$_\mathrm{i}$ ($}$$

*where $q_i, h_i \in C(\langle 0,\infty), \mathbb{R})$, $q_i(t) > 0$, $\lim\limits_{t\to\infty} h_i(t) = \infty$ and*

$$h_1(t) \le h_2(t), \quad q_1(t) \le q_2(t), \quad \textit{for } t > t_0 \ge 0\,.$$

*If* $(L, h_1)_1$ *has property A then* $(L, h_2)_2$ *has property A.*
*If* $(L^{\mathcal{A}}, h_1)_1$ *has property B then* $(L^{\mathcal{A}}, h_2)_2$ *has property B.*

Independently on properties A and B, it is easy to show the following:

**Lemma 2.2** *([3], Lemma 1.1) It holds:*

*i) Any solution* $x$ *of (L,h) [(N,h)] from* $\mathcal{N}_0$ *satisfies* $\lim\limits_{t\to\infty} x^{[i]}(t) = 0,\ i = 1, 2.$

*ii) Any solution* $z$ *of* $(L^{\mathcal{A}}, h)$ $[(N^{\mathcal{A}}, h)]$ *from* $\mathcal{M}_3$ *satisfies* $\lim\limits_{t\to\infty} |z^{[i]}(t)| = \infty,\ i = 0, 1.$

# 3 Comparison results

We begin our consideration with the following comparison theorem.

**Theorem 3.1** *Assume* $(H5)$, $(H^*)$ *and* $h(t) \geq t$. *If* $(L_K)$ *has property A for some* $K > 0$, *then* $(N, h)$ *has property A and* $(N^{\mathcal{A}}, h)$ *has property B.*

**Proof:** *a)* Let us prove that $(N, h)$ has property A.
Let $x$ be a proper nonoscillatory solution of $(N, h)$. We may assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. The case $x(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $x \in \mathcal{N}_0 \cup \mathcal{N}_2$. Now we assume that $(N, h)$ does not have property A. By Lemma 2.2 there are two possibilities:
*I.* $x \in \mathcal{N}_2$,
*II.* $x \in \mathcal{N}_0$ such that $\lim\limits_{t\to\infty} x(t) = l > 0$.
*Case I.* Let $x \in \mathcal{N}_2$. We consider linearized differential equation with deviating argument

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F_1(t)w(h(t)) = 0\,, \qquad (\mathrm{L_{F_1}}, \mathrm{h})$$

where $F_1(t) = \dfrac{f(x(h(t)))}{x(h(t))}$. Then $w \equiv x$ is an its nonoscillatory solution. In view of the fact $x \in \mathcal{N}_2$ we have that $(L_{F_1}, h)$ does not have property A.
Because $x^{[1]}$ is an eventually positive increasing function, there exists $T \geq 0$ such that $x^{[1]}(t) \geq x^{[1]}(T)$ for all $t \geq T$. Integrating this inequality in $(T, t)$ we get

$$x(t) \geq x(T) + x^{[1]}(T) \int_T^t r(s)\, ds.$$

As $t \to \infty$ we get that function $x(t)$ is unbounded.
In view of the facts $x(\infty) = \infty$ and assumption $(H5)$, there exist positive constant $k_1$ and $T_1 \geq 0$ such that $F_1(t) > k_1$ for all $t \geq T_1$. Hence by Theorem 2.3 for

$q_1(t) = q(t)k_1$, $q_2(t) = q(t)F_1(t)$, $h_1(t) = t$, $h_2(t) = h(t)$ we obtain that linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + k_1 q(t)w(t) = 0 \tag{$\mathrm{L_{k_1}}$}$$

does not have property A. But on other hand, by Theorem 2.2 equation $(L_k)$ has property A for all $k > 0$, which is a contradiction.
*Case II.* Let $x \in \mathcal{N}_0$ and $\lim_{t\to\infty} x(t) = l > 0$. Hence, there exists positive constant c such that

$$x(t) \geq c > 0 \qquad \text{for t sufficiently large.} \tag{1}$$

We consider linearized differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F_2(t)w(t) = 0\,, \tag{$\mathrm{L_{F_2}}$}$$

where $F_2(t) = \dfrac{f(x(h(t)))}{x(t)}$. Because $w \equiv x$ is an its nonoscillatory solution such that $x \in \mathcal{N}_0$ and $x(\infty) > 0$, $(L_{F_2})$ does not have property A. In view of continuity of function $f$ and (1), there exist positive constant $k_2$ and $T_2 \geq 0$ such that $F_2(t) > k_2$ for all $t \geq T_2$. Hence by Theorem 2.3 for $q_1(t) = q(t)k_2$, $q_2(t) = q(t)F_2(t)$, $h_1(t) = h_2(t) = t$ we obtain that linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + kq(t)w(t) = 0 \tag{$\mathrm{L_{k_2}}$}$$

does not have property A. But on other hand, by Theorem 2.2 equation $(L_k)$ has property A for all $k > 0$, which is a contradiction.

