STRUCTURE OF THE RK-ORDER OF P-POINTS

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Our work starts with the following question, posed by A. Blass:

Question ([1]). *What ordinals can be embedded into the RK-ordering of P-points?*

Of course, for this question to make any sense, one needs to assume some axiom which guarantees that there are (sufficiently many) P-points. Typically, axioms such as *MA* or *CH* are assumed. Since the RK-order is witnessed by functions from ω to ω it is immediately obvious that every ultrafilter can have at most $\mathfrak{c}$-many RK-predecessors and that the largest ordinal one can hope to embed is $\mathfrak{c}^+$. The following result of A. Blass shows that $\mathfrak{c}$ is possible:

Theorem ([2]). *Assume MA. The ordinal $\mathfrak{c}$ can be embedded into the RK-ordering of P-points.*

The next step is given by the following theorem of D. Raghavan and S. Shelah whose immediate corollary is that, under MA every $\alpha < \mathfrak{c}^+$ embeds into the RK-ordering of P-points.

Theorem ([4]). *Assume MA. The ordering $\mathcal{P}(\omega)/fin$ can be embedded into the RK-ordering of P-points.*

B Kuzeljević and D. Raghavan ([3]) were then able use CH together with a complicated construction to embed $\mathfrak{c}^+$ into the RK-order of (rapid) P-points. Finally, the D. Raghavan together with the author have shown that the ordering of rapid P-points is, in fact, strongly closed:

Theorem ([5]). *Assume CH. The RK-order of rapid P-points is upwards $\mathfrak{c}^+$-closed!*

We aim to present a complementary theorem due to B. Kuzeljević, D. Raghavan and the author, showing that the ordering is also strongly closed in the other direction:

Theorem. *Assume MA. The RK-order below a $P_\mathfrak{c}$-point is $\mathfrak{c}$-closed.*

References

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