RANDOM FORCING

by

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Boolean algebras allow us to construct models of set theory and to obtain in this way some results concerning Lebesgue measure (mainly independence results). On the other hand, using measure one can construct a Boolean algebra. In this note I want to present some rather elementary results of this kind. Owing to the duality between measure and category the great majority of our results can be almost immediately dualised,

Our terminology and notation are standard. Mostly, we shall follow [2]. We recall some notions.

If B is Boolean algebra then $\mathcal{S}(B)$ is the Stone space of all ultrafilters on B with the topology induced by $\{s(a);\ a \in B\}$, where $s(a) = \{j \in \mathcal{S}(B);\ a \in j\}$. The completion Com (B) of B is the complete Boolean algebra RO($\mathcal{S}$ (B)) of all regular open subsets of $\mathcal{S}$ (B). s is the embedding of B into Com (B).

If B is a complete Boolean algebra then the Boolean universe V^B is defined by $\in$-induction as $V^B = \bigcup_{\alpha \in On} V^B_\alpha$, where

$$V^B_0 = \emptyset$$

$$V^B_\alpha = \bigcup_{\xi<\lambda} V^B_\xi \quad \text{for } \lambda \text{ limit,}$$

$$V_{\alpha+1} = \{\ f;\ dom\,(f) \subseteq V^B_\alpha\ \&\ rng(f) \subseteq B\}$$

The natural embedding " $\vee$ " of V into V^B is defined by induction:

$$\check{x}(\check{y}) = 1 \quad \text{for} \quad y \in x.$$

The Boolean value $\|\varphi(x_1,\ldots,x_n)\|_B$ is defined in [2] for $x_1,\ldots,x_n \in V^B$ (the subscript B is usually omitted). $V^B \models \varphi$ means that $\|\varphi\|_B = 1$. Instead of $V^B \models \varphi(\check{x}_1,\ldots,\check{x}_n)$ we write simply $V^B \models \varphi(x_1,\ldots,x_n)$ (for $x_1,\ldots,x_n \in V$). V is often considered as a subclass of V^B.

A class M is called a model-class iff all the axioms of set theory (including the axiom of choice are true in the model $M, \in \restriction M$. A notion $\square$ relativised to this model is denoted by $\square^M$.

If D is a complete subalgebra of B then V^D is a model-class in V^B..

If M is a model-class, $B \in M$ is a Boolean algebra, then B is said to be M-complete iff the union $\vee X$ exists for every $X \subseteq B$, $X \in M$. In a similar way the notion of a M-complete homomorphism can be defined.

§ 1. ITERATION AND FORCING

a) Minimal product of complete Boolean algebras

Let $j_i : B_i \to B$ be a complete embedding of a complete Boolean algebra B_i into a complete Boolean algebra B, i = 1,2. The triple B, j_1, j_2 (or simply B, when j_1, j_2 are understood) is called a product of B_1, B_2 iff

i) $j_1(B_1)$, $j_2(B_2)$ are independent, i.e.

$j_1(a_1) \wedge j_2(a_2) \neq 0$ for $a_1 \in B_1, a_2 \in B_2, a_1 \neq 0$, $a_2 \neq 0$.

ii) $j_1(B_1) \cup j_2(B_2)$ (completely) generates B.

If moreover

iii) $\{j_1(a_1) \wedge j_2(a_2),\ a_1 \in B_1 \ \& \ a_2 \in B_2\}$ is a dense subset of B, then B, j_1, j_2 is called the minimal product.

One can easily see that the minimal product is (up to isomorphism) unique. The existence is also clear, just set

$$B = RO(\mathcal{S}(B_1) \times \mathcal{S}(B_2)),$$

$$j_1(a) = s(a) \times \mathcal{S}(B_2),$$

$$j_2(a) = \mathcal{S}(B_1) \times s(a)$$

We shall often identify B_i with $j_i(B_i)$. The minimal product of B_1 and B_2 will be simply denoted by $B_1 \otimes B_2$.

The reader may compare some details with [2] (where the product is denoted by $B_1 \oplus B_2$) and [6].

b) Iteration of forcing

Let B be a complete Boolean algebra, $D \in V^B$ such that $V^B \models$ D is a complete Boolean algebra.

Then we can construct the Boolean - valued model $(V^D)^{V^B}$ inside the model V^B. It is known that this model is again (isomorphic to) a Boolean - valued model V^{B*D}, where $B * D$ is a suitable complete Boolean algebra (see e.g. [2], [8]. In the following we shall need some further information and hence, we must describe such iteration. We use a method different from those known in the literature. As far as I know the method was used first by me in 1967 after appearing, in the paper [10] by P. Vopěnka and is based on a well-known idea by S. Kripke [3].