*b)* Let us prove that $(N^{\mathcal{A}}, h)$ has property B.
Let $z$ be a proper nonoscillatory solution of $(N^{\mathcal{A}}, h)$. We may assume that there exists $T \geq 0$ such that $z(t) > 0$ for all $t \geq T$. The case $z(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $z \in \mathcal{M}_1 \cup \mathcal{M}_3$. Now we assume that $(N^{\mathcal{A}}, h)$ does not have property B. By Lemma 2.2 there are two possibilities:
*I.* $z \in \mathcal{M}_3$ such that $\lim_{t\to\infty} z^{[2]}(t) \neq \infty$,
*II.* $z \in \mathcal{M}_1$.
*Case I.* We consider, for sufficiently large $t$, linearized differential equation with deviating argument

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)F_3(t)w(h(t)) = 0\,, \tag{$\mathrm{L^{\mathcal{A}}_{F_3}}$, h}$$

where $F_3(t) = \dfrac{f(z(h(t)))}{z(h(t))}$. Because $w \equiv z$ is an its nonoscillatory solution such that $z \in \mathcal{M}_3$ and $\lim\limits_{t\to\infty} z^{[2]}(t) \neq \infty$, $(L^{\mathcal{A}}_{F_3}, h)$ does not have property B. Taking into account that $z(\infty) = \infty$ and assumption $(H5)$, there exist positive constant $k_3$ and $T_3 \geq 0$ such that $F_3(t) > k_3$ for all $t \geq T_3$. Hence by Theorem 2.3 for $q_1(t) = q(t)k_3$, $q_2(t) = q(t)F_3(t)$, $h_1(t) = t$, $h_2(t) = h(t)$ we obtain that linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)k_3w(t) = 0 \qquad (\mathrm{L}^{\mathcal{A}}_{\mathrm{k}_3})$$

does not have property B. On the other hand, by Theorem 2.2 equation $(L_k)$ has property A for all $k > 0$ and thus by Theorem 2.1 equation $(L^{\mathcal{A}}_k)$ has property B for all $k > 0$, which is a contradiction.

*Case II.* Let $x \in \mathcal{M}_1$. Because $z$ is an eventually positive increasing function, there are two possibilities: $z(\infty) = \infty$ or $z(\infty) < \infty$.

If $z(\infty) = \infty$, the proof proceeds as in the *case I* and hence omitted.

Now, we suppose that $z(\infty) < \infty$ and consider linearized differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)F_4(t)w(t) = 0\,, \qquad (\mathrm{L}^{\mathcal{A}}_{\mathrm{F}_4})$$

where $F_4(t) = \dfrac{f(z(h(t)))}{z(t)}$. Because $w \equiv z$ is an its nonoscillatory solution such that $z \in \mathcal{M}_1$, $(L^{\mathcal{A}}_{F_4})$ does not have property B. In view of continuity of function $f$ and $z(\infty) < \infty$, there exist positive constant $k_4$ and $T_4 \geq 0$ such that $F_4(t) > k_4$ for all $t \geq T_4$. Hence by Theorem 2.3 for $q_1(t) = q(t)k_4$, $q_2(t) = q(t)F_4(t)$, $h_1(t) = h_2(t) = t$ we obtain that linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)k_4(t)w(t) = 0 \qquad (\mathrm{L}^{\mathcal{A}}_{\mathrm{k}_4})$$

does not have property B. On the other hand, by Theorem 2.2 equation $(L_k)$ has property A for all $k > 0$ and thus by Theorem 2.1 equation $(L^{\mathcal{A}}_k)$ has property B for all $k > 0$, which is a contradiction. The proof is complete.

■

**Remark 3.** Unlike other comparison results (see e.g., Theorem 1 in [6]), Theorem 3.1 does not require neither monotonicity assumptions of the nonlinearity in the whole $\mathbb{R}$ nor the domination of the nonlinearity $|f(u)|$ over the linear term $|u|$ in the whole $\mathbb{R}$. Theorem 3.1 will be valid even in the case of the substitution of assumptions $(H^*)$ and $(L_K)$ has property A for some $K > 0$ for the assumption

$(L_k)$ has property A for all $k > 0$. And thus the identity $h(t) \equiv t$ in Theorem 3.1 both gives Theorem 4 in [5] and extends Theorem 3 in [2].

Theorem 3.1 together with integral criteria ensuring property A for $(L_K)$ gives the following result.

**Corollary 3.1** *Let $h(t) \geq t$, $(H5)$ hold and one of the following conditions be satisfied:*

*(i)* $I(q,r) = I(q,p) = \infty$,

*(ii)* $I(q) = \infty$,

*(iii)* $I(q,p) < \infty$, $\displaystyle\int_0^\infty r(t) \left(\int_t^\infty q(s)\,ds\right)\left(\int_t^\infty p(s)\int_s^\infty q(a)\,da\,ds\right) dt = \infty.$

*Then $(N,h)$ has property A and $(N^{\mathcal{A}},h)$ has property B.*

**Proof:** From Theorems 4 and 5 in [4] and Proposition 1 in [4] it follows that $(L_k)$ has property A for all $k > 0$. Now, we get the assertion from Theorem 3.1 (see Remark 3). The proof is finished.

■

The following result also holds:

**Corollary 3.2** *Assume $(H5)$ and $h(t) \geq t$. If every nonoscillatory solution of $(L_k)$ is a Kneser solution for any $k > 0$ and $I(q,p,r) = \infty$, then $(N,h)$ has property A and $(N^{\mathcal{A}},h)$ has property B.*

**Proof:** First let us remark that if $I(q,p,r) = \infty$, then $I(kq,p,r) = \infty$ for any positive constant $k$. By Proposition 2.1 and Lemma 2.2, every Kneser solution $x$ of $(L_k)$ satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$, $i = 0, 1, 2$. Taking into account that every nonoscillatory solution of $(L_k)$ is a Kneser one, we get that $(L_k)$ has property A for any $k > 0$. Now, Theorem 3.1 yields the assertion (see Remark 3). This completes the proof.

■

Theorem 3.1 yields the following comparison result between nonlinear equations without and with deviating argument.

**Theorem 3.2** *Assume $(H5)$, $(H6)$, $h(t) \geq t$ and $(L_k)$ is oscillatory for all $k > 0$. If $(N)$ has property A, then $(N,h)$ has property A and $(N^{\mathcal{A}},h)$ has property B.*

**Proof:** To prove this assertion we will show that *a)* if $(N)$ has property A, then $(L_k)$ has property A for all $k > 0$ and *b)* if $(L_k)$ has property A for all $k > 0$, then $(N,h)$ has property A and $(N^{\mathcal{A}},h)$ has property B.

*a)* Assumption $(H6)$ implies $\int_0^1 \frac{1}{f(u)}\, du = \infty$. Hence by Proposition 1.1 in [1], there exists at least one Kneser solution $x$ of $(N)$. Because $(N)$ has property A, $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$. Let $F$ is the function given by

$$F(t) = \frac{f(x(t))}{x(t)}$$

and we consider for t sufficiently large linearized differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F(t)w(t) = 0. \tag{L$_\mathrm{F}$}$$

Since $w \equiv x$ is a Kneser solution of $(L_F)$ such that $w^{[i]}(\infty) = 0$, $i = 0, 1, 2$, Lemma 2.1 implies that

$$I(qF, p, r) = \infty. \tag{2}$$

Because $(H6)$ holds, there exists a positive constant $M$ such that

$$0 < F(t) = \frac{f(x(t))}{x(t)} < M \qquad \text{for all sufficiently large } t. \tag{3}$$

Because (3) implies $I(qF, p, r) \le M\, I(q, p, r)$, from (2) we have that $I(q, p, r) = \infty$. Now we assume that there exists a positive constant $k_0$ such that $(L_{k_0})$ does not have property A. Because $(L_{k_0})$ is oscillatory for all $k > 0$, from Theorem 2.1 we obtain that

$$k_0\, I(q, p, r) = I(k_0 q, p, r) < \infty\,,$$

which is a contradiction. Now part *a)* is proved.

*b)* Let $(L_k)$ has property A for all $k > 0$. From Theorem 3.1 we immediately get that $(N, h)$ has property A and $(N^{\mathcal{A}}, h)$ has property B (see Remark 3). Now part *b)* is proved. The proof is complete.

■

**Remark 4.** If $h(t) \equiv t$ in Theorem 3.2, we obtain known result concerning property A for $(N)$ and property B for $(N^{\mathcal{A}})$, see Theorem 4.1 in [1].

# 4 Properties of Kneser and strongly monotone solutions

The following results establish some asymptotic properties for Kneser and strongly monotone solutions of $(N, h)$ and $(N^{\mathcal{A}}, h)$, respectively.

**Theorem 4.1** *If $I(q, p, r) = \infty$, then every Kneser solution $x$ of equation $(N, h)$ satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$.*

**Proof:** By Lemma 2.2 every Kneser solution $x$ of $(N,h)$ satisfies $x^{[i]}(\infty)=0$ for i=1, 2. Suppose that there exists an eventually positive Kneser solution $x$ of $(N,h)$ such that

$$\lim_{t\to\infty} x(t) = c > 0. \tag{4}$$

We consider linearized differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F_2(t)w(t) = 0\,, \tag{$\mathrm{L_{F_2}}$}$$

where $F_2(t) = \dfrac{f(x(h(t)))}{x(t)}$. Because $w \equiv x$ is an its nonoscillatory solution, $(L_{F_2})$ has a Kneser solution such that (4) holds. From Proposition 2.1, we obtain

$$I(qF_2,p,r) < \infty. \tag{5}$$

Since $x$ is an eventually positive decreasing function, taking into account (4) and continuity of function $f$ there exists positive constant $k_1$ such that

$$F_2(t) > k_1 > 0 \qquad \text{for all sufficiently large } t.$$

Hence, by (5), we have that

$$k_1 I(q,p,r) < I(qF_2,p,r) < \infty\,,$$

which is a contradiction. The case $x(t)<0$ for all $t \ge T^*$ may be proved by using similar arguments. The proof is now complete.

■

**Theorem 4.2** *Assume $(H5)$, $h(t) \ge t$ and $(L_k)$ is oscillatory for all $k > 0$. If $I(q,p,r)=\infty$, then every strongly monotone solution $z$ of $(N^{\mathcal{A}},h)$ satisfies $\lim_{t\to\infty}|z^{[i]}(t)| = \infty$ for $i = 0,\ 1,\ 2$.*

**Proof:** By Lemma 2.2 every strongly monotone solution $z$ of $(N^{\mathcal{A}},h)$ satisfies $|z^{[i]}(\infty)|=\infty$ for i=0, 1. Suppose that there exists an eventually positive strongly monotone solution $z$ of $(N^{\mathcal{A}},h)$ such that $\lim_{t\to\infty} z^{[2]}(t) < \infty$. Hence, $(N^{\mathcal{A}},h)$ does not have property B.
We consider, for sufficiently large $t$, linearized differential equation with deviating argument