Let Col_α be the algebra $RO({}^{\omega}o_\alpha)$, the topology on ${}^{\omega}o_\alpha$ being the product topology. Thus, Col_α is the standard collapsing algebra. By the Kripke's theorem [3], [4], every complete Boolean algebra is (isomorphic to) a subalgebra of the algebra Col_α for a sufficiently large cardinal α.

Assume that α is sufficiently large. Then the algebra B is a subalgebra of Col_α. The model V^B is a model-class of the model V^{Col_α}. Moreover, the V^B - complete Boolean algebra D is small in V^{Col_α}. One can easily find an $f \in V^{Col_\alpha}$ such that for every $x \in V^B$, $V^B \models x \in D \,\&\, x \neq 0$ we have $\|f = x\|_{Col_\alpha} \neq 0$. By the Rasiowa-Sikorski lemma (applied inside V^{Col_α} - α is sufficiently large, thus "everything" is countable) there exists an $F \in V^{Col_\alpha}$ such that

$$V^{Col_\alpha} \models F \text{ is } V^B \text{ - generic ultrafilter on } D \text{ and } f \in F.$$

Now, we set (T. Jech denotes this algebra by $D * B$) :

$$B * D = \{ \|x \in F\|_{Col_\alpha} \; ; \quad x \in V^B \,\&\, \| x \in D\|_B = 1 \} .$$

For $a \in B$, let $\hat{a} \in V^{Col_\alpha}$ be such that $\|\hat{a} = 1\| = a$, $\|\hat{a} = 0\| = -a$, Then " ^ " induces a complete embedding of B into $B * D$.

One can easily check that $B * D$ is a complete Boolean algebra.

The isomorphism $x \to \bar{x}$ of $(V^D)^{V^B}$ onto $V^{B * D}$ is defined by $\in$-induction as follows.

Let $x \in (V^D)^{V^B}$, i.e. $x \in V^B$ and $\|x \in V^D\|_B = 1$.

We set

$$\bar{x}(\bar{y}) = \| a \in F\|_{Col_\alpha} \quad \text{for} \quad y \in \text{dom}(x),$$

where $\| x(y) = a\|_B = 1$ and $\| a \in D\|_B = 1$

c) Relation between * and $\otimes$

If B, D are complete Boolean algebras, then

$$V^B \models \check{D} \text{ is a Boolean algebra.}$$

We can construct the completion $Com(\check{D})$ in V^B and then $B * Com(\check{D})$. We show that

$$B * Com(\check{D}) = B \otimes D$$

We shall follow the notation of part b). The embedding j_1 is defined by

$$j_1(a) = \| \hat{a} \in F\| \quad \text{for} \quad a \in B$$

The embedding j_2 is defined by

$$j_2(a) = \| \check{a} \in F\| \quad \text{for} \quad a \in D$$

If $a \in B$, $a \neq 0$, $b \in D$, $b \neq 0$, then

$j_1(a) \wedge j_2(b) = \| \hat{a} \in F\| \wedge \| \check{b} \in F \| = \| \hat{a} \wedge \check{b} \in F\| \neq 0$,

because $\| \hat{a} \wedge \check{b} \in F\| \geq \| \hat{a} \wedge \check{b} = f \|$.

If $a \in B * Com(\check{D})$ then $a = \| x \in F\|$ for some x such that

$$V^B \models x \in Com(\check{D}).$$

Since $\check{D}$ is dense in $Com(\check{D})$, there exists a $y \in D$, $y \neq 0$ such that $\| \check{y} \leq x\|_B \neq 0$. Then

$$j_1(\| \check{y} \leq x\|_B) \wedge j_2(y) \leq \| \check{y} \leq x\|_{Col_\alpha} \wedge \| \check{y} \in F\| \leq \|x \in F\|$$

i.e. iii) of a) is true.

d) Solovay's lemma

We remind the reader of an important result of R.Solovay (see [2] and [8]) :

Assume that B_ξ, $\xi < \alpha$ are complete Boolean algebras, each B_ξ is C.C.C., B_ξ is a complete subalgebra of B_η for $\xi < \eta$ and $B_\lambda = Com(\bigcup_{\xi<\lambda} B_\xi)$ for λ limit. Then $\bigcup_{\xi<\alpha} B_\xi$ is C.C.C., moreover, if $cf(\alpha) > \aleph_0$ then $\bigcup_{\xi<\alpha} B_\xi$ is complete.

e) Solovay-Tennenbaum closure

R. Solovay and S. Tennenbaum in [8] constructed (for a given cardinal α such that $\alpha^\beta = \alpha$ for $\beta < \alpha$) a complete C.C.C. Boolean algebra D such that Martin's axiom MA is true in V^D and $2^{\aleph_0} = \alpha$ in V^D.