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)F_3(t)w(h(t)) = 0\,, \tag{$\mathrm{L^{\mathcal{A}}_{F_3}}$, h}$$

where $F_3(t) = \dfrac{f(z(h(t)))}{z(h(t))}$. Because $w \equiv z$ is an its nonoscillatory solution, $(L^{\mathcal{A}}_{F_3},h)$ does not have property B, too. Taking into account that $z(\infty)=\infty$ and assumption $(H5)$, there exists positive constant $k_2$ such that $F_3(t) > k_2 > 0$ for all sufficiently large $t$. Hence by Theorem 2.3 for $q_1(t) = q(t)k_2$, $q_2(t) = q(t)F_3(t)$, $h_1(t) = t$, $h_2(t) = h(t)$ we obtain that linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)k_2w(t) = 0 \qquad (\mathrm{L}^{\mathcal{A}}_{\mathrm{k}_2})$$

does not have property B. Since $(L_{k_2})$ is oscillatory and does not have property B, by Theorem 2.1, we have that

$$k_2I(q,p,r) = I(qk_2,p,r) < \infty\,,$$

which is a contradiction. The case $z(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. The proof is now finished.

■

**Theorem 4.3** *Assume (H6), $h(t) \geq t$. If there exists a Kneser solution $x$ of $(N,h)$ such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0,\, 1,\, 2$, then $I(q,p,r) = \infty$.*

**Proof:** Suppose that $I(q,p,r) < \infty$. Let $x$ be an eventually positive Kneser solution of $(N,h)$, thus there exists $T \geq 0$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T$, and satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ for i=$0,1,2$. The case $x(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. Let $T_1 > T$ be such that $h(t) > T$ for all $t \geq T_1$. Integrating $(N,h)$ three times in $(t,\infty)$ we obtain

$$x(t) = \int_t^\infty r(s)\int_s^\infty p(u)\int_u^\infty q(b)f(x(h(b)))\,db\,du\,ds\,. \qquad (6)$$

In view of the continuity of function $f$ and assumption of this assertion, there exists positive constant $k_3$ such that

$$0 < \frac{f(x(h(t)))}{x(h(t))} < k_3 \qquad \text{for all sufficiently large } t. \qquad (7)$$

Taking into account that $x$ is an eventually positive decreasing function and (7) holds, so from (6) we have

$$x(t) < k_3\int_t^\infty r(s)\int_s^\infty p(u)\int_u^\infty q(b)x(h(b))\,db\,du\,ds \leq$$
$$\leq k_3\,x(h(t))\int_t^\infty r(s)\int_s^\infty p(u)\int_u^\infty q(b)\,db\,du\,ds\,.$$

Thus

$$0 < \frac{1}{k_3} \leq \frac{x(t)}{k_3\,x(h(t))} < \int_t^\infty r(s)\int_s^\infty p(u)\int_u^\infty q(b)\,db\,du\,ds\,,$$

by interchanging the order of integration, we get a contradiction. This completes the proof.

■

**Corollary 4.1** *Assume* $(H6)$, $h(t) \geq t$. *If there exists a Kneser solution* $x_*$ *of* $(N, h)$ *such that* $\lim_{t\to\infty} x_*^{[i]}(t) = 0$ *for* $i = 0, 1, 2$, *then every Kneser solution* $x$ *of* $(N, h)$ *satisfies* $\lim_{t\to\infty} x^{[i]}(t) = 0$ *for* $i = 0, 1, 2$.

**Proof:** The assertion immediately follows from Theorem 4.1 and Theorem 4.3.

■

The following two examples illustrate the meaning of Theorems 4.1–4.3.
**Example 1.** We consider the differential equation

$$\left(\frac{1}{t^2}\left(\frac{1}{t}x'(t)\right)'\right)' + \frac{18}{t+t^5}\left[x^3(t^2) + x(t^2)\right] = 0, \qquad t \geq 1. \tag{8}$$

This is the equation of the form $(N, h)$, where $r(t) = t$, $p(t) = t^2$, $q(t) = \dfrac{18}{t+t^5}$, $h(t) = t^2$ and $f(u) = u^3 + u$. The assumption of Theorem 4.1 holds and hence we know that every Kneser solution $x$ of equation (8) satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$. One such solution is the function $x(t) = \frac{1}{t}$.

□

**Remark 5.** The differential equation (8) in the Example 1 satisfies also assumptions of Theorem 4.3 and so we know that existence of Kneser solution $x$ of equation (8) such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$ (it is the function $x(t) = \frac{1}{t}$) implies $I(q, p, r) = \infty$.

**Example 2.** We consider the differential equation

$$z'''(t) - \frac{e^t}{arctg\, e^{t+1} + e^{t+1}} \cdot [arctg\, z(t+1) + z(t+1)] = 0, \qquad t \geq 0. \tag{9}$$

This is the equation of the form $(N^{\mathcal{A}}, h)$, where $r(t) = p(t) = 1$, $h(t) = t + 1$, $q(t) = e^t/(arctg\, e^{t+1} + e^{t+1})$ and $f(u) = arctg\, u + u$. It is easy to verify that assumptions of Theorem 4.2 are fulfilled and so every strongly monotone solution $z$ of equation (9) satisfies $\lim_{t\to\infty} |z^{[i]}(t)| = \infty$ for $i = 0, 1, 2$. One such solution is the function $z(t) = e^t$.

□

# 5 Sufficient conditions for properties A and B

In the proof of Theorem 5.1 and Theorem 5.2 we will need the next two lemmas. They deal with some asymptotic properties of solutions of $(N, h)$ $[(N^{\mathcal{A}}, h)]$ which belong to the class $\mathcal{N}_2$ $[\mathcal{M}_1]$. Note that these results are evident when $p(t) \equiv r(t) \equiv 1$.

**Lemma 5.1** *Let* $x$ *be a solution of* $(N, h)$ *in the class* $\mathcal{N}_2$. *Then the following assertions hold:*

*a)* $\lim_{t\to\infty} |x(t)| = \infty$,

*b) If* $\lim_{t\to\infty} x^{[2]}(t) \neq 0$, *then* $\lim_{t\to\infty} |x^{[1]}(t)| = \infty$.

**Proof:** Because $x$ is nonoscillatory solution of $(N, h)$ in the class $\mathcal{N}_2$, there exists $T \geq 0$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments.

*a)* Because $x^{[1]}$ is an eventually positive increasing function, we have $x^{[1]}(t) \geq x^{[1]}(T)$ for all $t \geq T$. By integrating we obtain

$$x(t) \geq x(T) + x^{[1]}(T) \int_T^t r(s)\, ds.$$

As $t \to \infty$ we get the first assertion.

*b)* Since $x^{[2]}(t) = \dfrac{1}{p(t)} \left(x^{[1]}(t)\right)'$, integrating in $(T, t)$ we obtain

$$x^{[1]}(t) = x^{[1]}(T) + \int_T^t x^{[2]}(s) p(s)\, ds.$$

Taking into account that $x^{[2]}(t)$ is an eventually positive decreasing function, we get

$$x^{[1]}(t) \geq x^{[1]}(T) + x^{[2]}(t) \int_T^t p(s)\, ds.$$

As $t \to \infty$, assumption implies the second assertion.

■

**Remark 6.** It is easy to prove that for any fixed $t \geq 0$ holds

$$\int_t^\infty r_1(s) \int_t^s r_2(u) \int_t^u r_3(a)\, da\, du\, ds < \infty \, \textit{if and only if} \, I(r_1, r_2, r_3) < \infty\,. \tag{10}$$

To prove (10), the following auxiliary result will be needed:

If $\int_t^\infty r_1(s) \int_t^s r_2(u) \int_t^u r_3(a)\, da\, du\, ds < \infty$, then

$$\int_t^\infty r_1(s) \int_t^s r_2(u)\, du\, ds < \infty \textit{ and } \int_t^\infty r_1(s)\, ds < \infty. \tag{11}$$

This assertion follows immediately from the fact that

$$\begin{aligned} &\int_t^\infty r_1(s) \int_t^s r_2(u) \int_t^u r_3(a)\, da\, du\, ds \geq \\ &\qquad \geq \left(\int_t^a r_3(b)\, db\right) \left(\int_t^\infty r_1(s) \int_t^s r_2(u)\, du\, ds\right) \geq \\ &\qquad\qquad \geq \left(\int_t^a r_3(b)\, db\right) \left(\int_t^u r_2(c)\, dc\right) \left(\int_t^\infty r_1(s)\, ds\right). \end{aligned}$$

Now, we prove (10). By easy computation we obtain

$$I(r_1, r_2, r_3) = \int_0^\infty r_1(s) \int_0^s r_2(u) \int_0^u r_3(a)\, da\, du\, ds =$$

$$= \int_0^t r_1(s) \int_0^s r_2(u) \int_0^u r_3(a)\, da\, du\, ds + \int_t^\infty r_1(s) \int_t^s r_2(u) \int_t^u r_3(a)\, da\, du\, ds +$$
$$+ \left( \int_0^t r_2(u) \int_0^u r_3(a)\, da\, du \right) \left( \int_t^\infty r_1(s)\, ds \right) +$$
$$+ \left( \int_0^t r_3(a)\, da \right) \left( \int_t^\infty r_1(s) \int_t^s r_2(u)\, du\, ds \right) .$$

And thus (11) implies immediately the assertion (10).
Analogous results hold also for two-dimensional integrals.

**Lemma 5.2** *Let $z$ be a solution of $(N^{\mathcal{A}}, h)$ in the class $\mathcal{M}_1$. Then the following assertions hold:*

*a) If $\lim_{t\to\infty} z^{[1]}(t) \neq 0$, then $\lim_{t\to\infty} |z(t)| = \infty$,*

*b) $\lim_{t\to\infty} z^{[2]}(t) = 0$,*

*c) If $I(q, r, p) = \infty$, then $\lim_{t\to\infty} |z(t)| = \infty$.*

**Proof:** Because $z$ is nonoscillatory solution of $(N^{\mathcal{A}}, h)$ in the class $\mathcal{M}_1$, there exists $T \geq 0$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) < 0$ for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ may be proved by using similar arguments.

*a)* Since $z^{[1]}(t)$ is an eventually positive decreasing function, we have $0 < z^{[1]}(\infty) \leq z^{[1]}(t)$ for all $t \geq T$. Integrating this inequality in $(T, t)$, we obtain

$$z^{[1]}(\infty) \int_T^t p(s)\, ds + z(T) \leq z(t) .$$

As $t \to \infty$, assumption implies the first assertion.

*b)* We assume $\lim_{t\to\infty} z^{[2]}(t) < 0$. This limit exists, because $(N^{\mathcal{A}}, h)$ implies $\left(z^{[2]}(t)\right)' > 0$ for all $t \geq T$, so $z^{[2]}(t)$ is an eventually increasing function. Since $z^{[2]}(t)$ is also negative, we have $0 < -z^{[2]}(\infty) \leq -z^{[2]}(t)$ for all $t \geq T$. Integrating this inequality in $(T, t)$ we obtain

$$-z^{[2]}(\infty) \int_T^t r(s)\, ds \leq z^{[1]}(T) - z^{[1]}(t) ,$$

as $t \to \infty$ we get a contradiction and thus $\lim_{t\to\infty} z^{[2]}(t) = 0$.