We can realize this construction inside a model V^B. Thus, let B be a C.C.C. complete Boolean algebra, α be a cardinal such that

$$V^B \models (\forall \beta < \check{\alpha})\ \check{\alpha}^\beta = \check{\alpha}$$

Let $D \in V^B$ be such that

$$V^B \models (V^D \models \text{MA} \ \& \ 2^{\aleph_0} = \check{\alpha}).$$

We write

$$ST_\alpha(B) = B * D .$$

$ST_\alpha(B)$ is a C.C.C. complete Boolean algebra such that

$$V^{ST_\alpha(B)} \models 2^{\aleph_0} = \alpha \ \& \ \text{MA}.$$

In this definition the cardinal α can be replaced by an $f \in V^B$ such that

$$V^B \models f \text{ is a cardinal } \& \ (\forall \beta < f)\ f^\beta = f$$

Evidently, there exists a first cardinal $f \in V^B$ such that

$$V^B \models (\forall \beta < f)\ f^\beta = f.$$

In this case the algebra $ST_f(B)$ is denoted for short by $ST(B)$ and is called the Solovay-Tennenbaum closure of B.

The algebra $ST_\alpha(B)$ may be defined similarly as is done in [8]. We define the sequence B_ξ, $\xi < \alpha$ as in [2] or [8], just changing

$$B_0 = B.$$

§ 2. RANDOM AND COHEN REALS

a) The Boolean algebras R and C

For any finite set X we denote by $\mathbb{I}_X$ the generalized Cantor set ${}^X2$ endowed with the product topology. Let B_X denote the smallest σ-field of subsets of $\mathbb{I}_X$ containing the open sets, i.e. B_X is the σ-field of Borel subsets of $\mathbb{I}_X$.

From the measure v on 2 ($v(\{0\})=v(\{1\})=\frac{1}{2}$), one can easily construct a σ-additive measure μ_X on $\mathbb{I}_X$ such that every Borel set is μ_X-measurable.

If $\varphi : A \to 2$, $A \subseteq X$, we write

$$\theta_\varphi = \{ x \in {}^X2 ; \varphi \subseteq x \}$$

The set $\{\theta_\varphi ; \operatorname{dom}(\varphi)$ is a finite subset of $X\}$ is a basis for the topology on $\mathbb{I}_X$. The μ_X-measure of such a set θ_φ is equal to

$$\mu_X(\theta_\varphi) = (\tfrac{1}{2})^{\overline{\overline{\operatorname{dom}(\varphi)}}}$$

For $\xi \in X$ we write

$$\mathcal{G}_\xi = \{ x \in \mathbb{I}_X , x(\xi) = 1 \} = \theta_{\{\langle\xi,1\rangle\}}$$

C_X denotes the complete Boolean algebra of regular open subsets of $\mathbb{I}_X$. Evidently C_X is isomorphic to the algebra B_X modulo the ideal of meager sets. R_X denotes the complete Boolean algebra B_X modulo the ideal of set μ_X-zero sets.
Both algebras are C.C.C.
We shall idenfity elements of B_X with the corresponding elements of C_X and R_X, respectively.

If $X_1 \subseteq X_2$ then one can naturally define a mapping from $\mathbb{I}_{X_2}$ onto $\mathbb{I}_{X_1}$, which induces complete embeddings of C_{X_1} into C_{X_2} and R_{X_1} into R_{X_2}.

If $X = \omega_0$ we shall often omit the subscript X and simply write $\mathbb{I}$, B, μ, C, R. Thus, e.g. $\mathbb{I}$ is the set of reals and μ is Lebesque measure on $\mathbb{I}$.

b) Solovay's coding of Borel sets

R. Solovay [7] defined a natural coding of Barel subsets of $\mathbb{I}$ by the reals: for each $a \in \mathbb{I}$ a Borel set B_a is defined.

If M is a model-class, one can define a mapping $\#$ from B^M into B as follows :

$$\#(B_a^M) = B_a \quad \text{for} \quad a \in \mathbb{I} \cap M.$$

R. Solovay has shown that for any $a, b, b_n \in M$, $\{b_n ; n \in \omega_0\} \in M$ the following holds :

1) $M \models B_a^M$ is meager $\equiv$ B_a is meager,

2) $M \models B_a^M$ has measure zero $\equiv$ B_a has measure zero,

3) $M \models B_a^M = - B_b^M \equiv B_a = - B_b$,

4) $M \models B_a^M \subseteq B_b^M \equiv B_a \subseteq B_b$,

5) $M \models B_b^M = \bigcup_n B_{b_n}^M \equiv B_b = \bigcup_n B_{b_n}$

Thus, the mapping $\#$ induces an M-complete embedding of C^M (R^M) into C (in R). Since every element $a \in C_X$ belongs to some $C_Y \subsetneq C_X$ with Y countable, $Y \subsetneq X$, one can easily define an M-complete embedding of C_X^M into C_X. Similarly for R_X^M and R_X.

Since every Borel set is equal to a G_δ-set up to a measure-zero set, it suffices to deal with G_δ-sets. We describe a coding for such sets.

The countable set $\{\theta_\varphi ; \varphi \in \bigcup_n {}^n 2\}$ is a basis for the topology on $\mathbb{I}$. Let $\{\Xi_n ; n \in \omega_0\}$ be a fixed enumeration of this basis. For a real $a \in \mathbb{I}$ we set

$$\Omega_a = \bigcup \{\Xi_n ; a(n) = 1\}$$

The set Ω_a is open and every open subset of $\mathbb{I}$ is equal to an Ω_a for some $a \in \mathbb{I}$.

Let π be a fixed one-to-one mapping from $\omega_0 \times \omega_0$ onto ω_0. We set

$$\Gamma_a = \bigcap_n \bigcup_m \{\Xi_m ; a(\pi(n,m) = 1\}$$

Then Γ_a is G_δ-set and every G_δ-subset of $\mathbb{I}$ is of this form.

Evidently $\#(\Omega_a^M) = \Omega_a$, $\#(\Gamma_a^M) = \Gamma_a$ for $a \in M$.

c) <u>Random and Cohen reals</u>

Let M be a model-class. A real $r \in \mathbb{I}$ is said to be <u>M-random</u> or random over M (or simply random when the class M is understood) iff for every $x \in \mathbb{I}^M = \mathbb{I} \cap M$ for which $\mu(\Gamma_x)=0$ we have $r \notin \Gamma_x$. Similarly, a real $c \in \mathbb{I}$ is said to be <u>M-Cohen</u> (Cohen over M, Cohen) iff for every $x \in \mathbb{I}^M$ for which $\mathbb{I} - \Gamma_x$ is meager, we have $c \in \Gamma_x$.

If G is an M-generic ultrafilter on C^M (on R^M, then the real x defined by $x(n)=1 \equiv g_n \in G$ is an M-Cohen

(M-random) real and vice versa, if x is an M-Cohen (M-random) real then there exists an M-generic ultrafilter G on C^M (on R^M) such that $x(n)=1 \equiv g_n \in G$ (just set $G = \{X \in B^M;\ x \in \#(X)\}$),

If y is a real, the real x is said to be M-random over y (M-Cohen over y) iff x is $M(y)$-random ($M(y)$-Cohen). Let us remark that as usual $M(A)$ denotes the smallest model-class such that $M \subseteq M(A)$, $A \in M(A)$,

If G is an M-generic ultrafilter over C_Y^M and $X \subseteq Y$ is countably infinite, one can define an M-Cohen real C_X as follows :

$$C_X(n) = 1 \equiv g_{\xi_n} \in G_Y \text{ where } X = \{\xi_n\ ;\ n \in \omega_0\}$$

Similarly for R_Y^M and r_X .

It is well known that in the model V^{C_X} the following holds

$\mathbb{I} \cap V$ has measure zero and does not possess the Baire property.

On the other hand, the following holds in V^{R_X} :

$\mathbb{I} \cap V$ is meager and non-measurable

(see e.g. [1], [11]).

d) Products

Let B be a complete Boolean algebra. We can construct $C_X^{V^B}$ and $R_X^{V^B}$ in the model V^B. The algebras $B * C_X^{V^B}$, $B * R_X^{V^B}$ will be simply denoted by $B * C_X$, $B * R_X$ respectively.

We already know that $\check{C}_X$ is a V-subalgebra of $C_X^{V^B}$. One can prove that $B \cup C_X$ generates $B * C_X$, thus $B * C_X$ is a product of B and C_X. Similarly, $B * R_X$ is a product of B and R_X.