*c)* We assume $\lim_{t\to\infty} z(t) < \infty$. This limit exists, because $z^{[1]}(t) > 0$ for all $t \ge T$, so $z(t)$ is an eventually increasing function). Integrating $(N^{\mathcal{A}}, h)$ three times in $(t, \infty)$ and using assertions a), b) of Lemma 5.2, we obtain

$$z(\infty) = z(t) + \int_t^\infty p(s) \int_s^\infty r(u) \int_u^\infty q(b) f(z(h(b)))\, db\, du\, ds\,.$$

Since $0 < z(\infty) < \infty$, in view of the fact that $f$ is a continuous function, there exists positive constant $K$ such that $f(z(h(t))) > K$ for all $t$ sufficiently large, and so we get

$$z(\infty) > z(t) + K \textstyle\int_t^\infty p(s) \int_s^\infty r(u) \int_u^\infty q(b)\, db\, du\, ds =$$

$$= z(t) + K \textstyle\int_t^\infty q(s) \int_t^s r(u) \int_t^u p(b)\, db\, du\, ds\,,$$

which is a contradiction with $I(q, r, p) = \infty$ (see Remark 6) and thus $\lim_{t\to\infty} z(t) = \infty$.

■

Now we state some integral criteria ensuring that $(N, h)$ has property A a $(N^{\mathcal{A}}, h)$ has property B.

**Theorem 5.1** *Assume (H5) and $I(q) = \infty$. Then $(N, h)$ has property A and $(N^{\mathcal{A}}, h)$ has property B.*

**Proof:** *a)* Let us prove that $(N, h)$ has property A.
Let $x$ be a proper nonoscillatory solution of $(N, h)$. We may assume that there exists $T \ge 0$ such that $x(t) > 0$ for all $t \ge T$. The case $x(t) < 0$ for all $t \ge T^*$ may be proved by using similar arguments. We know that $x \in \mathcal{N}_0 \cup \mathcal{N}_2$. Now we assume that $(N, h)$ does not have property A. By Lemma 2.2 there are two possibilities:
*I.* $x \in \mathcal{N}_2$,
*II.* $x \in \mathcal{N}_0$ such that $\lim\limits_{t\to\infty} x(t) = l > 0$.
*Case I.* Since $x$ is positive nonoscillatory solution of $(N, h)$ in the class $\mathcal{N}_2$, there exists $T_1 \ge T$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \ge T_1$. Because $\left(x^{[2]}(t)\right)' = -q(t) f(x(h(t))) < 0$ for all $t \ge T_1$, $x^{[2]}(t)$ is an eventually positive decreasing function and thus $0 < x^{[2]}(\infty) < \infty$. Let $T_2 > T_1$ be such that $h(t) > T_1$ for all $t \ge T_2$. Integrating $(N, h)$ in $(T_2, \infty)$, we obtain

$$x^{[2]}(T_2) - x^{[2]}(\infty) = \int_{T_2}^\infty q(t) f(x(h(t)))\, dt\,.$$

In view of the fact $x^{[2]}(\infty) < \infty$, there exists positive constant $c$ such that

$$c = \int_{T_2}^\infty q(t) f(x(h(t)))\, dt\,. \tag{12}$$

Assertion a) of Lemma 5.1 allows us to use assumption (H5), which implies, there exists positive constant $K_1$ such that $f(x(h(t))) > K_1 x(h(t))$ for all $t \geq T_2$ and thus from (12) we get

$$c > K_1 \int_{T_2}^{\infty} q(t)x(h(t))\,dt\,. \tag{13}$$

Because $x$ is an eventually positive increasing function, so from (13), we obtain

$$c > K_1 x(T_1) \int_{T_2}^{\infty} q(t)\,dt\,,$$

which is a contradiction.
*Case II.* Because $x$ is positive nonoscillatory solution $(N,h)$ in the class $\mathcal{N}_0$, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T_1$. Integrating $(N,h)$ in $(T_1,t)$, we obtain

$$x^{[2]}(t) = x^{[2]}(T_1) - \int_{T_1}^{t} q(s)f(x(h(s)))\,ds\,.$$

Since $0 < x(\infty) < \infty$, in view of the fact that $f$ is a continuous function, there exists positive constant $K$ such that $f(x(h(t))) > K$ for all $t$ sufficiently large, and so, we get

$$x^{[2]}(t) < x^{[2]}(T_1) - K \int_{T_1}^{t} q(s)\,ds\,,$$

which gives a contradiction as $t \to \infty$, because $x^{[2]}(t)$ is positive.