Since $\check{C}_X$ is a dense subalgebra of $C_X^{V^B}$ (a basis of C_X is defined by finite sequence of zero and ones !), we obtain

$$B * C_X = B \otimes C_X$$

The similar equality for R_X need not be true. The set $\{s(a)\ s(b):\ a \in C_X, b \in C_Y\}$ is a basis for the product topology on $\mathcal{S}(C_X) \times \mathcal{S}(C_Y)$. Thus, $C_{X \cup Y}$ is the minimal product $C_X \otimes C_Y$ $(X \cap Y = \emptyset)$.
The algebra $R_{X \cup Y}$ need not be the minimal product of R_X and R_Y. We sketch a proof of the inequality

$$R_{\omega_0+\omega_0} \neq R_{\omega_0} \otimes R_{\omega_0} .$$

The idea of the proof is based on an observation made by R. Sikowski [6], p. 189–190.

H. Steinhaus [9] has proved the following theorem: if $A,B \subseteq \mathbb{I}$, $\mu(A) > 0$, $\mu(B) > 0$, then there exists a non-empty open set C such that $C \subseteq \{|x - y|;\ x \in A,\ y \in B\}$

Now, let $D \subseteq \mathbb{I}$ be a closed nowhere dense subset of $\mu(D) > 0$. We set

$$X = \{[x_1,x_2] \in \mathbb{I}_{\omega_0+\omega_0},\ |x_1 - x_2| \in D\}$$

Then X is a non-zero element of $R_{\omega_0+\omega_0}$. Assume that $\{[a,b];\ a,b \in R\}$ is dense in $R_{\omega_0+\omega_0}$. Then there exists two sets $A,B \subseteq \mathbb{I}$, $\mu(A) > 0$, $\mu(B) > 0$ such that

$$\mu_{\omega_0+\omega_0}(X - A \times B) = 0$$

Write
$$A' = A - \bigcup \{\Xi_n,\ \mu(A \cap \Xi_n) = 0\}$$

$$B' = B - \bigcup \{\Xi_n,\ \mu(B \cap \Xi_n) = 0\}$$

By the Steinhaus theorem there exists a non-empty open set C such that

$$C \subseteq \{|x - y|,\ x \in A',\ y \in B'\}$$

Since

$$\{|x - y|;\ [x,y] \in X\} = D,$$

D is nowhere dense, C is not a subset of D.

Therefore

$$A' \times B' \not\subseteq X.$$

The closed X may expressed as

$$X = \mathbb{I}_{\omega_0+\omega_0} - \bigcup \{\Xi_n \times \Xi_m;\ [n,m] \in W\}$$

Thus, there exists a couple $[n,m] \in W$ such that $(A' \times B') \cap \cap (\Xi_n \times \Xi_m) \neq \emptyset$

From the definition of A' and B' we obtain

$$\mu(A' \cap \Xi_n) > 0, \quad \mu(B' \cap \Xi_m) > 0$$

and therefore

$$\mu(A \times B - X) \geq \mu(A' \times B' - X) \geq \mu((A' \cap \Xi_n) \times (B' \cap \Xi_m)) > 0$$

– a contradiction.

e) Relations between reals

Let a be a real, X being infinite subset of ω_0 . We define a new real a_X as follows.

Let $X = \{k_n, n \in \omega_0\}$ be the order preserving enumeration of X. We define

$$a_X(n) = a(k_n).$$

If $a|X$ denotes the restriction of the function a to the set X then difference between $a|X$ and a_X is inessential; in fact, for any model class M containing the set X, we have

$$M(a|X) = M(a_X)$$

If c is an M-Cohen real, $X \in M$, $X \subseteq \omega_0$, X infinite, then c_X is also an M-Cohen real - if $c(n)=1 \equiv g_n \in G$, where G is an M-generic ultrafilter over C, then

$$c_X(n) = 1 \equiv g_{k_n} \in G \cap C_X .$$

If $\omega_0 = X \cup Y$, $X \cap Y = \emptyset$, X,Y being infinite, then both c_X and c_Y are Cohen reals. By the results of the part d), we have

$$C_{\omega_0} = C_{X \cup Y} = C_X \otimes C_Y = C_X * C_Y.$$

Thus, c_Y is also Cohen over c_X.

Let us replace the condition " $X \cap Y = \emptyset$ " by "$X \cap Y$ is finite". Then $M(c_X) = M(c_{X-Y})$, c_Y is Cohen c_{X-Y}. Thus, c_Y is also Cohen over c_X.

If $\mathfrak{F}$ is a family of almost disjoint subsets of ω_0 then $\{c_X;\ X \in \mathfrak{F}\}$ is a family of mutually Cohen reals.

Similar results hold for random reals.

It suffices to show the following :

if $\omega_0 = X \cup Y$, $X \cap Y = \emptyset$, X, Y infinite, r is random, then r_X is random over r_Y.