*b)* Let us prove that $(N^{\mathcal{A}},h)$ has property B.
Let $z$ be a proper nonoscillatory solution of $(N^{\mathcal{A}},h)$. We may assume that there exists $T \geq 0$ such that $z(t) > 0$ for all $t \geq T$. The case $z(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $z \in \mathcal{M}_1 \cup \mathcal{M}_3$. Now we assume that $(N^{\mathcal{A}},h)$ does not have property B. By Lemma 2.2 there are two possibilities:
*I.* $z \in \mathcal{M}_3$ such that $\lim\limits_{t\to\infty} z^{[2]}(t) < \infty$,
*II.* $z \in \mathcal{M}_1$.
*Case I.* Since $z$ is positive nonoscillatory solution of $(N^{\mathcal{A}},h)$ in the class $\mathcal{M}_3$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) > 0$ for all $t \geq T_1$. Taking into account that $z^{[2]}(\infty) < \infty$, Lemma 2.2 allows us to use assumption (H5) and $z(t)$ is an eventually positive increasing function, the proof proceeds as in the case I of part a) and hence omitted.
*Case II.* Since $z$ is positive nonoscillatory solution of $(N^{\mathcal{A}},h)$ in the class $\mathcal{M}_1$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) < 0$ for all $t \geq T_1$. Because $\left(z^{[2]}(t)\right)' = q(t)f(z(h(t))) > 0$ for all $t \geq T_1$, so then $z^{[2]}(t)$ is an eventually negative increasing function and thus $-\infty < z^{[2]}(\infty) \leq 0$. Taking

into account that $-\infty < z^{[2]}(\infty) \leq 0$, assertion c) of Lemma 5.2 allows us to use assumption (H5) and $z(t)$ is an eventually positive increasing function, the proof proceeds as in the case I of part a) and hence omitted. The proof is now complete.

■

**Example 3.** We consider the differential equation

$$\left(\frac{1}{t^3}\left(\frac{1}{t^2}x'(t)\right)'\right)' + 90t^2x^3(t^2) = 0\,, \quad t \geq 1 \tag{14}$$

This is the equation of the form $(N, h)$, where $r(t) = t^2$, $p(t) = t^3$, $q(t) = 90t^2$, $h(t) = t^2$ and $f(u) = u^3$. Assumptions of Theorem 5.1 hold and so we know that equation (14) has property A. One nonoscillatory solution of equation (14) such that $|x^{[i]}(t)| \downarrow 0$ *as* $t \to \infty$, $i = 0, 1, 2$ is the function $x(t) = \frac{1}{t^2}$.

□

**Theorem 5.2** *Assume (H5).*

*a) If* $I(q, p, r) = \infty$ *and* $\int_T^\infty q(t) \int_T^{h(t)} r(s)\,ds\,dt = \infty$, *then* $(N, h)$ *has property A.*

*b) If* $I(q, r) = \infty$ *and* $\int_T^\infty q(t) \int_T^{h(t)} p(s)\,ds\,dt = \infty$, *then* $(N^{\mathcal{A}}, h)$ *has property B.*

**Proof:** *a)* Let $x$ be a proper nonoscillatory solution of $(N, h)$. We may assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. The case $x(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $x \in \mathcal{N}_0 \cup \mathcal{N}_2$. Now we assume that $(N, h)$ does not have property A. By Lemma 2.2 there are two possibilities:

*I.* $x \in \mathcal{N}_2$,

*II.* $x \in \mathcal{N}_0$ such that $\lim\limits_{t\to\infty} x(t) = l > 0$.

*Case I.* Since $x$ is positive nonoscillatory solution of $(N, h)$ in the class $\mathcal{N}_2$, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T_1$. Because $\left(x^{[2]}(t)\right)' = -q(t)f(x(h(t))) < 0$ for all $t \geq T_1$, $x^{[2]}(t)$ is an eventually positive decreasing function and thus $0 < x^{[2]}(\infty) < \infty$. Let $T_2 > T_1$ be such that $h(t) > T_1$ for all $t \geq T_2$. Integrating $(N, h)$ in $(T_2, \infty)$, we obtain

$$x^{[2]}(T_2) - x^{[2]}(\infty) = \int_{T_2}^\infty q(t)f(x(h(t)))\,dt\,.$$

In view of the fact $x^{[2]}(\infty) < \infty$, there exists positive constant $c$ such that

$$c = \int_{T_2}^\infty q(t)f(x(h(t)))\,dt\,. \tag{15}$$

Assertion a) of Lemma 5.1 allows us to use assumption (H5), which implies, there exists positive constant $K_1$ such that $f(x(h(t))) > K_1 x(h(t))$ for all $t \geq T_2$ and thus from (15) we get

$$c > K_1 \int_{T_2}^{\infty} q(t) x(h(t))\, dt\,. \tag{16}$$

Because $x^{[1]}(t)$ is an eventually positive increasing function, we have $x^{[1]}(t) \geq x^{[1]}(T_1)$ for all $t \geq T_1$. Integrating this inequality in $(T_1, t)$, we get

$$x(t) \geq x(T_1) + x^{[1]}(T_1) \int_{T_1}^{t} r(s)\, ds > x^{[1]}(T_1) \int_{T_1}^{t} r(s)\, ds \qquad \text{for all } t \geq T_1$$

or

$$x(h(t)) > x^{[1]}(T_1) \int_{T_1}^{h(t)} r(s)\, ds > x^{[1]}(T_1) \int_{T_2}^{h(t)} r(s)\, ds \qquad \text{for all } t \geq T_2$$

Substituting into (16) we obtain

$$c > K_1 x^{[1]}(T_1) \int_{T_2}^{\infty} q(t) \int_{T_2}^{h(t)} r(s)\, ds\, dt\,,$$

which is a contradiction.
*Case II.* Because $x$ is positive nonoscillatory solution $(N, h)$ in the class $\mathcal{N}_0$, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T_1$. Integrating $(N, h)$ three times in $(t, \infty)$, we obtain

$$x(t) = x(\infty) + \int_{t}^{\infty} r(s) \int_{s}^{\infty} p(u) \int_{u}^{\infty} q(a) f(x(h(a)))\, da\, du\, ds\,.$$

Since $0 < x(\infty) < \infty$, in view of the fact that $f$ is a continuous function, there exists positive constant $K_2$ such that $f(x(h(t))) > K_2$ for all $t$ sufficiently large, and so, we get

$$x(t) > x(\infty) + K_2 \int_{t}^{\infty} r(s) \int_{s}^{\infty} p(u) \int_{u}^{\infty} q(a)\, da\, du\, ds =$$

$$= x(\infty) + K_2 \int_{t}^{\infty} q(s) \int_{t}^{s} p(u) \int_{t}^{u} r(a)\, da\, du\, ds\,,$$

which is a contradiction with $I(q, p, r) = \infty$ (see Remark 6).