Assume not. Then there exists a real $Z \in V^{R_Y}$ such that

$$V^R \models \mu(\Omega_Z) < \tfrac{1}{2} \ \&\ r_X \in \Omega_Z$$

By definition

$$\| r_X \in \Omega_Z \|_R = \bigvee_n \| Z(n) = 1 \|_R \wedge \| r_X \in \Xi_n \|_R$$

Since ${}^{\omega_0}2 = {}^{X}2 \times {}^{Y}2$, $\| Z(n) = 1 \|_R \in R_Y$, there are sets $A_n \subseteq {}^{Y}2$ such that $\| Z(n) = 1 \| = {}^{X}2 \times A_n \subseteq {}^{\omega_0}2$.

Evidently,

$$\| r_X \in \Xi_n \|_R = \| r_X \in \Xi_n \|_{R_X} \times {}^{Y}2 = \Xi_n \times {}^{Y}2$$

Thus

$$\| r_X \in \Omega_Z \|_R = \bigcup_n \Xi_n \times A_n$$

The measure μ_{ω_0} is a product of measures. The product of measures is associative, thus

$$\mu_{\omega_0} = \mu_X \times \mu_Y$$

We have assumed $\| r_X \in \Omega_Z \| = 1$, i.e.

$$\mu_{\omega_0}\Big(\bigcup_n \Xi_n \times A_n\Big) = 1$$

Let

$$A'_n = A_n - \bigcup\{\bigcap_{i \in W} A_i,\ W \subseteq \{0,\dots,n\}\ \&\ \mu\Big(\bigcap_{i \in W} A_i\Big) = 0\}$$

Then $\mu(A'_n) = \mu(A_n)$ and also

$$\mu\Big(\bigcup_n \Xi_n \times A'_n\Big) = 1$$

There exists a point $y \in {}^{Y}2$ such that

$$\mu_X(\{x,\ [x,y] \in \bigcup_n \Xi_n \times A'_n\}) = 1$$

Let k be such that

$$\mu_X(\{x;\ [x,y] \in \bigcup_{n=0}^{k} \Xi_n \times A'_n\}) > \frac{1}{2}.$$

Let $W = \{n \leq k;\ y \in A'_n\}$. Then

$$\mu_X(\bigcup\{\Xi_n;\ n \in W\}) > \frac{1}{2}.$$

But

$$\| \mu(\Omega_Z) > \tfrac{1}{2} \|_R \geq \| (\forall n \in W) Z(n) = 1 \| = \bigcap_{n \in W} \| Z(n) = 1 \| =$$

$$= {}^{X}2 \times \bigcap_{n \in W} A_n .$$

Since $\bigcap_{n\in W} A'_n \neq \emptyset$, we obtain $\mu(\bigcap_{n\in W} A_n) = \mu(\bigcap_{n\in W} A'_n) > 0$ – a contradiction.

f) A random real produces a new Cohen real

From d) one can easily see that in $M(r)$, r being M-random, there is no M-Cohen real and vice versa, in $M(c)$, c being M-Cohen, there is no M-random real. However we can prove the following :

Let G be an M-generic ultrafilter on $R \otimes C$, r, c being the corresponding M-random and M-Cohen reals, respectively. Then there exists an M-Cohen real $\bar{c} \in M(r,c) = M(G)$ such that $\bar{c} \notin M(c)$.

We start with $M^{R_\alpha * C}$, α being sufficiently large. Then in M^{R_α}, there exists a family $\mathcal{F}$ of almost disjoint subsets of ω_0 of cardinality α. Therefore, the family of M-Cohen reals $\{c_X ; X \in \mathcal{F}\}$ has cardinality α. Since in $M(c)$ there is only a small number of M-Cohen reals, there exists an M-Cohen real $\bar{c} \in M^{R_\alpha * C}$ such that $\bar{c} \notin M(c)$. Evidently, $\bar{c} \in M^{R_X * C}$ for some countable $X \subseteq \alpha$.

§ 3. CARDINAL CHARACTERISTICS OF LEBESGUE MEASURE AND THE BAIRE PROPERTY

a) Additivity of measure

A measure ν is said to be α - additivite iff for every sequence $\{ A_\xi, \xi < \alpha \}$ of pairwise disjoint measurable sets, the union $\bigcup_{\xi<\alpha} A_\xi$ is measurable and

$$\nu(\bigcup_{\xi<\alpha} A_\xi) = \sum_{\xi<\alpha} \nu(A_\xi) .$$

Since at most countable number of A_ξ's may have positive measure, one can easily see that ν is α - additive iff for every sequence $\{ A_\xi, \xi < \alpha \}$ of measure zero sets the union $\bigcup_{\xi<\alpha} A_\xi$ has measure zero.

We write

ALM = the first α such that Lebesgue measure μ is not α- additive.