*b)* Let $z$ be a proper nonoscillatory solution of $(N^{\mathcal{A}}, h)$. We may assume that there exists $T \geq 0$ such that $z(t) > 0$ for all $t \geq T$. The case $z(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $z \in \mathcal{M}_1 \cup \mathcal{M}_3$. Now we assume that $(N^{\mathcal{A}}, h)$ does not have property B. By Lemma 2.2 there are two possibilities:

*I.* $z \in \mathcal{M}_3$ such that $\lim\limits_{t\to\infty} z^{[2]}(t) < \infty$,
*II.* $z \in \mathcal{M}_1$.
*Case I.* Since $z$ is positive nonoscillatory solution of $(N^{\mathcal{A}}, h)$ in the class $\mathcal{M}_3$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) > 0$ for all $t \geq T_1$. Taking into account that $z^{[2]}(\infty) < \infty$, Lemma 2.2 allows us to use assumption (H5) and $z^{[1]}(t)$ is an eventually positive increasing function, in the same way as in the proof the case I of part a) we get a contradiction.
*Case II.* Since $z$ is positive nonoscillatory solution of $(N^{\mathcal{A}}, h)$ in the class $\mathcal{M}_1$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) < 0$ for all $t \geq T_1$. In virtue of $z^{[1]}(t)$ is an eventually positive decreasing and $z^{[2]}(t)$ is an eventually negative increasing, we have

$$0 \leq z^{[1]}(\infty) < \infty \quad \textit{and} \quad 0 \leq -z^{[2]}(\infty) < \infty \,. \tag{17}$$

Integrating $(N^{\mathcal{A}}, h)$ twice in $(t, \infty)$ and from (17) we obtain

$$z^{[1]}(t) = z^{[1]}(\infty) + \int_t^\infty r(s) \int_s^\infty q(u) f(z(h(u)))\, du\, ds \geq$$

$$\geq \int_t^\infty r(s) \int_s^\infty q(u) f(z(h(u)))\, du\, ds \,.$$

Assertion c) of Lemma 5.2 allows us to use assumption (H5), which implies, there exists positive constant $K_3$ such that $f(z(h(t))) > K_3 z(h(t))$ for $t$ sufficiently large and thus we have

$$z^{[1]}(t) > K_3 \int_t^\infty r(s) \int_s^\infty q(u) z(h(u))\, du\, ds >$$

$$> K_3 z(h(t)) \int_t^\infty r(s) \int_s^\infty q(u)\, du\, ds = K_3 z(h(t)) \int_t^\infty q(s) \int_t^s r(u)\, du\, ds \,,$$

which gives a contradiction with $I(q, r) = \infty$ (see Remark 6). The proof is now complete.

$\blacksquare$

The following example illustrates the meaning of Theorem 5.2 .
**Example 4.** We consider the differential equation

$$x'''(t) + \frac{6}{t^2} x(t^2) = 0 \,, \quad t \geq 1 \tag{18}$$

This is the equation of the form $(N, h)$, where $r(t) = p(t) = 1$, $q(t) = 6/t^2$, $h(t) = t^2$ and $f(u) = u$. In this case $I(q) < \infty$ and thus Theorem 5.1 is not applicable. But it is easy to verify that conditions of Theorem 5.2-a) are fulfilled and so we get that equation (18) has property A. One nonoscillatory solution of equation (18) such that $|x^{[i]}(t)| \downarrow 0 \; as \; t \to \infty, \; i = 0, 1, 2$ is the function $x(t) = \frac{1}{t}$.
$\square$

# References

[1] Cecchi M., Došlá Z., Marini M., *On nonlinear oscillations for equations associated to disconjugate operators,* Nonlinear Analysis, Theory, Methods & Applications, Vol. **30**, No.3 (1997), 1583–1594.

[2] Cecchi M., Došlá Z., Marini M., *Comparison theorems for third order differential equations,* Proceeding of Dynamic Systems and Applications, Vol. **2** (1996), 99–106.

[3] Cecchi M., Došlá Z., Marini M., *Asymptotic behavior of solutions of third order delay differential equations,* Archivum Mathematicum(Brno), Tomus **33** (1997), 99–108.

[4] Cecchi M., Došlá Z., Marini M., *Some properties of third order differential operators,* Czech. Math. J. **47** (122) (1997), 729–748.

[5] Cecchi M., Došlá Z., Marini M., *An Equivalence Theorem on Properties A, B for Third Order Differential Equations,* Annali di Matematica pura ed applicata (IV), Vol. CLXXIII, (1997), 373–389.

[6] Kusano T., Naito M., *Comparison theorems for functional differential equations with deviating arguments,* J. Math. Soc. Japan, Vol. **33**, No.3 (1981), 509–532.

**Author's addresses:**
Ivan Mojsej and Ján Ohriska, Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: mojsej@science.upjs.sk
e-mail: ohriska@kosice.upjs.sk