Evidently ALM is regular and $\aleph_0 < \text{ALM} \leq 2^{\aleph_0}$.

Let us remark that Martin's axiom AM implies $\text{ALM} = 2^{\aleph_0}$ (see Martin-Solovay [5]).

Similarly, we define

AFC = the first α such that there exists a sequence $\{A_\xi ; \xi \in \alpha\}$ of meager subsets of $\mathbb{I}$ such that $\bigcup_{\xi<\alpha} A_\xi$ is not meager.

Then AFC is regular and $\aleph_0 < \text{AFC} \leq 2^{\aleph_0}$.

Also

$$\text{MA} \longrightarrow \text{AFC} = 2^{\aleph_0}$$

b) Smallest nonmeasurable set and partition

We define

SNM = the first α such that there is a nonmeasurable set $X \subseteq \mathbb{I}$ of cardinality α,

SNB = the first α such that there is a set $X \subseteq \mathbb{I}$, $\overline{\overline{X}} = \alpha$ such that X does not possess the Baire property,

PZS = the first α such that there is a sequence $\{A_\xi ; \xi < \alpha\}$ of measure zero subsets of $\mathbb{I}$ such that $\bigcup_{\xi<\alpha} A_\xi = \mathbb{I}$

PFC = the first α such that there is a sequence $\{A_\xi ; \xi < \alpha\}$ of meager subsets of $\mathbb{I}$ such that $\bigcup_{\xi<\alpha} A_\xi = \mathbb{I}$.

c) Classical inequalities

One can easily prove :

ALM is regular,

AFC is regular,

$$\aleph_0 < \text{ALM} \leq \text{cf(SNM)} \leq \text{SNM} \leq 2^{\aleph_0}$$

$$\aleph_0 < \text{AFC} \leq \text{cf(SNB)} \leq \text{SNB} \leq 2^{\aleph_0}$$

$$\text{ALM} \leq \text{cf(PZS)} \leq \text{PZS} \leq 2^{\aleph_0}$$

$$\text{AFC} \leq \text{cf(PFC)} \leq \text{PFC} \leq 2^{\aleph_0}$$

d) Standard mistake

At first sight it seems that ALM = PZS. It is easy to show the following fact :

if $\{ A_\xi ; \xi < \alpha \}$ is a sequence of measure zero sets such that $\mu(\bigcup_{\xi<\alpha} A_\xi) > 0$, then PZS $\leq \alpha$.
(For every n, there exists an interval B_n such that

$$\mu(B_n \cap \bigcup_{\xi<\alpha} A_\xi) \geq \frac{n}{n+1} \mu(B_n) ;$$

let f_n be a linear mapping of B_n onto $\mathbb{I}$. Then

$$\mu(\bigcup_{\xi<\alpha} \bigcup_{n<\omega_0} f_n(A_\xi \cap B_n)) = 1.)$$

However from ALM = α does not follow the existence of such a sequence : $\bigcup_{\xi<\alpha} A_\xi$ does not have measure zero but may be nonmeasurable (that is the case !).

§4. CONSISTENCY RESULTS

In this paragraph we assume the generalized continuum hypothesis.

a) ALM can be any regular cardinal

Let $\aleph_0 < \alpha \leq \beta$, $cf(\beta) > \aleph_0$, α being regular. We construct a model in which ALM = α and $2^{\aleph_0} = \beta$.

Set
$$\mathfrak{A}^0 = C_\beta$$
$$\mathfrak{A}^{\xi+1} = ST(\mathfrak{A}^\xi * C)$$
$$\mathfrak{A}^\lambda = Com(\bigcup_{\xi<\alpha}\mathfrak{A}^\xi) \text{ for } \lambda \text{ limit,}$$
and
$$\mathfrak{A}_{\alpha,\beta} = \bigcup_{\xi<\alpha}\mathfrak{A}^\xi$$

One can easily show by induction that $\mathfrak{A}^\xi$ is C.C.C. and $V^{\mathfrak{A}^\xi} \models 2^{\aleph_0} = \beta$. Thus, $\mathfrak{A}_{\alpha,\beta}$ is a complete C.C.C. Boolean algebra and $2^{\aleph_0} = \beta$ is true in $V^{\mathfrak{A}_{\alpha,\beta}}$

Evidently

$$\mathbb{I} = \bigcup_{\xi<\alpha} \mathbb{I} \cap V^{\mathfrak{A}^\xi}$$

Since $\mathfrak{A}^{\xi} * C \subseteq \mathfrak{A}^{\xi+1} \subseteq \mathfrak{A}_{\alpha,\beta}$ and $V^{\mathfrak{A}^{\xi} * C} \models \mu(\mathbb{I} \cap V^{\mathfrak{A}^{\xi}}) = 0$, we obtain

$$\mu(\mathbb{I} \cap V^{\mathfrak{A}^{\xi}}) = 0 \quad (\text{in } V^{\mathfrak{A}_{\alpha,\beta}})$$

and hence

$$\text{ALM} \leq \alpha$$

On the other hand, let $A_{\xi} \subseteq \mathbb{I}$, $\xi < \gamma < \alpha$, $\mu(A_{\xi}) = 0$ (in $V^{\mathfrak{A}_{\alpha,\beta}}$). We assume that $A_{\xi} \subseteq \Gamma a_{\xi}$, $\mu(\Gamma a_{\xi}) = 0$, $a_{\xi} \in \Gamma$. Every a_{ξ} (as a Boolean function) is a function from ω_0 into $\mathfrak{A}_{\alpha,\beta}$. Since $\alpha > \aleph_0$, α regular, $\gamma < \alpha$, there exists a $\xi_0 < \alpha$ such that

$$a_{\xi} \in V^{\mathfrak{A}^{\xi}} \text{ for each } \xi < \gamma$$

In the model $V^{\mathfrak{A}^{\xi+1}}$, Martin's axiom is true; thus (as Lebesgue measure is $<\beta$-additive) there exists a real a such that $\mu(\Gamma_a) = 0$ and $\Gamma_{a_{\xi}} \subseteq \Gamma_a$ for each $\xi < \gamma$ (everything in $V^{\mathfrak{A}^{\xi_0+1}}$). Then also in $V^{\mathfrak{A}_{\alpha,\beta}}$ the union $\bigcup_{\xi<\gamma} A_{\xi} \subseteq \bigcup_{\xi<\gamma} \Gamma_{a_{\xi}} \subseteq \Gamma_a$ has measure zero.

Thus

$$\text{ALM} = \alpha$$

Let us remark that in this model PZS = ALM is true.

b) <u>AFC can be any regular cardinal</u>

The construction is same as in a), just replace C by R, i.e.

$$\mathfrak{B}^{0} = R_{\beta},$$

$$\mathfrak{B}^{\xi+1} = ST(\mathfrak{B}^{\xi} * R),$$

$$\mathfrak{B}^{\lambda} = Com(\bigcup_{\xi<\lambda} \mathfrak{B}^{\xi}), \quad \lambda \text{ limit}$$

and

$$\mathfrak{B}_{\alpha,\beta} = \bigcup_{\xi<\alpha} \mathfrak{B}^{\xi}$$

Also AFC = PFC.

c) <u>ALM $\neq$ PZS</u>

Let $cf(\beta) > \aleph_0$. Then in $V^{R_{\beta}}$, $2^{\aleph_0} = \beta$ and PMZ $\leq \beta$. We show that actually PZS = β.

Let $\{x_\xi, \xi < \gamma\}$, $\gamma < \beta$ be a sequence of reals such that $\mu(\Gamma_{x_\xi}) = 0$ for each $\xi < \gamma$. Since $\gamma < \beta$, there exists an infinite countable set $T \subseteq \beta$ such that $x_\xi \in V^{\mathbb{R}_{\beta - T}}$ for each $\xi < \gamma$. Let T_ξ denote an infinite countable set such that $x_\xi \in V^{\mathbb{R}_{T_\xi}}$.

Since r_T is random over r_{T_ξ}, we obtain

$$r_T \notin \Gamma_{x_\xi}$$

Thus

$$\bigcup_{\xi<\gamma} \Gamma_{x_\xi} \neq \mathbb{I}$$

In the model $V^{\mathbb{R}_\beta}$, the set $\mathbb{I} \cap V$ is nonmeasurable and $\overline{\overline{\mathbb{I} \cap V}} = \aleph_1$. Thus

$$ALM = \aleph_1$$

d) AFC $\neq$ PFC

Construct the model V^{C_β} and use arguments dual to those of part e) ❙

e) Relation between PZS and SNM

In V^{R_β}, $SNM = \aleph_1$, $PZS = \beta$ is true. Thus

$$SNM < PZS$$

for $\beta > \aleph_1$.

On the other hand, it is well known (see Martin-Solovay [5]) that

$$V^{C_\beta} = SNM = \beta \ \& \ PZS = \aleph_1$$

i.e.

$$SNM > PZS.$$

f) ALM $\neq$ SNM

Evidently ALM $\leqslant$ SNM. In the model V^{C_β} (from e)), we have SNM = β and ALM = $\aleph_1$ ($\leqslant$ PZS).

Thus

$$ALM < SNM.$$

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